DOI: 10.15393/j3.art.2014.2569

UDC 517.518

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ANALOG OF AN INEQUALITY OF BOHR FOR INTEGRALS OF FUNCTIONS FROM $L^p(\mathbb{R}^n)$. II

Abstract. Let $p \in (2, +\infty]$, $n \geq 1$ and $\Delta = (\Delta_1, \dots, \Delta_n)$, $\Delta_k > 0$, $1 \leq k \leq n$. It is proved that for the functions $\gamma(t) \in$ $\in L^p(\mathbb{R}^n)$ with spectrum at distance at least Δ_k from each of the n coordinate hyperplanes, $1 \leq k \leq n$ respectively, the following inequality is valid:

$$\left\| \int\limits_{E_t} \gamma(\tau)\, d\tau \right\|_{L^\infty(\mathbb{R}^n)} \leq C^n(q) \left[\prod_{k=1}^{n} \frac{1}{\Delta_k^{1/q}} \right] \|\gamma(\tau)\|_{L^p(\mathbb{R}^n)},$$

where $t = (t_1, \dots, t_n) \in \mathbb{R}^n$, $E_t = \{\tau \,|\, \tau = (\tau_1, \dots, \tau_n) \in \mathbb{R}^n$, $\tau_j \in [0, t_j]$, if $t_j \geq 0$, and $\tau_j \in [t_j, 0]$, if $t_j < 0$, $1 \leq j \leq n\}$, and the constant $C(q) > 0$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$ does not depend on $\gamma(\tau)$ and Δ.

Key words: *inequality of Bohr*

2010 Mathematical Subject Classification: *26D99*

This work is the second and the final part of the authors' article [1]. It contains the proof of the statement announced in [1]. This statement provides conditions sufficient for validity of a new analog of the Bohr inequality [2]. (Find more details about inequality of Bohr and it's generalizations in the author's paper [3]).

Let $p \in (2, +\infty]$, $n \geq 1$, $\Delta = (\Delta_1, \Delta_2, \dots, \Delta_n)$ be any vector with positive coordinates and $t = (t_1, t_2, \dots, t_n) \in \mathbb{R}^n$.

Let us introduce the following notation:

1) $Q(\Delta) = \bigcup\limits_{k=1}^{n} \{y \,|\, y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n,\ |y_k| < \Delta_k\}$, i.e. $Q(\Delta)$ is the "crosslike" neigborhood of zero in $\mathbb{R}^n$;

2) $\Gamma(\mathbb{R}^n \setminus Q(\Delta), p)$ is the set of all functions $\gamma(t) \in L^p(\mathbb{R}^n)$ whose Fourier transformation support belongs to $\mathbb{R}^n \setminus Q(\Delta)$;
3) $E_t = \{\tau \mid \tau = (\tau_1, \tau_2, \ldots, \tau_n) \in \mathbb{R}^n,\ \tau_j \in [0, t_j]$, if $t_j \geq 0$, and $\tau_j \in [t_j, 0]$, if $t_j < 0,\ 1 \leq j \leq n\}$ is the parallelepiped in $\mathbb{R}^n$.

The main result of this paper is Theorem 3.2 from § 3, which is the following.

Theorem 3.2. *Let $n \geq 1$, $p \in (2, +\infty]$ and $\Delta = (\Delta_1, \ldots, \Delta_n)$, $\Delta_k > 0$, $1 \leq k \leq n$. Then for any function $\gamma(\tau) \in \Gamma(\mathbb{R}^n \setminus Q(\Delta), p)$ the following inequality is fulfilled:*

$$\left\| \int\limits_{E_t} \gamma(\tau) d\tau \right\|_{L^\infty(\mathbb{R}^n)} \leq C^n(q) \left[\prod_{k=1}^{n} \frac{1}{\Delta_k^{1/q}} \right] \|\gamma(\tau)\|_{L^p(\mathbb{R}^n)},$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, and the constant $C(q) > 0$ does not depend on $\gamma(\tau)$ and vector Δ.

Recall some notions and statements from the first part of the work. For simplicity, the author keeps numbering of lemmas and formulas used there.

