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DIRICHLET BOUNDARY-VALUE PROBLEM FOR THE POLYANALYTIC EQUATION IN A RECTANGULAR DOMAIN

Abstract. In this paper, we consider the Dirichlet boundary-value problem for higher-order polyanalytic equations in a rectangular domain. First, the parqueting-reflection principle is used to construct an explicit solution to the Dirichlet problem. The boundary behavior of the related integral operators is then studied in detail. Moreover, we investigate the Dirichlet boundary-value problems for both homogeneous and inhomogeneous Cauchy–Riemann equations. Finally, we discuss the solvability conditions and solve the Dirichlet boundary-value problem for the inhomogeneous polyanalytic equation in a rectangular domain.

Key words: *boundary-value problem, Dirichlet problem, polyanalytic equation, rectangular domain*

2020 Mathematical Subject Classification: *30E25, 31A10, 35C15, 35F15, 35J25, 45E05*

1. Introduction. Boundary-value problems for partial differential equations constitute a fundamental area of mathematical research, with important applications in diverse fields, such as complex analysis, functional analysis, potential theory, elasticity theory, shell theory, quantum mechanics, thermodynamics, and [1]–[21]. In general, a boundary-value problem consists of determining a solution to a partial differential equation that satisfies prescribed conditions on the boundary of a given domain. Due to its significant role in both theory and applications, this topic has remained an active area of research, especially in the context of complex partial differential equations. The primary objectives in this field include establishing the existence and uniqueness of solutions, as well as investigating their qualitative behavior under various boundary conditions.

Among higher-order differential operators, the polyanalytic operator $\partial_{\bar{z}}^n$ is of particular importance, as it represents a natural generalization of

the classical Cauchy–Riemann operator $\partial_{\bar{z}}$. The Cauchy–Riemann operator forms the foundation of complex analysis and the theory of analytic functions, whereas the polyanalytic operator extends this framework to a broader class of functions known as polyanalytic functions. Owing to its rich mathematical structure, this operator plays an important role in the analysis of boundary-value problems and has found numerous applications in mathematical analysis and mathematical physics [3], [4].

In recent years, many researchers have investigated boundary-value problems for polyanalytic equations in various domains [5]–[14]. For example, in [5], Begehr discussed the Dirichlet boundary-value problem for the polyanalytic equation in the unit disc. The Dirichlet problem for the polyanalytic equation in the circular ring was presented by Begehr and Shupeyeva [6]. Furthermore, in [7], Gökgöz studied the Dirichlet problem for the polyanalytic equation in the upper half plane. In our previous works [8], [9], [10], [11], we investigated Schwarz boundary value problems for polyanalytic equations in different domains and obtained their solutions using properties of Schwarz-type and T -type operators.

In this paper, we extend the boundary conditions and consider the Dirichlet problem for the polyanalytic equation in the rectangular domain. The parqueting-reflection principle is used to construct explicit solutions of Dirichlet boundary-value problem for the generalized Cauchy-Riemann equations. In other words, in addition to the description of the rectangular domain, we want to express the reflections, coverages, and points we gain at each stage and obtain a unique solution to the Dirichlet problem for the Cauchy-Riemann equation. We also investigate the boundary behavior of integral operators, with particular emphasis on their behavior at the corner points of the rectangular domain. Finally, we discuss the solvability conditions and present an explicit solution to the Dirichlet boundary-value problem for the inhomogeneous polyanalytic equation in the rectangular domain.

2. Integral representation for the rectangular domain. The parqueting-reflection principle provides a powerful and effective framework for solving boundary-value problems associated with complex partial differential equations, particularly in domains bounded by circular arcs and composed of circular polygons and straight-line segments [15], [16]. When the boundary of a domain possesses these geometric characteristics, a new planar region can be generated by reflecting the original domain across its boundary. Repeated reflections of the resulting regions across their

respective boundaries produce an expanding collection of planar domains. By iterating this process, one obtains a family of regions whose union forms a covering of the complex plane. Depending on the geometry of the domain, this covering may be generated through a single reflection, a finite sequence of reflections, or an infinite reflection process. Consequently, successive reflections of the domain across its boundary yield a covering structure for the complex plane $\mathbb{C}$. The points generated by the parqueting-reflection principle constitute a fundamental tool for the construction of solutions and the development of special kernel functions in the context of boundary-value problems.

In what follows, we introduce a rectangular domain $\mathbb{D}$, whose boundary consists of four circular arcs meeting at four corner points. The domain is precisely defined as follows.

Let $\mathbb{D}$ be the rectangular domain in the complex plane $\mathbb{C}_\infty$ defined by

$$\mathbb{D} = \left\{ z \in \mathbb{C} : |z \pm 1| < \sqrt{2} < |z \pm i\sqrt{3}| \right\}$$

with boundary $\partial\mathbb{D}$ that consists of the four circular arcs, each contained in one of the circles

$$C_j = \left\{ z \in \mathbb{C} : |z - a_j| = \sqrt{2} \right\}, \quad j = 1, 2, 3, 4,$$

where $a_1 = 1$, $a_2 = -1$, $a_3 = i\sqrt{3}$, $a_4 = -i\sqrt{3}$ and

$$b_j = \frac{\sqrt{3}-1}{2}(\pm 1 \pm i), \quad j = 1, 2, 3, 4,$$

are the corner points of $\mathbb{D}$. The aim of this paper is to present an explicit solution and solvability conditions for the higher-order Dirichlet boundary-value problem in the rectangular domain.

In this section, by using the parqueting-reflection principle for the introduced domain $\mathbb{D}$, we achieve coverage for the entire complex plane $\mathbb{C}$. This cover is gained from reflections with eleven repetitions.

Reflection of any $z = z_1 \in \mathbb{D}$ across C_j , $j = 1, 2, 3, 4$ gives

$$|z - a_j| = \sqrt{2} \Rightarrow (z - a_j)(\bar{z} - \bar{a}_j) = 2 \Rightarrow z_{re}^* = \frac{a_j \bar{z} - a_j \bar{a}_j + 2}{\bar{z} - \bar{a}_j}.$$

The principle works as follows. The rectangular domain is reflected across its four sides. The four resulting areas are reflected across their sides, etc. The reflection of the point $z \in \mathbb{D}$ with respect to C_1 , is given by

$$\frac{\bar{z} + 1}{\bar{z} - 1},$$

and produces area

$$\mathbb{D}_1 = \left\{ z \in \mathbb{C}: |z - a_2| < \sqrt{2}, |z - a_1| > \sqrt{2}, |z - a_3| > \sqrt{2}, |z - a_4| > \sqrt{2} \right\}.$$

