DOI: 10.15393/j3.art.2026.21210

UDC 517.54

S. Yu. Graf, Yu. D. Yakhova

UNIVALENCE OF HARMONIC MAPPINGS OF THE DISK ONTO n-CONVEX DOMAINS

Abstract. In this article, we introduce the concept of special n-convex domains. We prove a sufficient condition for the univalence of the Poisson integral with a boundary function that maps the unit circle $\mathbb{T}$ homeomorphically onto the boundary of a special n-convex domain $\Omega \subset \mathbb{C}$. Corollaries describing classes of homeomorphisms to which this univalence condition applies are given, along with examples illustrating its application.

Key words: *harmonic mappings, Poisson integral, Radó-Kneser-Choquet theorem*

2020 Mathematical Subject Classification: *30C55*

1. Introduction. Recall that a complex-valued function $f = u + iv$ is harmonic in a domain $D \subset \mathbb{C}$ if u and v are real-valued harmonic functions in D, i.e., belong to the class $C^2(D)$ and satisfy the Laplace equation $\Delta u = \Delta v = 0$ in D. In a simply connected domain D, every harmonic function f can be represented as $f = h + \overline{g}$, where h, g are analytic functions.

Lewy's theorem (see, for example, [6]) allows us to assert that a harmonic function f in a planar domain D is locally univalent if and only if its Jacobian

$$J_f = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = |h'|^2 - |g'|^2$$

preserves sign in D. In particular, orientation-preserving locally univalent harmonic mappings have positive Jacobians $J_f(z) > 0$.

The question of global univalence of harmonic mappings is substantially more complex. An overview of key properties of univalent harmonic mappings and fundamental problems of this branch of geometric function theory can be found in P. Duren's monograph [6]. A large number of

© Petrozavodsk State University, 2026

(cc) BY-NC

publications are devoted to various conditions for univalence of harmonic mappings (see, for example, [4], [11], [2], [9], [18]).

Despite the obvious differences between harmonic and analytic functions, a number of classical principles of geometric function theory remain valid in the harmonic case as well. In particular, for orientation-preserving harmonic mappings f in a simply connected bounded Jordan domain D with piecewise smooth boundary ∂D, such that f is continuous in the closure of D and $f \neq 0$ on ∂D, the topological version of the argument principle remains valid:

$$\frac{1}{2\pi} \oint\limits_{\partial D^+} d\,\mathtt{Arg}\, f(z) = N,$$

where N is the number of zeros of f in D (see, for example, [6], [5]). Thus, if the restriction of a harmonic function f to the boundary ∂D is known to be a homeomorphism, then the univalence of f in D follows from its local univalence. The question of zeros of the Jacobian of a harmonic mapping is therefore central in verifying global univalence. Some properties of Jacobians of harmonic mappings are discussed in [10].

We denote by $\mathbb{D}$ the unit disk $\{z \in \mathbb{C}\colon |z| < 1\}$ in the complex plane $\mathbb{C}$. Recall that every continuous function φ on the unit circle $\mathbb{T} = \partial\mathbb{D}$ has a unique harmonic extension to the disk $\mathbb{D}$, representable as the Poisson integral

$$f_\varphi(z) = \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2} \varphi(e^{it}) dt.$$

Note that for continuous boundary functions φ, the Poisson integral f_φ is continuous in the closure $\overline{\mathbb{D}}$ of the unit disk (see, for example, [6], [7]).

One of the key results in the theory of univalent harmonic mappings, which has had a significant influence on the development of this direction, is the Radó-Kneser-Choquet theorem:

Theorem A (Radó-Kneser-Choquet theorem) [17], [6]. *Let $\Omega \subset \mathbb{C}$ be a convex bounded Jordan domain. Let φ be a homeomorphism of $\mathbb{T}$ onto $\partial\Omega$. Then the Poisson integral f_φ is univalent in $\mathbb{D}$ and defines a harmonic mapping of $\mathbb{D}$ onto Ω.*

For an arbitrary non-convex domain Ω, Theorem A is not valid. Moreover, G. Choquet proved [3] that for any non-convex domain Ω there exists

a homeomorphism φ of the circle $\mathbb{T}$ onto $\partial\Omega$, for which the Poisson integral f_φ is not univalent.

An elegant geometric proof of Theorem A belongs to H. Kneser [14]. The key point of his argument is the possibility of proving local univalence of the Poisson integral in the disk $\mathbb{D}$ and applying the argument principle. Moreover, as Kneser noted [14], Theorem A remains valid even for non-convex domains Ω if it is known that $f_\varphi(\mathbb{D}) \subset \Omega$, i.e., if the Poisson integral takes values inside the domain Ω. It is known (see, for example, [6]) that the values of the harmonic function $f_\varphi(z)$ for $z \in \mathbb{D}$ lie in the open convex hull of the values of the function $\varphi(\zeta)$, $\zeta \in \mathbb{T}$. This result follows from the fact that if, for example, $\operatorname{Re}\varphi(\zeta) \geqslant 0$ for all $\zeta \in \mathbb{T}$, then by the mean value theorem for any $z \in \mathbb{D}$ there exists a point $\xi \in [0; 2\pi]$, such that

$$\operatorname{Re} f_\varphi(z) = \frac{1}{2\pi}\frac{1-|z|^2}{|e^{i\xi}-z|^2}\int\limits_0^{2\pi} \operatorname{Re}\varphi(e^{it})dt > 0$$

provided that $\operatorname{Re}\varphi$ is not identically zero.

Several various generalizations of the Radó-Kneser-Choquet theorem are known. For example [6], if φ is a weak homeomorphism of the unit circle $\mathbb{T}$ onto the boundary of a convex bounded domain Ω (i.e., φ is the pointwise limit of a sequence of equally oriented homeomorphisms φ_n of the circle $\mathbb{T}$ onto $\partial\Omega$), then the Poisson integral f_φ is univalent in $\mathbb{D}$ and defines a harmonic mapping of $\mathbb{D}$ onto Ω. There are also versions of Theorem A for harmonic mappings of multiply connected domains [15].

In view of the results proved in the present article, it is important to mention a generalization of the Radó-Kneser-Choquet theorem related to boundary local univalence of the Poisson integral, obtained relatively recently by G. Alessandrini and V. Nesi [1].