Let $n \geq 1$. Denote the Fourier transformation of a function $u(t) \in L^1(\mathbb{R}^n)$ as $\hat{u}(y)$, where $y \in \mathbb{R}^n$, and choose $\hat{u}(y)$ in the following form:

$$\hat{u}(y) = \int\limits_{\mathbb{R}^n} e^{-i(y,t)} u(t) dt.$$

Let us denote the inverse Fourier transformation of a function $v(y) \in L^1(\mathbb{R}^n)$, by $\tilde{v}(t)$, $t \in \mathbb{R}^n$. Then $\tilde{v}(t)$ has the form:

$$\tilde{v}(t) = \left(\frac{1}{2\pi} \right)^n \int\limits_{\mathbb{R}^n} e^{i(y,t)} v(y) dy.$$

Introduce the following notation. Let $S(\mathbb{R}^n)$ be the space of infinitely differentiable functions which are rapidly decreasing at infinity, $S'(\mathbb{R}^n)$ be the space of slowly increasing distributions (also called the space of distributions of slow growth), i. e. the space of linear continuous functionals on $S(\mathbb{R}^n)$. Let us denote the space of finite infinitely differentiable functions on $\mathbb{R}^n$ by $D(\mathbb{R}^n)$ and the space of linear continuous functionals on $D(\mathbb{R}^n)$ by $D'(\mathbb{R}^n)$.

For every complex valued locally integrable function $\gamma(t)$, $t \in \mathbb{R}^n$, we put into correspondence [4, p. 30, p. 32] the functional

$$(\gamma, \varphi) = \int\limits_{\mathbb{R}^n} \overline{\gamma(t)}\varphi(t)dt, \quad \varphi(t) \in D(\mathbb{R}^n).$$

The distributions from $D'(\mathbb{R}^n)$, generated by locally integrable functions, are called regular functions. Since $D(\mathbb{R}^n)$ is densely embedded in $S(\mathbb{R}^n)$, then $S'(\mathbb{R}^n) \subset D'(\mathbb{R}^n)$. It is known, for example, that the functionals generated by functions $\gamma(t) \in L^p(\mathbb{R}^n)$, $p \geq 1$ are regular slowly increasing distributions.

A linear continuous functional on $S(\mathbb{R}^n)$, designated as $\hat{\gamma}(y)$ and defined (with regard to the choice of the definition for (γ, φ) and the form of Fourier transformation) as

$$(\hat{\gamma}, \hat{\varphi}) = (2\pi)^n(\gamma, \varphi)$$

is called the slowly increasing function Fourier transformation.

Owing to the introduced notion we have the following version of well known formulas:

$$\left.\begin{array}{l} \{\gamma_1(t)\cdot\gamma_2(t)\}\hat{}\,(y) = \left(\dfrac{1}{2\pi}\right)^n \hat{\gamma}_1(y) * \hat{\gamma}_2(y) \\ \gamma(\tau) \equiv 1,\ \hat{\gamma}(y) = 2\pi\delta(y), \quad \tau, y \in \mathbb{R}^1 \end{array}\right\} \tag{1.1}$$

Let $n \geq 2$, $1 \leq k < n$ and $\Delta_{k+1}, \ldots, \Delta_n > 0$. Denote $G(\Delta_{k+1}, \ldots, \Delta_n) = \bigcup\limits_{\beta=k+1}^{n} \{y \,|\, y = (y_1, \ldots, y_n) \in \mathbb{R}^n,\ |y_\beta| < \Delta_\beta\}$.

If $n-k = 1$, then $G(\Delta_n)$ is the direct product of $\mathbb{R}^{n-1}$ and the interval $(-\Delta_n, \Delta_n)$. If $n-k > 1$, $G(\Delta_{k+1}, \ldots, \Delta_n)$ is the direct product $\mathbb{R}^k$ and the "crosslike" neighborhood of zero in $\mathbb{R}^{n-k} = \{y \,|\, y = (y_{k+1}, \ldots, y_n)$, $y_j \in \mathbb{R}^1$, $k+1 \leq j \leq n\}$.