The reflection of the point $z \in \mathbb{D}$ with respect to C_2 , is given by

$$\frac{-\bar{z} + 1}{\bar{z} + 1},$$

and produces area

$$\mathbb{D}_2 = \left\{ z \in \mathbb{C}: |z - a_1| < \sqrt{2}, |z - a_2| > \sqrt{2}, |z - a_3| > \sqrt{2}, |z - a_4| > \sqrt{2} \right\}.$$

The reflection of the point $z \in \mathbb{D}$ with respect to C_3 , is given by

$$\frac{i\sqrt{3}\bar{z} - 1}{\bar{z} + i\sqrt{3}},$$

and produces area

$$\mathbb{D}_3 = \left\{ z \in \mathbb{C}: |z - a_3| < \sqrt{2}, |z - a_1| < \sqrt{2}, |z - a_2| < \sqrt{2}, |z - a_4| > \sqrt{2} \right\}.$$

The reflection of the point $z \in \mathbb{D}$ with respect to C_4 , is given by

$$\frac{-i\sqrt{3}\bar{z} - 1}{\bar{z} - i\sqrt{3}},$$

and produces area

$$\mathbb{D}_4 = \left\{ z \in \mathbb{C}: |z - a_4| < \sqrt{2}, |z - a_1| < \sqrt{2}, |z - a_2| < \sqrt{2}, |z - a_3| > \sqrt{2} \right\}.$$

The reflection of the point $\frac{i\sqrt{3}\bar{z} - 1}{\bar{z} + i\sqrt{3}} \in \mathbb{D}_3$ with respect to C_1 , is given by

$$\frac{(1 - i\sqrt{3})z - 1 - i\sqrt{3}}{-(1 + i\sqrt{3})z - 1 + i\sqrt{3}},$$

and produces area

$$\mathbb{D}_5 = \left\{ z \in \mathbb{C}: |z - a_2| < \sqrt{2}, |z - a_3| < \sqrt{2}, |z - a_1| > \sqrt{2}, |z - a_4| > \sqrt{2} \right\}.$$

The reflection of the point $\frac{i\sqrt{3}\bar{z} - 1}{\bar{z} + i\sqrt{3}} \in \mathbb{D}_3$ with respect to C_2 , is given by

$$\frac{(1 + i\sqrt{3})z + 1 - i\sqrt{3}}{(1 - i\sqrt{3})z - 1 - i\sqrt{3}},$$

and produces area

$$\mathbb{D}_6 = \left\{ z \in \mathbb{C} : |z - a_1| < \sqrt{2}, |z - a_3| < \sqrt{2}, |z - a_2| > \sqrt{2}, |z - a_4| > \sqrt{2} \right\}.$$

The reflection of the point $\frac{-i\sqrt{3}\bar{z} - 1}{\bar{z} - i\sqrt{3}} \in \mathbb{D}_4$ with respect to C_1 , is given by

$$\frac{(1 + i\sqrt{3})z - 1 + i\sqrt{3}}{(i\sqrt{3} - 1)z - 1 - i\sqrt{3}},$$

and produces area

$$\mathbb{D}_7 = \left\{ z \in \mathbb{C} : |z - a_2| < \sqrt{2}, |z - a_4| < \sqrt{2}, |z - a_1| > \sqrt{2}, |z - a_3| > \sqrt{2} \right\}.$$

The reflection of the point $\frac{-i\sqrt{3}\bar{z} - 1}{\bar{z} - i\sqrt{3}} \in \mathbb{D}_4$ with respect to C_2 , is given by

$$\frac{(1 - i\sqrt{3})z + 1 + i\sqrt{3}}{(1 + i\sqrt{3})z - 1 + i\sqrt{3}},$$

and produces area

$$\mathbb{D}_8 = \left\{ z \in \mathbb{C} : |z - a_1| < \sqrt{2}, |z - a_4| < \sqrt{2}, |z - a_2| > \sqrt{2}, |z - a_3| > \sqrt{2} \right\}.$$

The reflection of the point $\frac{(1 + i\sqrt{3})z - 1 + i\sqrt{3}}{(i\sqrt{3} - 1)z - 1 - i\sqrt{3}} \in \mathbb{D}_6$ with respect to C_1 , is given by

$$\frac{\bar{z} + i\sqrt{3}}{1 - i\sqrt{3}\bar{z}},$$

and produces area

$$\mathbb{D}_9 = \left\{ z \in \mathbb{C} : |z - a_3| < \sqrt{2}, |z - a_1| > \sqrt{2}, |z - a_2| > \sqrt{2}, |z - a_4| > \sqrt{2} \right\}.$$

The reflection of the point $\frac{(1 + i\sqrt{3})z - 1 + i\sqrt{3}}{(i\sqrt{3} - 1)z - 1 - i\sqrt{3}} \in \mathbb{D}_7$ with respect to C_2 , is given by

$$\frac{\bar{z} - i\sqrt{3}}{i\sqrt{3}\bar{z} + 1},$$

and produces area

$$\mathbb{D}_{10} = \left\{ z \in \mathbb{C} : |z - a_4| < \sqrt{2}, |z - a_1| > \sqrt{2}, |z - a_2| > \sqrt{2}, |z - a_3| > \sqrt{2} \right\}.$$

And finally, the reflection of the point $\frac{\bar{z} + 1}{\bar{z} - 1} \in \mathbb{D}_1$ with respect to C_2 , is given by

$$-\frac{1}{z}$$

and produces area

$$\mathbb{D}_{11} = \left\{ z \in \mathbb{C} : |z - a_1| > \sqrt{2}, |z - a_2| > \sqrt{2}, |z - a_3| > \sqrt{2}, |z - a_4| > \sqrt{2} \right\}.$$

The union of closures of domains $\mathbb{D}, \mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_3, \mathbb{D}_4, \mathbb{D}_5, \mathbb{D}_6, \mathbb{D}_7, \mathbb{D}_8, \mathbb{D}_9, \mathbb{D}_{10}, \mathbb{D}_{11}$ are a cover for the complex domain. In other words,

$$\mathbb{C} = \overline{\mathbb{D}} \cup \overline{\mathbb{D}_1} \cup \overline{\mathbb{D}_2} \cup \overline{\mathbb{D}_3} \cup \overline{\mathbb{D}_4} \cup \overline{\mathbb{D}_5} \cup \overline{\mathbb{D}_6} \cup \overline{\mathbb{D}_7} \cup \overline{\mathbb{D}_8} \cup \overline{\mathbb{D}_9} \cup \overline{\mathbb{D}_{10}} \cup \overline{\mathbb{D}_{11}}.$$