A planar mapping f continuous on the closed disk $\overline{\mathbb{D}}$ is said to be *locally homeomorphic at a boundary point* $\zeta \in \mathbb{T}$ if there exists a neighborhood $U_\delta(\zeta) = \{z \in \mathbb{C}\colon |z-\zeta| < \delta\}$ of ζ, such that f is univalent in $U_\delta(\zeta) \cap \overline{\mathbb{D}}$. The univalence condition obtained by Alessandrini and Nesi can be formulated in the following form, which differs somewhat from the authors' formulation but is more natural in the context of the present work:

Theorem B [1]. *Let φ be a homeomorphism of the circle $\mathbb{T}$ onto the boundary of a bounded Jordan domain Ω. The Poisson integral f_φ is a univalent mapping of the closed disk $\overline{\mathbb{D}}$ onto $\overline{\Omega}$ if and only if f_φ is locally*

homeomorphic at every boundary point $\zeta \in \mathbb{T}$.

Taking into account Kneser's remark above that the univalence of the Poisson integral with a homeomorphic boundary function can be ensured by the condition $f_\varphi(\mathbb{D}) \subset \Omega$, it seems possible to weaken the condition of Theorem B, requiring boundary local univalence not for all points $\zeta \in \mathbb{T}$, but only for those whose images under the mapping φ correspond to the non-convex part of the boundary of Ω.

The main result of this paper is a sufficient condition for univalence of the Poisson integral with boundary homeomorphisms φ mapping the circle $\mathbb{T}$ onto the boundary of a non-convex bounded domain of a special type.

2. Univalence of harmonic mappings onto n-convex domains. The concept of a convex set in function theory has been generalized repeatedly in various senses and directions, which led to the emergence of such notions as absolutely convex, weak convex, R-convex sets, and many others (see, for example, [12], [16], [19], [8]). In 1957, F. A. Valentine introduced the concept of three-point convexity [20], which was subsequently generalized in 1970 by D. C. Kay and M. D. Guay [13] to the case of a larger number of points: a set $A \subset \mathbb{R}^2$ is called *n-convex* if in any collection of n distinct points belonging to A, there exist two points that can be joined by a line segment lying in A. In particular, every 2-convex set is convex.

In the present paper, we introduce the notion of a *special n-convex domain*, related to the definition of n-convex set given above.

Recall that a boundary point $\zeta_0 \in \partial\Omega$ is called a support point for the domain Ω if there exists a line l passing through ζ_0, such that $l \cap \Omega = \varnothing$.

Definition 1. *A bounded simply connected Jordan domain Ω is called special n-convex if there exist distinct straight lines $\Gamma_k, k = 1, \dots, n-1$, such that*

1) each line Γ_k intersects $\partial\Omega$ at two points $\omega_{k,0}$ and $\omega_{k,1}$, where $\omega_{k,0}$ is a support point for Ω;

2) any straight line l passing through two distinct points $\omega_{k,i}, \omega_{l,j}$, $k \neq l$, $i, j = 0, 1$, intersects the boundary $\partial\Omega$ of the domain Ω in no more than two points;

3) the family of lines Γ_k, $k = 1, \dots, n-1$, partitions Ω into n convex subdomains, i.e., $\Omega \setminus \cup_{k=1}^{n-1} \Gamma_k$ is a union of n convex domains.

The points $\omega_{k,1}$ will be called vertices of the domain Ω, and the lines Γ_k

will be called partition lines. The partition lines may intersect at vertices or at support points of the domain Ω, but not inside the domain, since otherwise the number of connected components of the set $\Omega\backslash\cup_{k=1}^{n-1}\Gamma_k$ would be greater than n. Throughout what follows, we assume that the number n in the definition of a special n-convex domain is chosen to be minimal among those for which conditions 1) – 3) are satisfied. The partition of a special n-convex domain into convex components may not be unique.

From a comparison of this definition with the definition of an n-convex set in the sense of Kay and Guay, it follows that every special n-convex domain Ω is an $(n+1)$-convex set. Indeed, consider an arbitrary collection of $n+1$ distinct points $w_j \in \Omega, j = 1, \ldots, n+1$. Let $\Omega_m, m = 1, \ldots, n$, be the convex subdomains into which the lines Γ_k partition the domain Ω. Denote by Ω_m^* the convex sets $\overline{\Omega_m}\backslash\partial\Omega$. Then at least two of the $n+1$ points w_j fall into one of the n convex sets Ω_m^* and can be joined by a line segment lying in Ω. It is clear that for n points w_j this is no longer the case, i.e., a special n-convex domain need not be an n-convex set.

We also note that any convex domain can be considered special 1-convex in the sense of the definition above, since it consists of one convex component and the lines Γ_k for its partition are not required. Moreover, if Ω is a special 2-convex domain, then condition 2) in the Definition is a consequence of conditions 1) and 3). An example of a special 2-convex domain that is not convex is the interior of a cardioid.

Every special 2-convex domain is starlike with respect to any point of the partition line (recall that a domain Ω is called starlike if there exists a point $w_0 \in \Omega$, such that for any $w \in \Omega$, the line segment $[w_0, w]$ lies in Ω; see, for example, [6]). Indeed, for any point $w_0 \in \Omega$ lying on the partition line Γ of a special 2-convex domain, the rays emanating from w_0 have a connected intersection with Ω. Thus, Ω is starlike with respect to w_0.

Note also that the term "n-convex domain" in the sense of a domain that can be cut into n convex components in the context of another problem was used in [21].

Throughout the rest of the paper, we assume that $n \geqslant 2$, i.e., the domain Ω is not convex.

In this paper, we derive the sufficient conditions for the univalence of harmonic mappings f_φ of the disk $\mathbb{D}$ onto special n-convex domains.

Condition 2) in the definition above implies the following lemma.

Lemma. *Let Ω be a special n-convex domain with partition lines Γ_k, $k = 1, \ldots, n-1$. Then any straight line l passing through an arbitrary*

point $w_0 \in \Gamma_k \cap \Omega$ intersects the boundary $\partial\Omega$ of the domain Ω in exactly two points.

Proof. Suppose the domain Ω is special n-convex, but on some partition line Γ_k there is a point $w_0 \in \Gamma_k \cap \Omega$, such that there exists a line l_0 passing through w_0 that intersects $\partial\Omega$ in at least three points. Obviously, $l_0 \neq \Gamma_k$, since by condition 1) of the definition of a special n-convex domain, the line Γ_k intersects $\partial\Omega$ in only two points.

Choose two points ζ', ζ'' from the set $l_0 \cap \partial\Omega$ lying on the same side of the point w_0 and, consequently, on the same side of the line Γ_k. Without loss of generality, we may assume that ζ' is the closest point to w_0 among all points of $l_0 \cap \partial\Omega$ lying on the same side of Γ_k, and ζ'' is the farthest from w_0 on that same side. Thus, the interval $(w_0; \zeta') \subset l_0$ is contained in Ω, and the ray of the line l_0 emanating from ζ'' in the direction opposite to w_0 does not intersect Ω.