Lemma 1.1. *Let* $n \geq 2$, $1 \leq k < n$, $p \in [1, +\infty]$, *the functions* $g_\lambda(\theta) \in L^q(\mathbb{R}^1)$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $1 \leq \lambda \leq k$, $\gamma(\tau) \in L^p(\mathbb{R}^n)$, $\Delta = (\Delta_1, \ldots, \Delta_n)$ *is the vector with positive coordinates and* supp $\hat{\gamma}(y) \cap Q(\Delta) = \varnothing$. *Then*

$$\left(\operatorname{supp}\left\{\gamma(\tau_1, \ldots, \tau_k, \tau_{k+1}, \ldots, \tau_n)\prod_{\lambda=1}^{k} g_\lambda(\tau_\lambda)\right\}\hat{}\,(y)\right)\bigcap$$

$$\bigcap G(\Delta_{k+1}, \dots, \Delta_n) = \varnothing.$$

Let us take any $\beta > 0$ and denote by $\omega(\tau, \beta)$ the function with the Fourier transformation

$$\widehat{\omega}(y, \beta) = \frac{1}{\beta^2}\, \xi_{[-\beta/2,\, \beta/2]}(y) * \xi_{[-\beta/2,\, \beta/2]}(y), \tag{2.1}$$

where $\xi_M(y)$ is the characteristic function of the set $M \subseteq \mathbb{R}^1$.

Let $\alpha > \beta$. Denote by $\Omega(\tau, [-\alpha, \alpha], \beta)$ the function with the Fourier transform

$$\widehat{\Omega}(y, [-\alpha, \alpha], \beta) = \xi_{[-\alpha,\alpha]}(y) * \widehat{\omega}(y, \beta). \tag{2.3}$$

For arbitrary vectors $b = (b_1, \dots, b_n)$ and $a = (a_1, \dots, a_n)$ such that $0 < b_k < a_k$, $1 \le k \le n$, let us denote by $K(t, a, b)$, $t \in \mathbb{R}^n$ the function with the Fourier transformation

$$\widehat{K}(y, a, b) = \prod_{k=1}^{n} \frac{1}{iy_k} \left[1 - \widehat{\Omega}(y_k, [-a_k, a_k], b_k)\right], \tag{2.5}$$

where $y = (y_1, \dots, y_n)$. From (2.1) and (2.3) it follows that $\widehat{K}(y, a, b) = 0$ for $y \in \overline{Q(a - b)}$.

Lemma 2.1. *Let $n \ge 1$, $t = (t_1, \dots, t_n) \in \mathbb{R}^n$ and $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$ be any vectors which coordinates satisfy the following condition: $0 < b_k < a_k$, $1 \le k \le n$. Then*

$$K(t, a, b) = \left(\frac{1}{\pi}\right)^n \prod_{k=1}^{n} \int\limits_{\frac{1}{2} b_k t_k}^{sign\, t_k \cdot \infty} \frac{\sin\left(2\, \frac{a_k}{b_k}\, \theta\right) \sin^2\theta}{\theta^3}\, d\theta. \tag{2.6}$$

Lemma 2.2. *Let $n = 1$, $a > 0$, $M \in (1, +\infty)$ and $q \in [1, +\infty)$, then:*

$$\left\| K\left(t, a, \frac{1}{M}\, a\right) \right\|_{L^q(\mathbb{R}^1)} = \frac{1}{a^{1/q}}\, C_1(M, q),$$

where

$$C_1(M, q) = \left\{ 4M \int_0^\infty \left| \frac{1}{\pi} \int\limits_x^\infty \frac{\sin 2M\theta \cdot \sin^2\theta}{\theta^3}\, d\theta \right|^q dx \right\}^{1/q}.$$