The corresponding reflected points are given by

$$z_2 = -\frac{1}{z}, z_3 = \frac{(1 - i\sqrt{3})z - 1 - i\sqrt{3}}{-(1 + i\sqrt{3})z - 1 + i\sqrt{3}}, z_4 = \frac{(1 + i\sqrt{3})z + 1 - i\sqrt{3}}{(1 - i\sqrt{3})z - 1 - i\sqrt{3}},$$

$$z_5 = \frac{(1 + i\sqrt{3})z - 1 + i\sqrt{3}}{(i\sqrt{3} - 1)z - 1 - i\sqrt{3}}, z_6 = \frac{(1 - i\sqrt{3})z + 1 + i\sqrt{3}}{(1 + i\sqrt{3})z - 1 + i\sqrt{3}}, z_7 = \frac{\bar{z} + 1}{\bar{z} - 1},$$

$$z_8 = \frac{-\bar{z} + 1}{\bar{z} + 1}, z_9 = \frac{i\sqrt{3}\bar{z} - 1}{\bar{z} + i\sqrt{3}}, z_{10} = \frac{-i\sqrt{3}\bar{z} - 1}{\bar{z} - i\sqrt{3}},$$

$$z_{11} = \frac{\bar{z} - i\sqrt{3}}{i\sqrt{3}\bar{z} + 1}, z_{12} = \frac{\bar{z} + i\sqrt{3}}{1 - i\sqrt{3}\bar{z}} \notin \overline{\mathbb{D}}.$$

These reflections generate a parqueting of the entire complex plane $\mathbb{C}$, and the corresponding points will also be essential for constructing integral operators and the development of special kernel functions in the context of boundary-value problems.

One of the fundamental tools for solving boundary-value problems for inhomogeneous Cauchy–Riemann equations is the integral representation formula. In particular, this formula provides an explicit solution to the Dirichlet boundary-value problem. In the following, we derive an integral representation formula in the rectangular domain.

Theorem 1. *Any $\omega \in C^1(\mathbb{D}, \mathbb{C}) \cap C(\overline{\mathbb{D}}, \mathbb{C})$ can be represented as*

$$\omega(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \omega(\zeta) G_1(z, \zeta) d\zeta - \frac{1}{\pi} \int_{\mathbb{D}} \omega_{\bar{\zeta}}(\zeta) G_1(z, \zeta) d\xi d\eta, \quad (1)$$

where $G_1(z, \zeta) = \sum_{k=1}^6 \frac{1}{(\zeta - z_k)}$ and $\zeta = \xi + i\eta$.

Proof. According to the Cauchy–Pompeiu formula [17], for $z_1 \in \mathbb{D}$, we have

$$\omega(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \omega(\zeta) \frac{d\zeta}{\zeta - z_1} - \frac{1}{\pi} \int_{\mathbb{D}} \omega_{\bar{\zeta}}(\zeta) \frac{d\xi d\eta}{\zeta - z_1}. \quad (2)$$

Since the points obtained through the parqueting-reflection principle are outside the domain $\overline{\mathbb{D}}$, the Cauchy–Pompeiu integral formula vanishes at each of these points. Therefore, since z_2, z_3, z_4, z_5 and $z_6 \notin \overline{\mathbb{D}}$, we have

$$0 = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \omega(\zeta) \frac{d\zeta}{\zeta - z_k} - \frac{1}{\pi} \int_{\mathbb{D}} \omega_{\bar{\zeta}}(\zeta) \frac{d\xi d\eta}{\zeta - z_k}, \quad 2 \leq k \leq 6. \quad (3)$$

Adding the resulting above relations leads to the claimed integral representation formula. $\square$

Formula (1) provides a solution to the Dirichlet boundary-value problem for the inhomogeneous Cauchy–Riemann equation in the rectangular domain.

3. Dirichlet problem in the rectangular domain. In this section, we investigate the Dirichlet boundary-value problem for complex partial differential equations in the rectangular domain. As the first step, we consider the homogeneous Cauchy–Riemann equation and study the corresponding Dirichlet problem. Particular attention is devoted to the boundary behavior of its solution.

Theorem 2. *The Dirichlet problem for the homogeneous Cauchy–Riemann equation in the rectangular domain*

$$\begin{cases} \partial_{\bar{z}}\omega(z) = 0, & \text{in } \mathbb{D}, \\ \omega(\zeta) = \gamma(\zeta), & \text{on } \partial\mathbb{D}, \end{cases} \quad (4)$$

with given $\gamma \in C(\partial\mathbb{D}; \mathbb{C})$, is solvable if and only if for $z \in \mathbb{D}$

$$\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma(\zeta) G_2(z, \zeta) d\zeta = 0, \quad (5)$$

and its solution is

$$\omega(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma(\zeta) G_1(z, \zeta) d\zeta, \quad (6)$$

where $G_1(z, \zeta) = \sum_{k=1}^6 \frac{1}{(\zeta - z_k)}$, $G_2(z, \zeta) = \sum_{k=7}^{12} \frac{1}{(\zeta - z_k)}$ and $\zeta = \xi + i\eta$.

Proof. First, the condition in (5) is shown to be necessary. Let ω be the solution to the Dirichlet problem for the homogeneous Cauchy–Riemann equation. Then it follows that

$$\omega(\zeta) = \gamma(\zeta) \quad \text{on } \partial\mathbb{D}. \quad (7)$$

Let us consider the function $\phi(z)$ given by

$$\phi(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma(\zeta) G_2(\zeta) d\zeta, \quad (8)$$

and take the difference

$$\begin{aligned} \omega(z) - \phi(z) &= + \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma(\zeta) G_1(z, \zeta) d\zeta - \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma(\zeta) G_2(z, \zeta) d\zeta \\ &= \sum_{j=1}^4 \left(\frac{1}{2\pi i} \int_{\partial\mathbb{D} \cap C_j} \gamma(\zeta) \left[\frac{\zeta - a_j}{\zeta - z_1} + \frac{\bar{\zeta} - \bar{a}_j}{\bar{\zeta} - \bar{z}_1} - 1 + \frac{\zeta - a_j}{\zeta - z_2} + \frac{\bar{\zeta} - \bar{a}_j}{\bar{\zeta} - \bar{z}_2} - 1 \right. \right. \\ &\quad \left. \left. + \frac{\zeta - a_j}{\zeta - z_3} + \frac{\bar{\zeta} - \bar{a}_j}{\bar{\zeta} - \bar{z}_3} - 1 + \frac{\zeta - a_j}{\zeta - z_4} + \frac{\bar{\zeta} - \bar{a}_j}{\bar{\zeta} - \bar{z}_4} - 1 \right] d\zeta \right) \end{aligned}$$

$$+ \frac{\zeta - a_j}{\zeta - z_5} + \frac{\bar{\zeta} - \bar{a}_j}{\bar{\zeta} - \bar{z}_5} - 1 + \frac{\zeta - a_j}{\zeta - z_6} + \frac{\bar{\zeta} - \bar{a}_j}{\bar{\zeta} - \bar{z}_6} - 1 \Big] \frac{d\zeta}{\zeta - a_j} \Big). \quad (9)$$

The boundary of the rectangular domain is composed of four distinct components. Consequently, the boundary integral is evaluated by examining the contribution from each component separately.