Denote the vertex and support point corresponding to the intersection of the partition line Γ_k with the boundary of Ω by ω_1 and ω_2 (regardless of which one is the support point or vertex). Without loss of generality, we may assume that the points $\omega_1, \zeta', \zeta''$ and ω_2 follow in the positive orientation of $\partial\Omega$.

If on the interval $(\zeta'; \zeta'')$ the line l_0 leaves the domain Ω, then the positively oriented arc of the boundary $\partial\Omega$ connecting the points ζ' and ζ'' is not convex with respect to Ω. Consequently, on this arc there is at least one vertex of the special n-convex domain Ω. Among these vertices, choose the one closest to ζ' along the arc connecting ζ' and ζ''. Denote it by ω''.

If, however, the interval $(\zeta'; \zeta'')$ of the line l_0 lies in Ω, then the point ζ' itself is a vertex of Ω, since $(w_0; \zeta'), (\zeta'; \zeta'') \subset \Omega$, but $\zeta' \notin \Omega$. In this case, denote by ω'' the point ζ'.

The points ω'' and ζ'' follow along $\partial\Omega$ in the positive direction and lie on the same side of the line Γ_k. Otherwise Γ_k would have more than two intersections with $\partial\Omega$, contradicting condition 1) of the definition of a special n-convex domain.

Starting at ζ' and moving along the arc $\partial\Omega$ in the negative direction toward ω_1, select the nearest to ζ' vertex or support point of Ω (different from ζ') and denote it by ω' (possibly ω' coincides with ω_1). Obviously, $\omega' \neq \omega''$ and there are no other vertices or support points of the domain Ω on the positively oriented boundary arc $\widetilde{\omega'\omega''}$. This arc $\widetilde{\omega'\omega''}$ is convex with respect to Ω.

Consider the line l passing through the points ω' and ω''. By condition 2) of the definition of a special n-convex domain, the only intersection points of $l \cap \partial\Omega$ are ω' and ω''. Moreover, the interval $(\omega'; \omega'')$ of the line l lies in Ω, as does the interval $(w_0; \zeta')$ of the line l_0. The point ζ' lies on the arc $\widetilde{\omega'\omega''}$ or coincides with the vertex ω'', and the point ω_1 lies outside this arc or coincides with ω'.

Consequently, the points $w_0 \in \Gamma_k$ and $\zeta'' \in l_0$ lie on opposite sides of l. Otherwise, the line Γ_k passing through $w_0 \in l_0$ would intersect the line l on the segment $[\omega'; \omega'']$ and would separate the points ω'' and ζ''.

If the points w_0 and ω_2 were on opposite sides of the line l, then l would intersect the segment $[w_0; \omega_2]$ at an interior point of Ω and, consequently, there would be another intersection point of l with the boundary of Ω, contradicting condition 2). For a similar reason, l cannot pass through ω_2.

It follows that the points ω_2 and w_0 lie on the same side of l, while the points ζ'' and ω_2 lie on opposite sides of this line.

Ultimately, the positively oriented boundary arc $\widetilde{\omega''\zeta''\omega_2}$ must intersect the line l in at least one more point ω'''. Thus, l has more than two intersection points with $\partial\Omega$, namely: ω', ω'' and ω''', contradicting condition 2) of Definition 1.

This contradiction proves the lemma. □

The main result of the present work is the following:

Theorem 1. *Let Ω be a special n-convex domain with vertices $\omega_{k,1}$, $k = 1, \dots, n-1$. Let φ map $\mathbb{T}$ homeomorphically onto $\partial\Omega$, with $\zeta_{k,1} = \varphi^{-1}(\omega_{k,1})$, $k = 1, \dots, n-1$, being the preimages of the vertices of Ω under φ. If there exists $\delta > 0$, such that the Poisson integral f_φ maps $U_\delta(\zeta_{k,1}) \cap \mathbb{D}$ into Ω for $k = 1, \dots, n-1$, then f_φ realizes a univalent mapping of $\mathbb{D}$ onto Ω.*

Proof. Let Ω be a special n-convex domain. Consider an arbitrary partition line Γ_k, $k = 1, \dots, n-1$. Let $\Gamma = [w_0; w_1]$ denote the intersection of the line Γ_k with the closure of Ω, where $w_0 = \omega_{k,0}$ is a support point of Ω, and $w_1 = \omega_{k,1}$ is the corresponding vertex. Without loss of generality, we may assume that $\Gamma \subset \mathbb{R}$. Otherwise, instead of the function f_φ we consider the Poisson integral $f_{\tilde{\varphi}}$ with the boundary function $\tilde{\varphi}(z) = e^{i\theta}(\varphi(z) + c)$, which homeomorphically maps $\mathbb{T}$ onto the boundary $\partial\tilde{\Omega}$ of the domain $\tilde{\Omega}$ obtained by a suitable translation and rotation of the original domain Ω, such that the corresponding partition line $\tilde{\Gamma}_k$ coincides with $\mathbb{R}$.

We shall prove that under the conditions of the theorem, the Poisson

integral f_φ bijectively maps some Jordan arc in the disk $\mathbb{D}$ with endpoints on the circle $\mathbb{T}$ onto the segment Γ.

Consider the level set

$$\Lambda = \{z \in \mathbb{D}\colon \operatorname{Im} f_\varphi(z) = 0\}$$

of the real-valued harmonic function $v(x, y) = \operatorname{Im} f_\varphi(z)$. Let Λ' denote the closure of an arbitrary connected component of the set Λ.

Note that $\operatorname{Im} f_\varphi \not\equiv \text{const}$ in $\mathbb{D}$, since otherwise by continuity $\operatorname{Im} \varphi|_{\mathbb{T}} \equiv \text{const}$, contradicting the fact that $\varphi = f_\varphi|_{\mathbb{T}}$ maps $\mathbb{T}$ homeomorphically onto $\partial\Omega$.