3. Main theorem. We start with an auxiliary result.

Theorem 3.1. *Let $n \geq 1$, $p \in (2, +\infty]$, $\Delta = (\Delta_1, \dots, \Delta_n)$, $\Delta_k > 0$, $1 \leq k \leq n$, $M \in (1, +\infty)$ and $t = (t_1, \dots, t_n) \in \mathbb{R}^n$. Then for any function $\gamma(\tau) \in \Gamma(\mathbb{R}^n \setminus Q(\Delta), p)$ equality*

$$(-1)^{n(t)} \int\limits_{E_t} \gamma(\tau)\, d\tau = \int\limits_{\mathbb{R}^n} K\left(\tau_1, \dots, \tau_n, \frac{M}{M+1}\,\Delta, \frac{1}{M+1}\,\Delta\right) \gamma(\tau)\, d\tau -$$

$$-\sum_{\alpha=1}^{n} \int\limits_{\mathbb{R}^n} K\left(\tau_1, \dots, \tau_\alpha - t_\alpha, \dots, \tau_n, \frac{M}{M+1}\,\Delta, \frac{1}{M+1}\,\Delta\right) \gamma(\tau)\, d\tau +$$

$$+\sum_{\substack{\alpha,\, \beta=1 \\ \alpha<\beta}}^{n} \int\limits_{\mathbb{R}^n} K\left(\tau_1, \dots, \tau_\alpha - t_\alpha, \dots, \tau_\beta - t_\beta, \dots, \tau_n, \frac{M}{M+1}\,\Delta, \frac{1}{M+1}\,\Delta\right) \times$$

$$\times \gamma(\tau)\, d\tau + \ldots + (-1)^n \int\limits_{\mathbb{R}^n} K\left(\tau_1 - t_1, \dots, \tau_n - \right.$$

$$\left. - t_n, \frac{M}{M+1}\,\Delta, \frac{1}{M+1}\,\Delta\right) \gamma(\tau)\, d\tau \tag{3.1}$$

is true, where $n(t)$ is a number of negative coordinates of a point t.

Proof. Choose an arbitrary point $t \in \mathbb{R}^n$ and consider a subsidiary integral

$$\int\limits_{E_t} e^{-i(u,\tau)} \gamma(\tau)\, d\tau, \quad u \in \mathbb{R}^n. \tag{3.2}$$

Since

$$\{\xi_{E_t}(\tau)\}\hat{}\,(y) = \int\limits_{\mathbb{R}^n} e^{-i(y,\tau)} \xi_{E_t}(\tau)\, d\tau =$$

$$= (-1)^{n(t)} \int\limits_0^{t_1} \int\limits_0^{t_2} \ldots \int\limits_0^{t_n} e^{-i(y_1\tau_1 + y_2\tau_2 + \ldots + y_n\tau_n)} d\tau_1 d\tau_2 \ldots d\tau_n =$$

$$= (-1)^{n(t)} \prod_{k=1}^{n} \frac{1 - e^{-iy_k t_k}}{iy_k},$$

where $y = (y_1, y_2, \dots, y_n)$ and $\tau = (\tau_1, \tau_2, \dots, \tau_n)$, we obtain from (3.2) that:

$$(-1)^{n(t)} \int\limits_{E_t} e^{-i(u,\tau)} \gamma(\tau)\, d\tau = (-1)^{n(t)} \int\limits_{\mathbb{R}^n} e^{-i(u,\tau)} \xi_{E_t}(\tau) \gamma(\tau)\, d\tau =$$

$$= \int\limits_{\mathbb{R}^n} e^{-i(u,\tau)} \left\{ \prod_{k=1}^{n} \frac{1 - e^{-iy_k t_k}}{iy_k} \right\}\tilde{}\,(\tau) \cdot \gamma(\tau)\, d\tau.$$

We choose arbitrary $\nu \in (1, +\infty)$ and designate $\widehat{\Omega}(y_k) = \widehat{\Omega}\left(y_k, \left[-\frac{M\nu}{(M+1)\nu + 1} \Delta_k; \frac{M\nu}{(M+1)\nu + 1} \Delta_k \right], \frac{\nu}{(M+1)\nu + 1} \Delta_k \right)$, $1 \le k \le n$. Then

$$(-1)^{n(t)} \int\limits_{E_t} e^{-i(u,\tau)} \gamma(\tau)\, d\tau =$$

$$= \int\limits_{\mathbb{R}^n} e^{-i(u,\tau)} \prod_{k=1}^{n} \left\{ \frac{1 - e^{-iy_k t_k}}{iy_k} \left[1 - \widehat{\Omega}(y_k) \right] + \right.$$