As an example, let $\tau \in \partial\mathbb{D} \cap C_1$, where $|\tau - 1| = \sqrt{2}$. Then, for $\zeta \in \partial\mathbb{D} \cap C_1$, it follows that

$$\frac{\bar{\tau}(\bar{\zeta} - 1)}{\bar{\zeta}\bar{\tau} + 1} = \frac{\left(\frac{\tau+1}{\tau-1}\right)\left(\frac{2}{\bar{\zeta}-1}\right)}{\left(\frac{\zeta+1}{\zeta-1}\right)\left(\frac{\tau+1}{\tau-1}\right) + 1} = \frac{\tau + 1}{\zeta\tau + 1},$$

which means

$$\frac{\zeta - a_1}{\zeta - z_2} + \frac{\bar{\zeta} - \bar{a}_1}{\bar{\zeta} - \bar{z}_2} - 1 = 0,$$

also,

$$\begin{aligned} & \frac{((i\sqrt{3} - 1)\bar{\tau} - 1 - i\sqrt{3})(\bar{\zeta} - 1)}{\bar{\zeta}((i\sqrt{3} - 1)\bar{\tau} - 1 - i\sqrt{3}) - (1 + i\sqrt{3})\bar{\tau} + 1 - i\sqrt{3}} \\ &= \frac{2(i\sqrt{3} - \tau)}{\zeta((- (1 + i\sqrt{3})\tau - 1 + i\sqrt{3}) + (i\sqrt{3} - 1)\tau + 1 + i\sqrt{3})}, \\ & \frac{((1 + i\sqrt{3})\bar{\tau} - 1 + i\sqrt{3})(\bar{\zeta} - 1)}{\bar{\zeta}((1 + i\sqrt{3})\bar{\tau} - 1 + i\sqrt{3}) + (i\sqrt{3} - 1)\bar{\tau} - 1 - i\sqrt{3}} \\ &= \frac{-2(1 + i\sqrt{3}\tau)}{\zeta((1 - i\sqrt{3})\tau - 1 - i\sqrt{3}) - (1 + i\sqrt{3})\tau - 1 + i\sqrt{3}}, \\ & \frac{(-(1 + i\sqrt{3})\bar{\tau} - 1 + i\sqrt{3})(\bar{\zeta} - 1)}{\bar{\zeta}((i\sqrt{3} - 1)\bar{\tau} - 1 + i\sqrt{3}) + (i\sqrt{3} - 1)\bar{\tau} + 1 + i\sqrt{3}} \\ &= \frac{-2(\tau + i\sqrt{3})}{\zeta((i\sqrt{3} - 1)\tau - 1 - i\sqrt{3}) - (1 + i\sqrt{3})\tau + 1 - i\sqrt{3}}, \end{aligned}$$

and

$$\begin{aligned} & \frac{((1 - i\sqrt{3})\bar{\tau} - 1 - i\sqrt{3})(\bar{\zeta} - 1)}{\bar{\zeta}((1 - i\sqrt{3})\bar{\tau} - 1 - i\sqrt{3}) - (1 + i\sqrt{3})\bar{\tau} - 1 + i\sqrt{3}} \\ &= \frac{2(i\sqrt{3}\tau - 1)}{\zeta((1 + i\sqrt{3})\tau - 1 + i\sqrt{3}) + (i\sqrt{3} - 1)\tau - 1 - i\sqrt{3}}. \end{aligned}$$

By repeating the same computations carried out above, for $\tau \in \partial\mathbb{D} \cap C_j$, ($j = 1, 2, 3, 4$), we obtain

$$\begin{aligned} \lim_{z \rightarrow t} (\omega(z) - \phi(z)) &= \lim_{z \rightarrow \zeta} \frac{1}{2\pi i} \int_{\partial\mathbb{D} \cap C_j} \gamma(\zeta) \left[\frac{\zeta - a_j}{\zeta - z_1} + \frac{\bar{\zeta} - \bar{a}_j}{\bar{\zeta} - \bar{z}_1} - 1 \right] \frac{d\zeta}{\zeta - a_j} \\ &= \lim_{z \rightarrow \zeta} \frac{1}{2\pi i} \int_{C_j} \Upsilon(\zeta) \left[\frac{\zeta - a_j}{\zeta - z_1} + \frac{\bar{\zeta} - \bar{a}_j}{\bar{\zeta} - \bar{z}_1} - 1 \right] \frac{d\zeta}{\zeta - a_j}, \end{aligned}$$

where

$$\Upsilon(\zeta) = \begin{cases} \gamma(\zeta) & \zeta \in \partial\mathbb{D} \cap C_j, \\ 0 & \zeta \in C_j \setminus (\partial\mathbb{D} \cap C_j). \end{cases}$$

From the properties of the Poisson kernel for C_j [17], it follows that

$$\lim_{z \rightarrow t} (\omega(z) - \phi(z)) = \gamma(t),$$

for all $t \in \partial\mathbb{D} \cap C_j$, except at the corner points b_j of the rectangular domain $\mathbb{D}$. At these points, continuity of Υ may fail, unless γ vanishes there accidentally.

We now study the boundary behavior at the corner points. Let $\gamma = 1$. Since $\partial\mathbb{D}$ consists of four circular arcs, a direct computation of the integral operator on $\partial\mathbb{D}$ gives

$$\begin{aligned} 1 &= \sum_{j=1}^4 \left(\frac{1}{2\pi i} \int_{\partial\mathbb{D} \cap C_j} \left[\frac{\zeta - a_j}{\zeta - z_1} + \frac{\bar{\zeta} - \bar{a}_j}{\bar{\zeta} - \bar{z}_1} - 1 + \frac{\zeta - a_j}{\zeta - z_2} + \frac{\bar{\zeta} - \bar{a}_j}{\bar{\zeta} - \bar{z}_2} - 1 \right. \right. \\ &\quad \left. \left. + \frac{\zeta - a_j}{\zeta - z_3} + \frac{\bar{\zeta} - \bar{a}_j}{\bar{\zeta} - \bar{z}_3} - 1 + \frac{\zeta - a_j}{\zeta - z_4} + \frac{\bar{\zeta} - \bar{a}_j}{\bar{\zeta} - \bar{z}_4} - 1 \right. \right. \\ &\quad \left. \left. + \frac{\zeta - a_j}{\zeta - z_5} + \frac{\bar{\zeta} - \bar{a}_j}{\bar{\zeta} - \bar{z}_5} - 1 + \frac{\zeta - a_j}{\zeta - z_6} + \frac{\bar{\zeta} - \bar{a}_j}{\bar{\zeta} - \bar{z}_6} - 1 \right] \frac{d\zeta}{\zeta - a_j} \right). \end{aligned}$$