Let us clarify the structure of the set Λ'. Note that $\Lambda' \cap \mathbb{T} = \{f_\varphi^{-1}(\omega_{k,j})\}_{j=0,1} = \{\varphi^{-1}(w_j)\}_{j=0,1}$. Denote the points $\varphi^{-1}(w_j)$ by ζ_j, $j = 0, 1$. We show that there exists a point $z^* \in \mathbb{D} \cap \Lambda'$. The points ζ_j, $j = 0, 1$, divide the circle $\mathbb{T}$ into two arcs. On each of them choose points ζ' and ζ''. The image of the chord $\gamma = [\zeta'; \zeta'']$ under the continuous mapping f_φ is a continuous curve $\tilde{\gamma}$ with endpoints $\varphi(\zeta') = \omega'$ and $\varphi(\zeta'') = \omega''$, located in the upper and lower half-planes. Consequently, the curve $\tilde{\gamma}$ intersects the real axis at some point w^*, for which there is at least one point $z^* \in \gamma$, such that $\operatorname{Im} f_\varphi(z^*) = 0$. Thus, $\mathbb{D} \cap \Lambda' \neq \varnothing$.

We show that Λ' consists of a single analytic arc in $\overline{\mathbb{D}}$ with beginning and end at ζ_0 and ζ_1, having no other intersection points with the circle $\mathbb{T}$. For this, we use fairly standard arguments going back to the work of Kneser (see, for example, [6]).

Recall that a point $z_0 = x_0 + iy_0$ is called a critical point of a real-valued continuously differentiable function $u = u(x, y)$ if both partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ vanish at (x_0, y_0).

It is known [6] that the level set $\{v(x, y) = v(x_0, y_0)\}$ of a nonconstant real-valued harmonic function v in a neighborhood of a critical point $z_0 = x_0 + iy_0$ consists of two or more analytic curves intersecting at equal angles at z_0. If z_0 is a non-critical point of a real-valued harmonic function, then the level set of this function locally consists of a single analytic arc γ passing through z_0. In particular, such an arc γ cannot end at any interior point of the domain of definition of v. Moreover, no arcs of the level set of a nonconstant real-valued harmonic function v can form closed loops in the closure of the simply connected domain of definition D of the function v. Indeed, the presence of closed loops of level contours would entail the existence of a closed Jordan curve γ in the closure of the simply connected domain D on which the harmonic function v is continuous and

constant. Then from the maximum principle for harmonic functions it would follow that $v(x,y) \equiv \text{const}$ everywhere inside the loop γ. However, this would imply constancy of the harmonic function v throughout the entire domain D by the uniqueness of recovering an analytic function from its real or imaginary parts in a simply connected domain. This contradicts the nonconstancy of v.

Thus, any connected component of the level set Λ defined above of the nonconstant real-valued harmonic function $\operatorname{Im} f_\varphi(z)$ is either an analytic Jordan arc that meets the circle $\mathbb{T}$ in exactly two points, or a union of analytic Jordan arcs intersecting at critical points of the function $\operatorname{Im} f_\varphi(z)$, and forming no closed loops in $\overline{\mathbb{D}}$. However, if the second situation occurs, or if Λ consists of more than one connected component, then the set $\overline{\Lambda}$ has at least three intersection points with the circle $\mathbb{T}$. In this case, under the homeomorphic mapping φ, these points correspond to three distinct points of the Jordan curve $\partial\Omega$, and by the definition of the set Λ, these points must lie on the real axis. But by condition 1) of a special n-convex domain, only two points of $\partial\Omega$ lie on the real line, contradicting the homeomorphic nature of φ.

Therefore, the set Λ is connected and consists of a single analytic Jordan arc with beginning and end at ζ_0 and ζ_1. The closure of the arc Λ will still be denoted by Λ'.

Clearly, $f_\varphi^{-1}(\Gamma) \subset \Lambda'$. We shall prove that $f_\varphi^{-1}(\Gamma) = \Lambda'$. For this, it is sufficient to prove that $f_\varphi(\Lambda') \subset \Gamma$.

Let the curve Λ' be given by a continuous bijective parametrization $z = z(t)$, $t \in [0;1]$, $z(0) = \zeta_0$, $z(1) = \zeta_1$. Note that by continuity of the Poisson integral, $f_\varphi(\Lambda')$ is a connected and bounded set, i.e., $f_\varphi(\Lambda') = [a;b] \subset \mathbb{R}$, where $a = \min\limits_{t\in[0;1]} f_\varphi(z(t))$, $b = \max\limits_{t\in[0;1]} f_\varphi(z(t))$. Then the condition $f_\varphi(\Lambda') \subset \Gamma$ is equivalent to $a \geqslant w_0$ and $b \leqslant w_1$.

However, the point $w_0 = \omega_{k,0}$ is a support point for Ω. Consequently, f_φ does not take values to the left of w_0, i.e., $a \geqslant w_0$.

Suppose that $b > w_1$. Define $s(t) = f_\varphi(z(t))$, $s(t) \in [a;b]$ for $t \in [0;1]$, with $s(1) = w_1 < b$. Then there exists a point $t^* \in (0;1)$, such that $b = \max\limits_{t\in[0;1]} f_\varphi(z(t)) = s(t^*)$. By the condition of the theorem, there exists $\delta > 0$, such that $f_\varphi(U_\delta(\zeta_1) \cap \mathbb{D}) \subset \Omega$. Consequently, there exists a value $t_\delta \in (0;1)$, such that $z(t) \in U_\delta(\zeta_1) \cap \mathbb{D}$ and, as a consequence, $s(t) = f_\varphi(z(t)) < w_1$ for all $t \in (t_\delta;1)$. It is clear that $t_\delta > t^*$.

It follows from these relations that, for $t \in (t_\delta;1)$, the inequalities

$s(t) < s(1) = w_1 < s(t^*) = b$ hold, with $t^* < t_\delta < 1$. By continuity of the function $s(t)$, we obtain that on the interval $(t^*; 1)$ there exists a local minimum point $t_0 \in (t^*; 1)$ of the function $s(t)$, such that $z(t_0) \in \mathbb{D}$ and $s(t_0) < w_1$. Moreover, $s(t_0) > w_0 = a$, since the point $w_0 = \omega_{k,0}$ is a support point for domain Ω and therefore lies on the boundary of the convex hull of the boundary values f_φ. Consequently, $f_\varphi(z(t_0)) \neq w_0$. Thus, the function $s(t) = f_\varphi(z(t))$ is nonmonotone in every neighborhood of t_0. This means that the harmonic function $f_\varphi(z)$ is not univalent in any neighborhood of $z_0 = z(t_0)$, since in any such neighborhood there exist points $z', z'' \in \Lambda'$ for which $f_\varphi(z') = f_\varphi(z'') \in [a; b]$. Consequently, f_φ is not locally univalent at z_0, whence by Lewy's theorem [6] we obtain $J_{f_\varphi}(z_0) = 0$. Let $f_\varphi = u + iv$. The condition $J_{f_\varphi}(z_0) = 0, z_0 = x_0 + iy_0$, is equivalent to

$$\begin{vmatrix} u_x(x_0, y_0) & v_x(x_0, y_0) \\ u_y(x_0, y_0) & v_y(x_0, y_0) \end{vmatrix} = 0.$$

