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INEQUALITIES CONNECTING GENERALIZED TRIGONOMETRIC FUNCTIONS WITH THEIR INVERSES

Abstract. Motivated by the recent work [1], in this paper we study the relations of generalized trigonometric and hyperbolic functions of two parameters with their inverse functions.

Key words: *Inequalities, generalized trigonometric functions, Eigenfunctions and Incomplete beta function.*

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§ 1. Introduction

In [2] P. Lindqvist studied generalized trigonometric and hyperbolic functions (p-functions) for a parameter $p > 1$, and for $p = 2$ they coincide with elementary functions. These p-functions were studied extensively, see for example [2 − 9] and their references. Recently these functions have been extended to (p, q)-functions with two parameters $p, q > 1$ in [10 − 13]. These functions coincide with the p-functions for $p = q$. For the historical background see the bibliography of these papers. In [14] and [1] authors have studied the inequalities involving elementary functions and their inverses. Thereafter in [14] Klén et al. studied those results in terms of p-functions. Here we generalized those inequalities for (p, q)-functions and establish double inequality for $\sin_p$ in terms of elementary functions, $\sin_p$ occurs as an eigenfunction of the Dirichlet problem for the one-dimensional p-Laplacian, see [6].

Before we formulate our main results we define the (p, q)-functions and some other notation. The increasing homeomorphism function $F_{p,q}$: $[0, 1] \to [0, \pi_{p,q}/2]$ is defined by

$$\operatorname{arcsin}_{p,q}(x) = \int_0^x (1 - t^q)^{-1/p}\, dt.$$

Letting $t = z^{1/q}$, we have

$$\operatorname{arcsin}_{p,q}(x) = \frac{1}{q}\int\limits_0^{x^q} z^{1/q-1}(1-z)^{-1/p}\,dz = \frac{1}{q}\tilde{B}\left(\frac{1}{q}, 1-\frac{1}{p}, x^q\right),$$

where $\tilde{B}(a,b,x)$ is incomplete beta function defined as

$$\tilde{B}(a,b,x) = \int\limits_0^x t^{a-1}(1-t)^{b-1}dt.$$

The inverse of $\operatorname{arcsin}_{p,q}$ is denoted by $\sin_{p,q}$, which is defined on the interval $[0, \pi_{p,q}/2]$, where

$$\pi_{p,q} = 2\operatorname{arcsin}_{p,q}(1) = \frac{2}{q}\tilde{B}\left(\frac{1}{q}, 1-\frac{1}{p}, 1\right) = \frac{2}{q}B\left(\frac{1}{q}, 1-\frac{1}{p}\right),$$

here $B(a,b)$ denote the beta function. We also define

$$\operatorname{arccos}_{p,q} x = \operatorname{arcsin}_{p,q}((1-x^p)^{1/q})$$

(see [11, Prop. 3.1]), and

$$\cos_{p,q}(x) = \frac{d}{dx}\sin_{p,q}(x), \quad x \in [0, \pi_{p,q}/2].$$

Letting $y = \sin_{p,q}(x)$, we get

$$\cos_{p,q}(x) = (1 - (\sin_{p,q}(x))^q)^{1/p},$$

and

$$|\cos_{p,q}(x)|^p + |\sin_{p,q}(x)|^q = 1. \tag{1}$$

The generalized tangent function $\tan_{p,q}(x)$ is defined as

$$\tan_{p,q}(x) = \frac{\sin_{p,q}(x)}{\cos_{p,q}(x)}.$$

For $x \in (0, \infty)$, the inverse of generalized hyperbolic sine function $\sinh_{p,q}(x)$ is defined by

$$\operatorname{arcsinh}_{p,q} x = \int\limits_0^x (1+t^q)^{-1/p}dt,$$

and generalized hyperbolic cosine and tangent functions are defined by

$$\cosh_{p,q}(x) = \frac{d}{dx}\sinh_{p,q}(x), \quad \tanh_{p,q}(x) = \frac{\sinh_{p,q}(x)}{\cosh_{p,q}(x)}, \quad x \geq 0$$

respectively. It follows from the definitions, that

$$|\cosh_{p,q}(x)|^p - |\sinh_{p,q}(x)|^q = 1, \quad x \geq 0. \tag{2}$$

The main results of the this paper reads as below.