$$\left. + \frac{1 - e^{-iy_k t_k}}{iy_k} \widehat{\Omega}(y_k) \right\}\tilde{}\,(\tau) \cdot \gamma(\tau)\, d\tau =$$

$$= \int\limits_{\mathbb{R}^n} e^{-i(u,\tau)} \left\{ \prod_{k=1}^{n} \frac{1 - e^{-iy_k t_k}}{iy_k} \left[1 - \widehat{\Omega}(y_k) \right] \right\}\tilde{}\,(\tau)\, \gamma(\tau)\, d\tau +$$

$$+ \int\limits_{\mathbb{R}^n} e^{-i(u,\tau)} \left\{ L(y,t) \right\}\tilde{}\,(\tau) \cdot \gamma(\tau)\, d\tau, \tag{3.3}$$

where $L(y,t)$ is the sum of $2^n - 1$ various items of the form:

$$\prod_{\alpha \in J_1} \frac{1 - e^{-iy_\alpha t_\alpha}}{iy_\alpha} \left[1 - \widehat{\Omega}(y_\alpha) \right] \cdot \prod_{\beta \in J_2} \frac{1 - e^{-iy_\beta t_\beta}}{iy_\beta} \widehat{\Omega}(y_\beta),$$

with $J_1 \cup J_2 = \{1, 2, \dots, n\}$, $J_1 \cap J_2 = \varnothing$, $J_2 \neq \varnothing$.

We consider the first term in the right hand part (3.3). For the item standing in braces in (2.5) we obtain:

$$\left\{\prod_{k=1}^{n}\frac{1-e^{-iy_kt_k}}{iy_k}\left[1-\widehat{\Omega}(y_k)\right]\right\}\tilde{}(\tau)=$$

$$=\left(\frac{1}{2\pi}\right)^n\int\limits_{\mathbb{R}^n}e^{i(y,\tau)}\prod_{k=1}^{n}(1-e^{-iy_kt_k})\prod_{k=1}^{n}\frac{1-\widehat{\Omega}(y_k)}{iy_k}\,dy=$$

$$=\left(\frac{1}{2\pi}\right)^n\int\limits_{\mathbb{R}^n}e^{i(y,\tau)}\prod_{k=1}^{n}\frac{1-\widehat{\Omega}(y_k)}{iy_k}\,dy-$$

$$-\sum_{\alpha=1}^{n}\left(\frac{1}{2\pi}\right)^n\int\limits_{\mathbb{R}^n}e^{i(y,\tau)}e^{-iy_\alpha t_\alpha}\prod_{k=1}^{n}\frac{1-\widehat{\Omega}(y_k)}{iy_k}\,dy+$$

$$+\sum_{\substack{\alpha,\,\beta=1\\ \alpha<\beta}}^{n}\left(\frac{1}{2\pi}\right)^n\int\limits_{\mathbb{R}^n}e^{i(y,\tau)}e^{-iy_\alpha t_\alpha-iy_\beta t_\beta}\prod_{k=1}^{n}\frac{1-\widehat{\Omega}(y_k)}{iy_k}\,dy+$$

$$+\ldots+(-1)^n\left(\frac{1}{2\pi}\right)^n\int\limits_{\mathbb{R}^n}e^{i(y,\tau)}e^{-iy_1t_1-\ldots-iy_nt_n}\prod_{k=1}^{n}\frac{1-\widehat{\Omega}(y_k)}{iy_k}\,dy=$$

$$=K\left(\tau_1,\ldots,\tau_n,\frac{M\nu}{(M+1)\nu+1}\,\Delta,\frac{\nu}{(M+1)\nu+1}\,\Delta\right)-$$

$$-\sum_{\alpha=1}^{n}K\left(\tau_1,\ldots,\tau_\alpha-t_\alpha,\ldots,\tau_n,\frac{M\nu}{(M+1)\nu+1}\,\Delta,\frac{\nu}{(M+1)\nu+1}\,\Delta\right)+$$

$$+\sum_{\substack{\alpha,\,\beta=1\\ \alpha<\beta}}^{n}K\left(\tau_1,\ldots,\tau_\alpha-t_\alpha,\ldots,\tau_\beta-t_\beta,\ldots,\tau_n,\frac{M\nu}{(M+1)\nu+1}\,\Delta,\right.$$