Vanishing of the determinant means linear dependence of its rows, i.e., there exist $c, d \in \mathbb{R}$, with $c^2 + d^2 > 0$, such that

$$\begin{cases} c\, u_x(x_0,y_0) + d\, v_x(x_0,y_0) = 0, \\ c\, u_y(x_0,y_0) + d\, v_y(x_0,y_0) = 0. \end{cases}$$

Define the harmonic function $\psi(x, y) = c\, u(x, y) + d\, v(x, y)$ in the disk $\mathbb{D}$. The point z_0 is critical for ψ, as follows from the above system. Note that in these relations $c \neq 0$. Indeed, otherwise $\psi(x, y) = d\, v(x, y)$, with $d \neq 0$ and $\psi(x_0, y_0) = d\, v(x_0, y_0) = 0$ since $z_0 \in \Lambda'$. Then the zero level line of ψ coincides with Λ':

$$\{z \in \overline{\mathbb{D}}\colon \psi(x, y) = 0\} = \{z \in \overline{\mathbb{D}}\colon v(x, y) = 0\} = \{z \in \overline{\mathbb{D}}\colon \operatorname{Im} f_\varphi(z) = 0\} = \Lambda'.$$

Thus, $z_0 \in \Lambda'$ is a critical point of the real-valued harmonic function ψ. Consequently, the set Λ' in a neighborhood of z_0 is a union of at least two arcs intersecting at z_0 at equal angles. On the other hand, it was shown above that Λ' consists of a single Jordan arc, whence we obtain a contradiction. Therefore, $c \neq 0$. Without loss of generality, we may assume that $c = 1$.

Note that the function ψ is not constant, since otherwise $u(x, y) = e - d\, v(x, y)$, for some constants $d, e \in \mathbb{R}$. But in this case, the Poisson integral

$$f_\varphi(z) = u(x, y) + iv(x, y) = e + (i - d)v(x, y)$$

maps the disk $\mathbb{D}$ into a straight line. Consequently, $f_\varphi(\mathbb{T}) = \partial\Omega$ lies on a straight line, contradicting the special n-convexity of Ω.

Thus, the function $\psi(x,y) = u(x,y) + d\,v(x,y)$ is nonconstant and has a critical point $z_0 = x_0 + iy_0 \in \Lambda' \cap \mathbb{D}$, with $v(x_0,y_0) = 0$. Consider the connected component Λ_0 of the level set

$$\{z \in \overline{\mathbb{D}}\colon \psi(x,y) = \psi(x_0,y_0)\} = \{z \in \overline{\mathbb{D}}\colon u(x,y) + d\,v(x,y) = u(x_0,y_0)\},$$

passing through $z_0 = x_0 + iy_0$.

Note that f_φ maps the set Λ_0 into the line $L\colon u + d\,v = u(x_0,y_0)$. Moreover, the line L intersects the segment Γ at the interior point $u(x_0,y_0) = f_\varphi(z_0)$. By the Lemma, from the special n-convexity of Ω it follows that the line L has only two intersection points with $\partial\Omega$.

On the other hand, the level set Λ_0 contains the critical point z_0 of the nonconstant harmonic function ψ. Consequently, Λ_0 in a neighborhood of z_0 is a union of at least two arcs intersecting at z_0 at equal angles and, as noted above, in this case Λ_0 has at least four common points with $\mathbb{T}$.

However, $f_\varphi|_{\mathbb{T}} = \varphi$. By the hypothesis of the theorem being proved, φ is a homeomorphism of $\mathbb{T}$ onto $\partial\Omega$. Consequently, the set $f_\varphi(\Lambda_0) \subset L$ has at least four distinct common points with $\partial\Omega$, contradicting the property of the line L.

Thus, $w_0 \leqslant a$, $b \leqslant w_1$ and $f_\varphi(\Lambda') \subset \Gamma$. Hence, $f_\varphi(\Lambda') = \Gamma$.

We now show that f_φ maps the curve Λ' onto the segment Γ strictly monotonically and, consequently, bijectively. Indeed, if this were not the case, then there would exist a point $z_0 \in \Lambda' \cap \mathbb{D}$ in every neighborhood of which f_φ takes the same value. It follows that f_φ is not locally univalent at z_0 and $J_{f_\varphi}(z_0) = 0$. Repeating the arguments given above leads to a contradiction with the bijectivity of φ.

Thus, we have proved that the Poisson integral f_φ homeomorphically maps the curve Λ' onto the segment Γ. Since these arguments apply to any of the partition lines Γ_k, $k = 1,\dots,n-1$, the function f_φ homeomorphically maps certain Jordan arcs Λ_k onto the segments $\Gamma_k \cap \overline{\Omega}$, $k = 1,\dots,n-1$. Moreover, the arcs Λ_k are pairwise disjoint in the disk $\mathbb{D}$, since the lines Γ_k have this property.

The family of straight lines Γ_k partitions the domain Ω into n convex subdomains. Consequently, the Jordan curves Λ_k partition the unit disk $\mathbb{D}$ into n subdomains.

Consider an arbitrary one of these subdomains D_m. It is bounded by some number of Jordan curves Λ_{m_k} and arcs γ_{m_k} of the unit circle $\mathbb{T}$.

The Poisson integral f_φ homeomorphically maps Λ_{m_k} onto segments of the corresponding partition lines Γ_{m_k}, and the arcs γ_{m_k} onto the corresponding arcs $\varphi(\gamma_{m_k})$ on the boundary of domain Ω. Consequently, f_φ maps the boundary of the Jordan domain $D_m \subset \mathbb{D}$ onto the boundary of the corresponding convex subdomain $\Omega_m \subset \Omega$ homeomorphically.

Consider a univalent conformal mapping Φ_m of the unit disk $\mathbb{D}$ onto the Jordan domain D_m. By Carathéodory's extension theorem, Φ_m extends to a homeomorphism of the closed disk onto $\overline{D_m}$. Define the boundary homeomorphism φ_m of the circle $\mathbb{T}$ onto the boundary of the convex domain Ω_m by the formula $\varphi_m(\zeta) = f_\varphi(\Phi_m(\zeta))$. By the Radó-Kneser-Choquet theorem, the Poisson integral f_{φ_m} univalently maps $\mathbb{D}$ onto Ω_m. Consequently, the original Poisson integral f_φ univalently maps each of the subdomains $D_m \subset \mathbb{D}$ onto the corresponding subdomain $\Omega_m \subset \Omega$. Moreover, the function f_φ is continuous on the arcs Λ_k, $k = 1, \dots, n-1$.