Multiplying this relation by $\gamma(b_j)$ and subtracting the resulting equation from $(\omega(z) - \phi(z))$, we obtain, for $\tau \in \partial\mathbb{D} \cap C_j$,

$$(\omega(z) - \phi(z)) - \gamma(b_j) = \frac{1}{2\pi i} \int_{\partial\mathbb{D} \cap C_j} \hat{\gamma}(\zeta) \left[\frac{\zeta - a_j}{\zeta - z_1} + \frac{\bar{\zeta} - \bar{a}_j}{\bar{\zeta} - \bar{z}_1} - 1 \right] \frac{d\zeta}{\zeta - a_j},$$

where $\hat{\gamma}(\zeta) = \gamma(\zeta) - \gamma(b_j)$ and $\hat{\gamma}(b_j) = 0$. So,

$$\lim_{z \rightarrow b_j} (\omega(z) - \phi(z) - \gamma(b_j)) = 0.$$

Therefore,

$$\lim_{z \rightarrow t} (\omega(z) - \phi(z)) = \gamma(t), \quad t \in \partial\mathbb{D}. \quad (10)$$

Hence, by (7) and (9), we obtain

$$\lim_{z \rightarrow t} \phi(z) = 0, \quad t \in \partial\mathbb{D}.$$

Since $\overline{\phi(z)}$ is analytic in $\mathbb{D}$, the maximum modulus principle implies that $\phi(z) = 0$, for all $z \in \mathbb{D}$, which is precisely condition (5). We now verify that condition (5) is sufficient. If condition (5) holds, then

$$\phi(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma(\zeta) G_2(z, \zeta) d\zeta = 0.$$

Therefore, the analytic function ω can be written as

$$\omega(z) = \omega(z) - \phi(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma(\zeta) G_1(z, \zeta) d\zeta - \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma(\zeta) G_2(z, \zeta) d\zeta.$$

Consequently,

$$\lim_{z \rightarrow \zeta} \omega(z) = \gamma(\zeta), \quad \zeta \in \partial\mathbb{D}.$$

Therefore, the proof is complete. $\square$

Next, we investigate the Dirichlet problem for the inhomogeneous Cauchy-Riemann equation in the rectangular domain. By using the definition and properties of the Pompeiu operator, we reduce the inhomogeneous problem to a homogeneous one and examine the associated solvability condition.

Theorem 3. *The Dirichlet boundary-value problem for the inhomogeneous Cauchy–Riemann equation in the rectangular domain*

$$\begin{cases} \partial_{\bar{z}}\omega(z) = f(z), & z \in \mathbb{D}, & f \in L_p(\mathbb{D}; \mathbb{C}), & p > 2, \\ \omega(\zeta) = \gamma(\zeta), & \zeta \in \partial\mathbb{D}, & \gamma \in C(\partial\mathbb{D}; \mathbb{C}), \\ \omega \in C^1(\mathbb{D}; \mathbb{C}) \cap C(\overline{\mathbb{D}}; \mathbb{C}). \end{cases} \quad (11)$$

is solvable if and only if for $z \in \mathbb{D}$,

$$\frac{1}{2\pi i} \int_{\partial \mathbb{D}} \gamma(\zeta) G_2(z, \zeta) d\zeta = \frac{1}{\pi} \int_{\mathbb{D}} f(\zeta) G_2(z, \zeta) d\xi d\eta, \quad (12)$$

and its solution can be uniquely expressed as

$$\omega(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \gamma(\zeta) G_1(z, \zeta) d\zeta - \frac{1}{\pi} \int_{\mathbb{D}} f(\zeta) G_1(z, \zeta) d\xi d\eta. \quad (13)$$

Proof. Based on Vekua's method [21], and Theorem 1, if the Dirichlet boundary-value problem (10) is solvable, then its solution can be represented as (12). Let

$$\varphi(z) = \omega(z) - Tf(z)$$

where

$$T[f](z) = -\frac{1}{\pi} \int_{\mathbb{D}} f(\zeta) \frac{d\xi d\eta}{\zeta - z}, \quad f \in L_p(\mathbb{D}, \mathbb{C}), \quad p > 2$$

is the classical Pompeiu operator [21]. Applying the operator $\partial_{\bar{z}}$ to φ , we obtain

$$\partial_{\bar{z}}\varphi = \partial_{\bar{z}}\omega - \partial_{\bar{z}}Tf \Rightarrow \partial_{\bar{z}}\varphi = f - f = 0,$$

Hence, φ is analytic in $\mathbb{D}$. Moreover, since $\omega = \gamma$ on $\partial \mathbb{D}$, we have

$$\varphi = \gamma - Tf, \quad \text{on } \partial \mathbb{D}.$$

Therefore, we obtain the equivalent boundary-value problem

$$\begin{cases} \partial_{\bar{z}}\varphi = 0, & z \in \mathbb{D}, \\ \varphi = \gamma - Tf, & \zeta \in \partial \mathbb{D}, \end{cases} \quad (14)$$

which is equivalent to (10). From Theorem 2, the solvability condition is given by

$$\frac{1}{2\pi i} \int_{\partial \mathbb{D}} (\gamma(\zeta) - Tf(\zeta)) G_2(z, \zeta) d\zeta = 0, \quad (15)$$

Expanding this condition, we obtain

$$\frac{1}{2\pi i} \int_{\partial \mathbb{D}} \gamma(\zeta) G_2(z, \zeta) d\zeta = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} Tf(\zeta) G_2(z, \zeta) d\zeta.$$

Since the points obtained from the parqueting-reflection principle lie outside $\mathbb{D}$, by the definition of the operator T and Fubini's theorem, we have

$$\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma(\zeta) G_2(z, \zeta) d\zeta = \frac{1}{\pi} \int_{\mathbb{D}} f(\hat{\zeta}) \frac{1}{2\pi i} \int_{\partial\mathbb{D}} G_2(z, \zeta) \frac{d\zeta}{\zeta - \hat{\zeta}} d\hat{\xi} d\hat{\eta}.$$