Theorem 1. *For $p, q > 1$ the following hold*

1) *For all $x \in (0,1)$ and $y \in (0, \pi_{p,q}/2)$ with $y < \mathrm{arcsin}_{p,q}(x)$ we have*
$$\mathrm{arcsin}_{p,q}(x)\sin_{p,q}(y) > xy.$$
2) *For all $x \in (0, \pi_{p,q}/2)$ and $y \in (0,1)$ with $\tan_{p,q}(x) > y$ we have*
$$\tan_{p,q}(x)\mathrm{arctan}_{p,q}(y) > xy.$$
3) *For all $x, y \in (0,\infty)$ with $y < \sinh_{p,q}(x)$ we have*
$$\sinh_{p,q}(x)\mathrm{arcsinh}_{p,q}(y) > xy.$$
4) *For all $x \in (0,1)$ and $y \in (0,\infty)$ with $\mathrm{arctah}_{p,q}(x) > y$ we have*
$$\mathrm{arctah}_{p,q}(x)\tanh_{p,q}(y) > xy.$$

Theorem 2. *For $p, q > 1$ the following hold*

1) $\dfrac{x}{\mathrm{arcsin}_{p,q}(x)} > \dfrac{\sin_{p,q}(\pi_{p,q}x/2)}{\pi_{p,q}x/2}, \quad x \in (0,1),$
2) $\dfrac{\tan_{p,q}(x)}{x} < \dfrac{bx}{\mathrm{arctan}_{p,q}(bx)}, \quad x \in (0,k),\ 0 < k < \dfrac{\pi_{p,q}}{2},$
$b = \tan_{p,q}(k)/k,$
3) $\dfrac{\sinh_{p,q}(x)}{x} < \dfrac{x}{a\,\mathrm{arctan}_{p,q}(x/a)}, \quad x \in (0,k),\ k > 0,\ a = \dfrac{k}{\sinh_{p,q}(k)}.$
4) $\dfrac{x}{\mathrm{arctanh}_{p,q}(x)} > \dfrac{\tanh_{p,q}(cx)}{cx}, \quad x \in (0,k),\ k \in (0,1),$
$c = k/\mathrm{arctanh}_{p,q}(k).$

§ 2. Preliminaries and proofs

The following derivative formulas will be used in our calculations, and they can be derived easily from the definition.

Lemma 1. *For all $x \in (0, \pi_{p,q}/2)$, we have*

1) $\dfrac{d}{dx}\cos_{p,q}(x) = -\dfrac{p}{q}(\cos_{p,q}(x))^{2-p}(\sin_{p,q}(x))^{q-1}\,,$

2) $\dfrac{d}{dx}\tan_{p,q}(x) = 1 + \dfrac{p}{q}\dfrac{(\sin_{p,q}(x))^{q}}{(\cos_{p,q}(x))^{p}}\,,$
and for all $x \in (0, \infty)$

3) $\dfrac{d}{dx}\cosh_{p,q}(x) = \dfrac{q}{p}(\cosh_{p,q}(x))^{2-p}(\sinh_{p,q}(x))^{q-1}\,,$

4) $\dfrac{d}{dx}\tanh_{p,q}(x) = 1 - \dfrac{q}{p}\dfrac{(\sinh_{p,q}(x))^{q}}{(\cosh_{p,q}(x))^{p}}\,.$

For the following *monotone l'Hospital rule* see [15, Theorem 1.25].

Lemma 2. *For $-\infty < a < b < \infty$, let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, and be differentiable on (a, b). Let $g'(x) \neq 0$ on (a, b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a, b), then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \textit{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

For the proof of following lemma see ([1]).