It remains to note that if f_φ were not globally univalent in $\mathbb{D}$, then f_φ would lose local univalence at some points of the arcs Λ_k. However, as proved above, the Jacobian J_{f_φ} cannot vanish on Λ_k, $k = 1, \dots, n-1$.

Thus, the Poisson integral f_φ realizes a univalent harmonic mapping of $\mathbb{D}$ onto the special n-convex domain Ω.

The theorem is proved. □

Remark 1. *From the proof of Theorem 1 it follows that if φ homeomorphically maps $\mathbb{T}$ onto the boundary $\partial\Omega$ of a special n-convex domain Ω, then for univalence of the Poisson integral it is sufficient to verify that f_φ maps the preimages of arbitrarily small intervals of the partition lines Γ_k, adjacent to the vertices $\omega_{k,1}$ into the interior points of Ω. That is, univalence of f_φ follows from the condition $f_\varphi(\Lambda_k \cap U_\delta(\zeta_{k,1})) \subset \Omega$ for all $k = 1, \dots, n-1$.*

3. Corollaries and examples. Remark 1 allows us to identify some classes of homeomorphisms φ to which Theorem 1 is applicable, and to provide corresponding examples.

The following result gives a way to construct univalent local boundary variations of harmonic mappings onto special 2-convex domains.

Theorem 2. *Let Ω be a special 2-convex domain with the partition line $\Gamma = \mathbb{R}$, the vertex at the origin, and the support point on the left real half-axis. Let f_φ be a univalent harmonic mapping of $\mathbb{D}$ onto Ω with boundary homeomorphism φ, such that $\varphi(1) = 0$, and suppose there exists $\delta > 0$, such that $\varphi(e^{it}) = \overline{\varphi(e^{-it})}$ for all $t \in (0; \delta]$. Suppose also that $f_\varphi(r) < 0$*

for all $r \in (1-\varepsilon; 1)$ for some $\varepsilon > 0$. Define the function

$$\tilde{\varphi}(e^{it}) = \begin{cases} \varphi(e^{it}), & \text{if } \delta < |t| \leqslant \pi, \\ \varphi(e^{i\,p(t)}), & \text{if } |t| \leqslant \delta, \end{cases}$$

where $p(t)$ is a continuous odd function on $[-\delta; \delta]$, such that $p(t)$ is strictly monotone on $[0; \delta]$, $|p(t)| \leqslant |t|$ and $p(\delta) = \delta$. Then the Poisson integral $f_{\tilde{\varphi}}$ realizes a univalent harmonic mapping of the disk $\mathbb{D}$ onto Ω.

Proof. First, note that the function $\tilde{\varphi}$ defines a homeomorphism of the unit circle $\mathbb{T}$ onto $\partial\Omega$. In accordance with Remark 1, it suffices to show that $f_{\tilde{\varphi}}(r) < 0$ for all $r \in (1-\varepsilon; 1)$. For such r, consider

$$f_{\tilde{\varphi}}(r) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \frac{1-r^2}{|e^{it}-r|^2} \tilde{\varphi}(e^{it}) dt =$$

$$= \frac{1}{2\pi} \int\limits_{\delta<|t|\leqslant\pi} \frac{1-r^2}{|e^{it}-r|^2} \varphi(e^{it}) dt + \frac{1}{2\pi} \int\limits_{-\delta}^{\delta} \frac{1-r^2}{|e^{it}-r|^2} \varphi(e^{i\,p(t)}) dt =$$

$$= f_{\varphi}(r) + \frac{1}{2\pi} \int\limits_{-\delta}^{\delta} \frac{1-r^2}{|e^{it}-r|^2} \left(\varphi(e^{i\,p(t)}) - \varphi(e^{it})\right) dt =$$

$$= f_{\varphi}(r) + \frac{1}{2\pi} \Big\{ \int\limits_{0}^{\delta} \frac{1-r^2}{|e^{it}-r|^2} \left(\varphi(e^{i\,p(t)}) - \varphi(e^{it})\right) dt +$$

$$+ \int\limits_{0}^{\delta} \frac{1-r^2}{|e^{-it}-r|^2} \left(\varphi(e^{i\,p(-t)}) - \varphi(e^{-it})\right) dt \Big\}.$$

By the condition $\varphi(e^{it}) = \overline{\varphi(e^{-it})}$ and since $p(t)$ is odd for $|t| \leqslant \delta$, we obtain

$$f_{\tilde{\varphi}}(r) = f_{\varphi}(r) + \frac{1}{\pi} \int\limits_{0}^{\delta} \frac{1-r^2}{|e^{it}-r|^2} \operatorname{Re}\left(\varphi(e^{i\,p(t)}) - \varphi(e^{it})\right) dt.$$

The point $\omega = 0$ is the vertex of Ω. Moreover, some left half-neighborhood of the origin lies in Ω. The arcs $\gamma_\delta := \{\varphi(e^{it}) \colon 0 \leqslant t \leqslant \delta\}$ and $\overline{\gamma_\delta}$ are fragments of the boundary curves of the convex subdomains

$\Omega_+ := \Omega \cap \{w : \operatorname{Im} w > 0\}$ and $\Omega_- := \Omega \cap \{w : \operatorname{Im} w < 0\}$, respectively. Consequently, for sufficiently small $\delta > 0$, the function $u(t) = \operatorname{Re} \varphi(e^{it})$, decreasing monotonically, tends to 0 as $t \to 0$, $t \in (0; \delta)$.

Since by the condition $p(t) \leqslant t$ for all $t \in (0; \delta)$, we obtain the estimate

$$\operatorname{Re}\left(\varphi(e^{i\,p(t)}) - \varphi(e^{it})\right) = u(p(t)) - u(t) \leqslant 0 \quad \text{for} \quad 0 \leqslant t \leqslant \delta.$$

Finally, $f_{\tilde{\varphi}}(r) \leqslant f_{\varphi}(r) < 0$ for all $r \in (1 - \varepsilon; 1)$. Then for the function $f_{\tilde{\varphi}}$ the conditions of Remark 1 are satisfied, and it maps $\mathbb{D}$ onto Ω univalently. $\square$

To illustrate Theorem 2, consider the function $\Phi(z) = -(z-1)^2$, which realizes a univalent conformal mapping of the disk $\mathbb{D}$ onto the interior Ω of a cardioid. The boundary homeomorphism $\varphi(e^{it}) = \Phi(e^{it}) = -(e^{it} - 1)^2$ and the domain Ω satisfy the conditions of Theorem 2. Then, defining, for example, the function $p(t) = t^3/\delta^2$ for $|t| \leqslant \delta$ for sufficiently small $\delta > 0$, we obtain a non-conformal harmonic univalent mapping $f_{\tilde{\varphi}}$ of the unit disk onto Ω.