Hence, we obtain

$$\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma(\zeta) G_2(z, \zeta) d\zeta = \frac{1}{\pi} \int_{\mathbb{D}} f(\hat{\zeta}) G_2(z, \hat{\zeta}) d\hat{\xi} d\hat{\eta},$$

which is precisely condition (11). Conversely, if the condition of solvability (11) is satisfied, (12) can be rewritten as

$$\begin{aligned} \omega(z) &= \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma(\zeta) G_1(z, \zeta) d\zeta - \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma(\zeta) G_2(z, \zeta) d\zeta \\ &\quad + \frac{1}{\pi} \int_{\mathbb{D}} f(\zeta) G_2(z, \zeta) d\xi d\eta - \frac{1}{\pi} \int_{\mathbb{D}} f(\zeta) G_1(z, \zeta) d\xi d\eta. \end{aligned} \quad (16)$$

For the boundary behavior of the area integral, let $H(z, \zeta) = G_1(z, \zeta) - G_2(z, \zeta)$. Then for $t \in \partial\mathbb{D}$, we have $H(t, \zeta) = 0$. Hence, the area integral tends to 0 as $z \rightarrow t$, where $t \in \partial\mathbb{D}$. It then follows from the proof of Theorem 2 that (15) implies

$$\lim_{z \rightarrow t} \omega(z) = \gamma(t), \quad t \in \partial\mathbb{D}.$$

To establish the uniqueness of the solution to the Dirichlet problem, suppose that ω and ψ are two solutions satisfying

$$\partial_{\bar{z}}\omega = f \quad \text{in } \mathbb{D}, \quad \omega = \gamma \quad \text{on } \partial\mathbb{D},$$

and

$$\partial_{\bar{z}}\psi = f \quad \text{in } \mathbb{D}, \quad \psi = \gamma \quad \text{on } \partial\mathbb{D}.$$

Define $u = \omega - \psi$. By subtracting the two equations, we obtain

$$\partial_{\bar{z}}u = 0 \quad \text{in } \mathbb{D}.$$

Hence, u is analytic in $\mathbb{D}$. Moreover, since both solutions satisfy the same boundary condition, it follows that

$$u = \omega - \psi = 0 \quad \text{on } \partial\mathbb{D}.$$

By the maximum modulus principle for analytic functions, the only analytic function that vanishes on the boundary $\mathbb{D}$ is the trivial function. Therefore,

$$u \equiv 0 \quad \text{in } \mathbb{D}.$$

Consequently,

$$\omega = \psi \quad \text{in } \mathbb{D},$$

which proves the uniqueness of the solution to the Dirichlet problem. $\square$

Theorem 3 establishes the solvability of the Dirichlet problem for the inhomogeneous Cauchy–Riemann equation and provides an explicit integral representation of its solution. This theorem corresponds to the case $n = 1$ in the theory of polyanalytic equations and constitutes a fundamental result for the study of higher-order polyanalytic boundary-value problems. In the next step, we study the Dirichlet problem for higher-order polyanalytic equation in the rectangular domain.

Theorem 4. *The Dirichlet boundary-value problem for the inhomogeneous polyanalytic equation in the rectangular domain*

$$\begin{cases} (\partial_{\bar{z}}^n \omega)(z) = f(z), & z \in \mathbb{D}, \quad f \in L_p(\mathbb{D}; \mathbb{C}), \quad p > 2, \\ (\partial_{\bar{z}}^k \omega)(\zeta) = \gamma_k(\zeta), & \zeta \in \partial\mathbb{D}, \quad \gamma_k \in C(\partial\mathbb{D}; \mathbb{C}), \\ \omega \in C^n(\mathbb{D}; \mathbb{C}) \cap C^{n-1}(\overline{\mathbb{D}}; \mathbb{C}), & k = 0, 1, 2, \dots, n-1, \end{cases} \quad (17)$$

is solvable if and only if for $k = 0, 1, 2, \dots, n-1$, and $z \in \mathbb{D}$

$$\begin{aligned} \sum_{l=k}^{n-1} \frac{(-1)^{l-k}}{2\pi i} \int_{\partial\mathbb{D}} \gamma_l(\zeta) \frac{\overline{(\zeta - z)}^{l-k}}{(l-k)!} G_2(z, \zeta) d\zeta \\ + \frac{(-1)^{n-k}}{\pi} \int_{\mathbb{D}} f(\zeta) \frac{\overline{(\zeta - z)}^{n-1-k}}{(n-1-k)!} G_2(z, \zeta) d\xi d\eta = 0, \end{aligned} \quad (18)$$

and its solution can be written as

$$\begin{aligned} \omega(z) = \sum_{k=0}^{n-1} \frac{(-1)^k}{2\pi i} \int_{\partial\mathbb{D}} \gamma_k(\zeta) \frac{\overline{(\zeta - z)}^k}{k!} G_1(z, \zeta) d\zeta \\ + \frac{(-1)^n}{\pi} \int_{\mathbb{D}} f(\zeta) \frac{\overline{(\zeta - z)}^{n-1}}{(n-1)!} G_1(z, \zeta) d\xi d\eta. \end{aligned} \quad (19)$$

Proof. The necessary and sufficient solvability condition, together with the corresponding integral representation of the solution, follows directly from Theorem 3. Hence, the statement of the theorem is valid for $n = 1$.

For convenience, define

$$u_k = \partial_{\bar{z}}^k \omega, \quad 0 \leq k \leq n,$$

so that

$$u_n = f, \quad \text{in } \mathbb{D},$$

while

$$u_k = \gamma_k \quad \text{on } \partial\mathbb{D}, \quad 0 \leq k \leq n-1.$$

Thus the original problem is equivalent to the system

$$\begin{cases} \partial_{\bar{z}} u_k = u_{k+1}, & z \in \mathbb{D}, \quad 0 \leq k \leq n-1, \\ u_k = \gamma_k, & z \in \partial\mathbb{D}, \quad 0 \leq k \leq n-1, \\ u_n = f. \end{cases} \quad (20)$$

Since

$$\omega \in C^m(\mathbb{D}; \mathbb{C}) \cap C^{m-1}(\overline{\mathbb{D}}; \mathbb{C}),$$

we have

$$u_k \in C^{m-k}(\mathbb{D}; \mathbb{C}) \cap C^{m-k-1}(\overline{\mathbb{D}}; \mathbb{C}), \quad 0 \leq k \leq n-1.$$

We shall solve the system (19) successively, starting with u_{n-1} and proceeding downwards to $u_0 = \omega$.