Lemma 3. *Let $f : I \to J$ be a injective function, where I, J are the subsets of $(0, \infty)$. Suppose that the function $g(x) = f(x)/x$, $x \in I$ is strictly increasing. Then for any $x \in I$, $y \in J$ such that $f(x) \geq y$ following holds*

$$f(x)f^{-1}(y) \geq xy\,,$$

where $f^{-1} : J \to I$ denotes the inverse function of f. Under the same condition if $f(x) \leq y$ then we have

$$f(x)f^{-1}(y) \leq xy\,. \tag{3}$$

For the following lemma see [16, Theorem 2, p. 151], [13, Theorem 1].

Lemma 4.

1) *Let $J \subset \mathbb{R}$ be an open interval, and $f : J \to \mathbb{R}$ be a strictly monotonic function. Let $f^{-1} : f(J) \to J$ be the inverse of f. If f is concave and increasing, then f^{-1} is convex.*
2) *For all $x \in (0,1)$, the functions $p \mapsto \operatorname{arcsin}_p(x)$ and $p \mapsto \operatorname{arctanh}_p(x)$ are strictly decreasing in $p \in (1,\infty)$.*

Lemma 5. *For $p, q > 1$, the following hold*

1) *the function $f(x) = \dfrac{\operatorname{arcsin}_{p,q}(x)}{x}$ is increasing in $x \in (0,1)$,*
2) *the function $g(x) = \dfrac{\tan_{p,q}(x)}{x}$ is increasing in $x \in (0, \pi_{p,q}/2)$,*
3) *the function $h(x) = \dfrac{\sinh_{p,q}(x)}{x}$ is increasing in $x \in (0,\infty)$,*
4) *the function $j(x) = \dfrac{\operatorname{arctah}_{p,q}(x)}{x}$ is increasing in $x \in (0,\infty)$ with $p > q$.*

Proof. Let $f(x) = \dfrac{\operatorname{arcsin}_{p,q}(x)}{x} = \dfrac{f_1(x)}{f_2(x)}$. Then $f_1'(x) = (1-x^q)^{-1/p} > 0$ and $f_2'(x) > 0$. Now it is clear by Lemma 2 that f is increasing. For the proof of part (2) and (3), let

$$g(x) = \frac{\tan_{p,q}(x)}{x} = \frac{g_1(x)}{g_2(x)},\ h(x) = \frac{\sinh_{p,q}(x)}{x} = \frac{h_1(x)}{h_2(x)}.$$

Differentiation gives

$$g_1'(x) = 1 + \frac{p}{q}\frac{(\sin_{p,q}(x))^q}{(\cos_{p,q}(x))^p} > 0, \quad \text{and} \quad h_1'(x) = \cosh_{p,q}(x) > 0,$$

and the proof is obvious from Lemma 2. For part (4), we get

$$\begin{aligned}\frac{d^2}{dx^2}\tanh_{p,q}(x) &= -\frac{q}{p}\left(\frac{q(\sinh_{p,q}(x))^{q-1}(\cosh_{p,q}(x))^{p+1} - q\cosh_{p,q}(x)}{(\sinh_{p,q}(x))^{2q-1}}\right) = \\ &= -\frac{q}{p}(\sinh_{p,q}(x))^{q-1}(\cosh_{p,q}(x))^{1-2p} < 0,\end{aligned}$$

since $\tanh_{p,q}(x)$ is concave, and clearly with $p > q$ it is increasing. By Lemma 4(1), $\text{arctah}_{p,q}(x)$ is convex, and from this fact we get, that

$$\frac{d}{dx}\text{arctah}_{p,q}(x)$$

is increasing. Hence the rest of proof follows from Lemma 2. $\square$

Proof of Theorem 1. The functions

$$\frac{\arcsin_{p,q}(x)}{x}, \quad \frac{\tan_{p,q}(x)}{x} \quad \frac{\sinh_{p,q}(x)}{x}, \quad \text{and} \quad \frac{\text{arctah}_{p,q}(x)}{x}$$

are increasing by Lemma 5. The rest of proof follows immediately from Lemma 3. $\square$

It is easy to check by using the derivative formulas that the following relations

$$x < \arcsin_{p,q}(x), \quad x \in (0,1),$$

$$x < \tan_{p,q}(x), \quad x \in (0, \pi_{p,q}/2),$$

$$x < \sinh_{p,q}(x), \quad x \in (0,\infty),$$

$$x > \tanh_{p,q}(x) \Rightarrow \text{arctanh}_{p,q}(x) > x, \quad x \in (0,1).$$

hold true for all $p, q > 1$.