The following result is formulated in terms of Fourier coefficients of f_{φ}. Recall that the Poisson integral f_{φ} for any $z = r\,e^{i\theta} \in \mathbb{D}$ can be represented by the Fourier series:

$$f_{\varphi}(z) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} \varphi(e^{i\,t}) dt =$$

$$= \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \sum_{k=-\infty}^{+\infty} r^{|k|} e^{i\,k(\theta - t)} \varphi(e^{i\,t}) dt = \sum_{k=-\infty}^{+\infty} c_k r^{|k|} e^{i\,k\theta},$$

where

$$c_k = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \varphi(e^{it}) \cdot e^{-i\,kt} dt.$$

Theorem 3. *Let Ω be a special 2-convex domain with the partition line $\Gamma = \mathbb{R}$, the vertex at the origin, and the support point on the left real half-axis. Let $f_{\varphi}(z)$ be the Poisson integral with boundary function φ homeomorphically mapping the circle $\mathbb{T}$ onto the boundary $\partial\Omega$, such that $0 > f_{\varphi}(0) \in \Omega$, $f_{\varphi}(1) = 0$.*

If the series $\sum\limits_{k=1}^{\infty} k(c_k + c_{-k}) r^k$ converges uniformly for r on the half-interval $(1 - \varepsilon; 1]$ for some $0 < \varepsilon < 1$ and the Fourier coefficients of the

function f_φ satisfy the conditions

$$(a)\ \ c_k + c_{-k} \in \mathbb{R} \ \ \textit{for all } k \in \mathbb{N},$$
$$(b)\ \ \sum_{k=1}^{\infty} k(c_k + c_{-k}) > 0,$$

then f_φ maps $\mathbb{D}$ onto Ω univalently.

Proof. Consider the continuous function $u(r) := f_\varphi(r)$ on $(1-\varepsilon; 1]$, with $u(1) = 0$. From condition (a) and the fact that $c_0 = f_\varphi(0) < 0$ it follows that for all $r \in (0; 1)$ the values

$$u(r) = c_0 + \sum_{k=1}^{\infty} r^k (c_k + c_{-k}) \in \mathbb{R}.$$

Taking into account the structure of the special 2-convex domain Ω, this means that for the partition line $\Gamma = \mathbb{R}$, its preimage $\Lambda = f_\varphi^{-1}(\Gamma)$ contains the half-interval $(1-\varepsilon; 1]$. Then, by Remark 1 to Theorem 1, it suffices to prove that $f_\varphi(r) = u(r) < 0$ for $r \in (1-\varepsilon; 1)$.

From the uniform convergence of the series $\sum\limits_{k=1}^{\infty} k(c_k + c_{-k}) r^k$ it follows that the function $u(r)$ is continuously differentiable on $(1-\varepsilon; 1]$, and from condition (b) we obtain:

$$u'(1) = \sum_{k=1}^{\infty} k(c_k + c_{-k}) > 0.$$

In some half-neighborhood of the point $r = 1$, the continuous function $u'(r)$ preserves sign. Without loss of generality, we may assume that this is the same half-neighborhood $(1-\epsilon; 1]$ as above. Thus, $u'(r) > 0$ for all $r \in (1-\epsilon; 1]$, that is, $u(r)$ is strictly increasing for $1 - \epsilon < r \leqslant 1$.

Consequently, $f_\varphi(r) = u(r) < u(1) = f_\varphi(1) = 0$ for all $r \in (1-\epsilon; 1]$.

Then for the function f_φ the conditions of Remark 1 are satisfied, and it maps $\mathbb{D}$ onto Ω univalently.

Note also that if, under the other conditions of Theorem 3, condition (b) is replaced by the inequality $\sum\limits_{k=1}^{\infty} k(c_k + c_{-k}) < 0$, then this will automatically mean that the Poisson integral f_φ is not univalent. Indeed, in this situation the function $u(r)$ is strictly decreasing in the half-neighborhood

$(1-\epsilon;1]$ and, consequently, f_φ takes values outside Ω. Thus, inequality (b) is sharp. □

Remark 2. *Recall that a harmonic polynomial of degree n is a function of the form $F = P + \overline{Q}$, where P, Q are analytic polynomials with $\max\{\deg P, \deg Q\} = n$. In the case when in Theorem 3 f_φ is a harmonic polynomial of degree n, the condition of uniform convergence of the series is not required, and univalence of f_φ is ensured by the conditions*

$$c_k + c_{-k} \in \mathbb{R} \ \text{ for all } k = 1, \dots, n, \ \text{ and } \ \sum_{k=1}^{n} k(c_k + c_{-k}) > 0.$$

An illustration of this remark is the harmonic polynomial

$$f_\varphi(z) = -1 + \frac{5}{4}z - \frac{1}{2}z^2 + \frac{1}{4}\overline{z}.$$

It satisfies all the conditions described in Remark 2: $c_1 + c_{-1} = 3/2$, $c_2 + c_{-2} = -1/2$ and $c_1 + c_{-1} + 2(c_2 + c_{-2}) = 1/2 > 0$. The boundary function $\varphi(e^{it})$ homeomorphically maps the circle $\mathbb{T}$ onto the boundary of a special 2-convex domain. To verify this, consider the function

$$r(t) = \varphi(e^{it}) = -\frac{1}{2} + \frac{3}{2}\cos t - \cos^2 t + i \sin t \cdot (1 - \cos t).$$

Direct computations establish that $r(0) = r(2\pi) = 0$ and

$$\frac{d}{dt} \arg r(t) = \frac{2 - 5\cos t + 4\cos^2 t - \cos^3 t}{2|r(t)|^2}.$$

The numerator of this fraction is positive for all $t \in (0; 2\pi)$, and it is easy to see that $\lim\limits_{t\to 0+} \arg r(t) = 0$, $\arg r(\pi) = \pi$ and $\lim\limits_{t\to 2\pi-} \arg r(t) = 2\pi$. Consequently, $r(t)$ is strictly increasing for $t \in [0; 2\pi]$ and defines a parametrization of a starlike Jordan curve $\partial\Omega$.