For $0 \leq k \leq n-1$, define

$$\begin{aligned} U_k(z) = & \sum_{l=k}^{n-1} \frac{(-1)^{l-k}}{2\pi i} \int_{\partial\mathbb{D}} \gamma_l(\zeta) \frac{\overline{(\zeta - z)}^{l-k}}{(l-k)!} G_1(z, \zeta) d\zeta \\ & + \frac{(-1)^{n-k}}{\pi} \int_{\mathbb{D}} f(\zeta) \frac{\overline{(\zeta - z)}^{n-1-k}}{(n-1-k)!} G_1(z, \zeta) d\xi d\eta. \end{aligned} \quad (21)$$

We first show that

$$\partial_{\bar{z}} U_k = U_{k+1}, \quad 0 \leq k \leq n-2, \quad (22)$$

and

$$\partial_{\bar{z}} U_{n-1} = f. \quad (23)$$

Since $G_1(z, \zeta)$ is analytic,

$$\partial_{\bar{z}} G_1(z, \zeta) = 0.$$

Moreover,

$$\partial_{\bar{z}} \overline{(\zeta - z)}^n = -n \overline{(\zeta - z)}^{n-1}.$$

Therefore, differentiating (20) with respect to $\bar{z}$, we obtain

$$\begin{aligned} \partial_{\bar{z}} U_k(z) &= \sum_{l=k+1}^{n-1} \frac{(-1)^{l-k}}{2\pi i} \left(-\frac{l-k}{(l-k)!} \right) \int_{\partial \mathbb{D}} \gamma_l(\zeta) \overline{(\zeta - z)}^{l-k-1} G_1(z, \zeta) d\zeta \\ &\quad + \frac{(-1)^{n-k}}{\pi} \left(-\frac{n-1-k}{(n-1-k)!} \right) \int_{\mathbb{D}} f(\zeta) \overline{(\zeta - z)}^{n-2-k} G_1(z, \zeta) d\xi d\eta. \end{aligned}$$

Hence,

$$\begin{aligned} \partial_{\bar{z}} U_k(z) &= \sum_{l=k+1}^{n-1} \frac{(-1)^{l-k-1}}{2\pi i} \int_{\partial \mathbb{D}} \gamma_l(\zeta) \frac{\overline{(\zeta - z)}^{l-k-1}}{(l-k-1)!} G_1(z, \zeta) d\zeta \\ &\quad + \frac{(-1)^{n-k-1}}{\pi} \int_{\mathbb{D}} f(\zeta) \frac{\overline{(\zeta - z)}^{n-k-2}}{(n-k-2)!} G_1(z, \zeta) d\xi d\eta. \end{aligned}$$

The right-hand side is precisely $U_{k+1}(z)$. Thus (21) follows.

For $k = n - 1$, all boundary terms except $l = n - 1$ disappear, and

$$U_{n-1}(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \gamma_{n-1}(\zeta) G_1(z, \zeta) d\zeta - \frac{1}{\pi} \int_{\mathbb{D}} f(\zeta) G_1(z, \zeta) d\xi d\eta.$$

By Theorem 3, $\partial_{\bar{z}} U_{n-1} = f$ if and only if

$$\frac{1}{2\pi i} \int_{\partial \mathbb{D}} \gamma_{n-1}(\zeta) G_2(z, \zeta) d\zeta - \frac{1}{\pi} \int_{\mathbb{D}} f(\zeta) G_2(z, \zeta) d\xi d\eta = 0. \quad (24)$$

This is exactly the asserted solvability condition for $k = n - 1$.

We now determine the compatibility condition at an arbitrary level k . Suppose that U_{k+1} has already been constructed. By (21), U_k has to solve the inhomogeneous Cauchy-Riemann problem

$$\begin{cases} \partial_{\bar{z}} U_k = U_{k+1}, & z \in \mathbb{D}, \\ U_k = \gamma_k, & z \in \partial \mathbb{D}. \end{cases} \quad (25)$$

Theorem 3 implies that (24) is solvable if and only if

$$\frac{1}{2\pi i} \int_{\partial \mathbb{D}} \gamma_k(\zeta) G_2(z, \zeta) d\zeta - \frac{1}{\pi} \int_{\mathbb{D}} U_{k+1}(\zeta) G_2(z, \zeta) d\xi d\eta = 0. \quad (26)$$

By the representation formula for U_{k+1} ,

$$\begin{aligned} U_{k+1}(\zeta) = & \sum_{l=k+1}^{n-1} \frac{(-1)^{l-k-1}}{2\pi i} \int_{\partial \mathbb{D}} \gamma_l(\hat{\zeta}) \frac{\overline{(\hat{\zeta} - \zeta)}^{l-k-1}}{(l-k-1)!} G_1(\zeta, \hat{\zeta}) d\hat{\zeta} \\ & + \frac{(-1)^{n-k-1}}{\pi} \int_{\mathbb{D}} f(\hat{\zeta}) \frac{\overline{(\hat{\zeta} - \zeta)}^{n-k-2}}{(n-k-2)!} G_1(\zeta, \hat{\zeta}) d\hat{\xi} d\hat{\eta}, \end{aligned} \quad (27)$$

where $\hat{\zeta} = \xi + i\eta$.

Substituting (26) into (25), and using the integral identities relating G_1 and G_2 from Theorem 3, we obtain

$$\begin{aligned} \frac{1}{\pi} \int_{\mathbb{D}} U_{k+1}(\zeta) G_2(z, \zeta) d\xi d\eta = & \sum_{l=k+1}^{n-1} \frac{(-1)^{l-k-1}}{2\pi i} \int_{\partial \mathbb{D}} \gamma_l(\hat{\zeta}) \frac{\overline{(\hat{\zeta} - z)}^{l-k-1}}{(l-k-1)!} G_2(z, \hat{\zeta}) d\hat{\zeta} \\ & + \frac{(-1)^{n-k-1}}{\pi} \int_{\mathbb{D}} f(\hat{\zeta}) \frac{\overline{(\hat{\zeta} - z)}^{n-k-2}}{(n-k-2)!} G_2(z, \hat{\zeta}) d\hat{\xi} d\hat{\eta}. \end{aligned} \quad (28)$$

Consequently, condition (25) becomes

$$\begin{aligned} 0 = & \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \gamma_k(\zeta) G_2(z, \zeta) d\zeta \\ & - \sum_{l=k+1}^{n-1} \frac{(-1)^{l-k-1}}{2\pi i} \int_{\partial \mathbb{D}} \gamma_l(\zeta) \frac{\overline{(\zeta - z)}^{l-k-1}}{(l-k-1)!} G_2(z, \zeta) d\zeta \\ & - \frac{(-1)^{n-k-1}}{\pi} \int_{\mathbb{D}} f(\zeta) \frac{\overline{(\zeta - z)}^{n-k-2}}{(n-k-2)!} G_2(z, \zeta) d\xi d\eta. \end{aligned}$$

Since $-(-1)^{l-k-1} = (-1)^{l-k}$ and $-(-1)^{n-k-1} = (-1)^{n-k}$, we arrive at

$$0 = \sum_{l=k}^{n-1} \frac{(-1)^{l-k}}{2\pi i} \int_{\partial\mathbb{D}} \gamma(\zeta) \frac{\overline{(\zeta - z)}^{l-k}}{(l-k)!} G_2(z, \zeta) d\zeta \\ + \frac{(-1)^{n-k}}{\pi} \int_{\mathbb{D}} f(\zeta) \frac{\overline{(\zeta - z)}^{n-1-k}}{(n-1-k)!} G_2(z, \zeta) d\xi d\eta. \quad (29)$$

This is precisely the condition stated in the theorem. Thus, the compatibility condition at every level $k = 0, \dots, n-1$, is necessary and sufficient for the successive Cauchy-Riemann problems (24) to be solvable.