By Theorem 1 and above relations we conclude the following corollary.

Corollary. *For $p, q > 1$ the following hold*

1) $\dfrac{x}{\arcsin_{p,q}(x)} < \dfrac{\sin_{p,q}(x)}{x}, \quad x \in (0,1),$

2) $\dfrac{x}{\arctan_{p,q}(x)} < \dfrac{\tan_{p,q}(x)}{x}, \quad x \in (0,1),$

3) $\dfrac{x}{\text{arcsinh}_{p,q}(x)} < \dfrac{\sinh_{p,q}(x)}{x}, \quad x \in (0,\infty),$

4) $\dfrac{x}{\text{arctanh}_{p,q}(x)} < \dfrac{\tanh_{p,q}(x)}{x}, \quad x \in (0,1).$

Proof of Theorem 2. The monotonicity of the functions

$$\frac{\operatorname{arcsin}_{p,q}(x)}{x}, \quad \frac{\tan_{p,q}(x)}{x} \quad \frac{\sinh_{p,q}(x)}{x}, \quad \frac{\operatorname{arctah}_{p,q}(x)}{x}$$

imply, that

$$f_1(x) = \frac{\pi_{p,q}}{2}\operatorname{arcsin}_{p,q}(x) < x,$$

$$f_2(x) = \frac{\tan_{p,q}(x)}{b} < x,$$

$$f_3(x) = a\sinh_{p,q}(x) < x,$$

$$\textit{and} \quad f_4(x) = c\operatorname{arctanh}_{p,q}(x) < x.$$

Hence

$$f_1^{-1}(x) = \sin_{p,q}(\pi_{p,q}x/2), \quad f_2^{-1}(x) = \operatorname{arctan}_{p,q}(bx),$$
$$f_3^{-1}(x) = \operatorname{arcsinh}_{p,q}(x/a), \quad f_4^{-1}(x) = \operatorname{arctanh}_{p,q}(cx),$$

and the proof follows from (3) if we let $y = x$. □

Corollary. *The following assertions hold true:*

1) $\dfrac{x}{\arcsin(x)} < \dfrac{\sin_p(x)}{x}, \quad \text{for}\, x \in (0,1),\ p \geq 2,$

2) $\dfrac{\sin_p(x)}{x} < \dfrac{2x/\pi_p}{\arcsin(2x/\pi_p)}, \quad \text{for}\, x \in (0,\pi_2),\ p \in (1,2],$

3) $\dfrac{x}{\arctan(x)} < \dfrac{\tan_p(x)}{x}, \quad \text{for}\, x \in (0,1),\ p \in (1,2],$

4) $\dfrac{\tan_p(x)}{x} < \dfrac{bx}{\arctan(bx)}, \quad \text{for}\, x \in (0,k),\ 0 < k < \pi_p/2,\ b = \dfrac{\tan(k)}{k}.$

The proof follows from Theorem 1, Lemma 4(2) and Corollary 2.

Remark. In [17, Theorem 2.3], the following inequalities was proved

$$\tilde{B}(a,b,x)\tilde{B}(a,b,y) \leq \tilde{B}(a,b,x+y-z)\tilde{B}(a,b,z)$$

for $a \in (0,1)$, $b > 0$ and $x, y > z$. Under the same assumption with $0 < x + y - z < 1$ and $x, y, z \in (0,1)$ one has

$$\operatorname{arcsin}_{p,q}(x)\operatorname{arcsin}_{p,q}(y) \leq \operatorname{arcsin}_{p,q}(x+y-z)\operatorname{arcsin}_{p,q}(z).$$

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