Moreover, the tangent vector to $\partial\Omega$ is given by

$$\tau(t) = -\frac{3}{2}\sin t + \sin(2t) + i(\cos t - \cos(2t)).$$

From this we obtain that

$$\frac{d}{dt} \arg \tau(t) = \frac{7 - 9\cos t + 2\cos^3 t}{2|\tau(t)|^2}.$$

The numerator of the last fraction is also positive for $t \in (0; 2\pi)$. Thus, the Jordan arc $\{r(t)\colon t \in (0; 2\pi)\}$ of the curve $\partial\Omega$ is convex. Convexity and regularity of $\partial\Omega$ are violated only at the point $r(0) = 0$, since $\tau(0) = \tau(2\pi) = 0$. As a result, we obtain that $\varphi(e^{it}) = r(t)$ homeomorphically maps $\mathbb{T}$ onto the boundary of a special 2-convex domain Ω with vertex $\omega = 0$ and partition line $\Gamma = \mathbb{R}$. Consequently, by Theorem 3, the harmonic polynomial f_φ univalently maps $\mathbb{D}$ onto Ω.

Note that in accordance with Choquet's result [3], the homeomorphic nature of the function φ in this example does not by itself guarantee univalence of the mapping f_φ.

Acknowledgment. The authors would like to thank the reviewers for their valuable comments, which made it possible to improve the article.

References

[1] Alessandrini G., Nesi V. *Invertible harmonic mappings, beyond Kneser.* Ann. Scuola Norm. Sup. Pisa Cl. Sci., 2009, vol. VII (5), pp. 451 – 468. DOI: https://doi.org/10.2422/2036-2145.2009.3.03

[2] Chen Sh. L., Ponnusamy S., Rasila A., Wang X. T. *Linear connectivity, Schwarz-Pick lemma and univalency criteria for planar harmonic mapping.* Acta Math. Sin., Engl. Ser., 2016, vol. 32(3), pp. 297–308. DOI: https://doi.org/10.1007/s10114-016-4259-3

[3] Choquet G. *Sur un type de transformation analytique generalisant la representation conforme et define au moyen de fonctions harmoniques.* Bull. Sci. Math., 1945, vol. 69, 156 – 165.

[4] Clunie J. G., Sheil-Small T. *Harmonic univalent functions.* Ann. Acad. Sci. Fenn. Ser. A I Math., 1984, vol. 9, no. 1, pp. 3 – 25. DOI: https://doi.org/10.5186/aasfm.1984.0905

[5] Cristea M. *A generalization of the argument principle.* Complex Variables, Theory and Application: An International Journal, 2000, vol. 42(4), pp. 333 – 345. DOI: https://doi.org/10.1080/17476930008815293

[6] Duren P. *Harmonic mappings in the plane.* Cambridge University Press, 2004, 214 p.

[7] Garnett J.B., Marshall D.E. *Harmonic measure* Cambridge University Press, 2005, 571 p.

[8] Graf S. Yu. *Harmonic mappings onto R-convex domains* Probl. Anal. Issues Anal., 2019, vol. 8(26), no. 2, pp. 37 – 50. DOI: https://doi.org/10.15393/j3.art.2019.6190

[9] Graf S. Yu., Ponnusamy S., Starkov V. V. *Univalence criterion for harmonic mappings and Φ-like functions.* Complex Variables and Elliptic Equations, 2018, vol. 63(12), pp. 1767 – 1779.
DOI: https://doi.org/10.1080/17476933.2017.1409741

[10] Graf S. Yu., Nikitin I. A. *Planar harmonic mappings with a given Jacobian.* Probl. Anal. Issues Anal., 2023, vol. 12 (30), no. 3, pp. 69 – 85.
DOI: https://doi.org/10.15393/j3.art.2023.14110

[11] Hernandez, R., Martin, M. J. *Criteria for univalence and quasiconformal extension of harmonic mappings in terms of the Schwarzian derivative.* Arch. Math., 2015, vol. 104 (1), pp. 53 – 59.
DOI: https://doi.org/10.1007/s00013-014-0714-5

[12] Ivanov G. E. *Weak convex sets and functions. Theory and applications.* Physmathlit, Moscow, 2006, 352 p. (in Russian)

[13] Kay D. C., Guay M. D. *Convexity and a certain property P_m.* Israel J. Math., 1970, vol. 8, pp. 39 – 52.
DOI: https://doi.org/10.1007/BF02771549

[14] Kneser H. *Losung der Aufgabe 41.* Jahresber. Deutsch. Math.-Verein., 1926, vol. 35, pp. 123 –124.

[15] Lyzzaik A. *Univalence Criteria for Harmonic Mappings in Multiply Connected Domains.* Journal of the London Mathematical Society, 1998, vol. 58(1), pp. 163 –171.
DOI: https://doi.org/10.1112/S0024610798006358

[16] Polovinkin E. S., Balashov M. V. *Elements of convex and strongly convex analysis.* Physmathlit, Moscow, 2004, 416 p. (in Russian)

[17] Radó T. *Aufgabe 41.* Jahresber. Deutsch. Math.-Verein., 1926, vol. 35, p. 49.

[18] Starkov V. V. *Univalence of harmonic functions, problem of Ponnusamy and Sairam, and constructions of univalent polynomials.* Probl. Anal. Issues Anal., 2014, vol. 3(21), no. 2, pp. 59 –73.
DOI: https://doi.org/10.15393/j3.art.2014.2729

[19] Starkov V. V., Shmelev N. A. *Biholomorphic mappings of the disk onto strongly convex domains.* Siberian Math. J., 2014, vol. 55, no. 4, pp. 869 – 875. DOI: https://doi.org/10.1134/S0037446614040132

[20] Valentine F. A. *A three point convexity property.* Pacific J. Math., 1957, vol. 7, no. 2, pp. 1227–1235.

[21] Xuan L. A. *Harmonic mappings and finite valency of analytic functions.* Uchenye Zapiski Kazanskogo Universiteta. Seriya Fiziko-Matematicheskie Nauki, 2018, vol. 160, no. 4, pp. 771–777. (In Russian)

Received July 09, 2026.
In revised form, September 26, 2026.
Accepted October 15, 2026.
Published online November 02, 2026.

S.Yu. Graf
Tver State University
35 Sadovyy per., Tver, 170002, Russia,
Petrozavodsk State University
33 Lenina pr., Petrozavodsk, 185910, Russia
E-mail: Graf.SY@tversu.ru

Yu.D. Yakhova
Lomonosov Moscow State University
1 Leninskie Gory, Moscow, 119991, Russia
Tver State University
35 Sadovyy per., Tver, 170002, Russia,
E-mail: juliayahova@mail.ru