Conversely, suppose that the compatibility conditions of the theorem hold for all $k = 0, \dots, n-1$. Define U_k by (20) as shown above:

$$\partial_{\bar{z}} U_k = U_{k+1}, \quad 0 \leq k \leq n-2, \quad (30)$$

and by Theorem 3, together with the compatibility condition for $k = n-1$

$$\partial_{\bar{z}} U_{n-1} = f. \quad (31)$$

Moreover, applying Theorem 3, successively and using the compatibility condition at each level k , we obtain

$$U_k = \gamma_k \text{ on } \partial\mathbb{D}, \quad k = 0, \dots, n-1.$$

Therefore, setting $\omega = U_0$, we obtain

$$\partial_{\bar{z}}^n \omega = f, \text{ in } \mathbb{D}, \quad \partial_{\bar{z}}^k \omega = \gamma_k, \text{ on } \partial\mathbb{D}, \quad k = 0, \dots, n-1.$$

Hence the original Dirichlet problem is solvable. $\square$

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References

- [1] Lu J. *Boundary value problems for analytic functions*. World Scientific Publication, Singapore, 1993.
- [2] Gakhov F. D. *Boundary value problems*. Dover, New York, 1990.

- [3] Balk M. B. *Polyanalytic functions*. Akademie Verlag, Berlin, 2001.
- [4] Abreu L. D. Feichtinger H. G. *Function spaces of polyanalytic functions*. Harmonic and Complex Analysis and its Applications. Trends in Mathematics. Birkhäuser, Cham., 2014, pp. 1-38.
DOI: <https://doi.org/10.1007/978-3-319-01806-5>
- [5] Begehr H. *Boundary value problems in complex analysis II*. Boletín de la Asociación Matemática Venezolana., 2005, vol. XII, pp. 217–250.
- [6] Begehr H., Shupeyeva B. *Polyanalytic boundary value problems for planar domains with harmonic Green function*. Anal.Math.Phys., 2021, vol. 11, pp. 1–22. DOI: <https://doi.org/10.1007/s13324-021-00569-2>
- [7] Gökgöz P. A. *Dirichlet boundary value problem for linear polyanalytic equation in upper half plane*. Complex Var. Elliptic Equ., 2025, vol. 70, pp. 549–554. DOI: <https://doi.org/10.1080/17476933.2024.2332317>
- [8] Darya A., Taghizadeh N. *Schwarz boundary value problem for the polyanalytic equation in a lens*. Sib. Math. J., 2025, vol. 66, pp. 486–493.
DOI: <https://doi.org/10.1134/S003744662502017X>
- [9] Darya A., Ebrahimzadeh, K., Darya N., Daria, N. *Schwarz problem for the inhomogeneous poly-analytic equation in a half-lens*. Probl. Anal. Issues Anal., 2025, vol. 14, pp. 3–22.
DOI: <https://doi.org/10.15393/j3.art.2025.18950>
- [10] Darya A., Taghizadeh N. *Schwarz boundary value problem for the polyanalytic equation in the partial eclipse domain*. Comput. Math. and Math. Phys., 2025, vol. 65, pp. 1815–1822.
DOI: <https://doi.org/10.1134/S0965542525700952>
- [11] Darya A., Ebrahimzadeh K., Darya N., Daria N. *Schwarz boundary value problem for higher-order complex partial differential equations on a triangle*. Eur. J. Math. Anal., 2026, vol. 6, pp. 1–11.
DOI: <https://doi.org/10.28924/ada/ma.6.5>
- [12] Begehr H., Schmiersau D. *The Schwarz problem for polyanalytic functions*. Z. Anal. Anwend., 2005, vol. 24, pp. 341–351.
DOI: <https://doi.org/10.4171/ZAA/1244>
- [13] Wang Y. *Schwarz-type boundary value problems for the polyanalytic equation in the half unit disc*. Complex Var. Elliptic Equ., 2012, vol. 57, pp. 983–993. DOI: <https://doi.org/10.1080/17476933.2010.534151>
- [14] Aksoy U., Begehr H., Celebi A. *Schwarz problem for higher-order complex partial differential equations in the upper half plane*. Math. Nachr., 2019, vol. 292, pp. 1183–1193.
DOI: <https://doi.org/10.1002/mana.201800028>

- [15] Begehr H., Vaitekhovich T. *Green functions, reflections, and plane parqueting*. Eurasian Math. J., 2010, vol. 1, pp. 17–31.
- [16] Begehr H., Vaitekhovich T. *The parqueting-reflection principle for constructing Green functions*. Analytic Methods of Analysis and Differential Equations: AMADE-2012, Cambridge Sci. Publ., Cottenham, 2013, pp. 11–20.
- [17] Begehr H. *Boundary value problems in complex analysis I*. Boletín de la Asociacion Matematica Venezolana., vol. XII, 2005, pp. 65–85.
- [18] Mazepa E. A., Ryaboshlykova D. K. *Boundary-value problems for the inhomogeneous Schrödinger equation with variations of its potential on non-compact Riemannian manifolds*. Probl. Anal. Issues Anal., 2021, vol. 10(28), no. 3, pp. 113–128.
DOI: <https://doi.org/10.15393/j3.art.2021.10911>
- [19] Darya A., Taghizadeh N. *Schwarz and Dirichlet Problems for $\bar{\partial}$ -Equation in a triangular domain*. Russ. Math., 2024, vol. 68, pp. 9–17.
DOI: <https://doi.org/10.3103/S1066369X24700853>
- [20] Ivanshin P. N., Shirokova E. A. *The solution of a mixed boundary value problem for the Laplace equation in a multiply connected domain*. Probl. Anal. Issues Anal. J., 2019, vol. 8(26), no. 2, pp. 51–66.
DOI: <https://doi.org/10.15393/j3.art.2019.5570>
- [21] Vekua I. N. *Generalized analytic functions*. Pergamon Press, Oxford, 1962.

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