$$\left.\frac{\nu}{(M+1)\nu+1}\,\Delta\right)+\ldots+(-1)^nK\left(\tau_1-t_1,\ldots,\tau_n-t_n,\right.$$

$$\left.\frac{M\nu}{(M+1)\nu+1}\,\Delta,\frac{\nu}{(M+1)\nu+1}\,\Delta\right),$$

where $\Delta = (\Delta_1, \ldots, \Delta_n)$,

$$K\left(\tau_1, \ldots, \tau_n, \frac{M\nu}{(M+1)\nu+1}\,\Delta, \frac{\nu}{(M+1)\nu+1}\,\Delta\right) =$$

$$= \prod_{k=1}^{n} K\left(\tau_k, \frac{M\nu}{(M+1)\nu+1}\,\Delta_k, \frac{\nu}{(M+1)\nu+1}\,\Delta_k\right) =$$

$$= \prod_{k=1}^{n} \left\{ \frac{1-\widehat{\Omega}(y_k)}{iy_k} \right\}^{\sim}(\tau_k).$$

Hence,

$$\int\limits_{\mathbb{R}^n} e^{-i(u,\tau)} \left\{ \prod_{k=1}^{n} \frac{1-e^{-iy_k t_k}}{iy_k} \left[1-\widehat{\Omega}(y_k)\right] \right\}^{\sim}(\tau)\,\gamma(\tau)\,d\tau =$$

$$= \int\limits_{\mathbb{R}^n} e^{-i(u,\tau)} K\left(\tau_1, \ldots, \tau_n, \frac{M\nu}{(M+1)\nu+1}\,\Delta, \frac{\nu}{(M+1)\nu+1}\,\Delta\right) \cdot \gamma(\tau)\,d\tau -$$

$$- \sum_{\alpha=1}^{n} \int\limits_{\mathbb{R}^n} e^{-i(u,\tau)} K\left(\tau_1, \ldots, \tau_\alpha - t_\alpha, \ldots, \tau_n, \frac{M\nu}{(M+1)\nu+1}\,\Delta,\right.$$

$$\left.\frac{\nu}{(M+1)\nu+1}\,\Delta\right) \cdot \gamma(\tau)\,d\tau + \sum_{\substack{\alpha,\,\beta=1\\ \alpha<\beta}}^{n} \int\limits_{\mathbb{R}^n} e^{-i(u,\tau)} K\left(\tau_1, \ldots, \tau_\alpha - t_\alpha, \ldots, \tau_\beta -\right.$$

$$\left.- t_\beta, \ldots, \tau_n, \frac{M\nu}{(M+1)\nu+1}\,\Delta, \frac{\nu}{(M+1)\nu+1}\,\Delta\right) \cdot \gamma(\tau)\,d\tau +$$

$$+ \ldots + (-1)^n \int\limits_{\mathbb{R}^n} e^{-i(u,\tau)} K\left(\tau_1 - t_1, \ldots, \tau_n - t_n, \frac{M\nu}{(M+1)\nu+1}\,\Delta,\right.$$

$$\left.\frac{\nu}{(M+1)\nu+1}\,\Delta\right) \cdot \gamma(\tau)\,d\tau. \tag{3.4}$$

As $\gamma(\tau) \in L^p(\mathbb{R}^n)$ and owing to Lemma 2.2

$$K\left(\tau, \frac{M\nu}{(M+1)\nu+1}\,\Delta, \frac{\nu}{(M+1)\nu+1}\,\Delta\right) \in L^q(\mathbb{R}^n),$$

M, $\nu \in (1, +\infty)$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then for every fixed $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$ all integrals given in the right part of (3.4), are continuous in point $u = 0$. So,

$$\lim_{\|u\|\to 0} \int\limits_{\mathbb{R}^n} e^{-i(u,\tau)} \left\{ \prod_{k=1}^{n} \frac{1 - e^{-iy_k t_k}}{iy_k} \left[1 - \widehat{\Omega}(y_k) \right] \right\}\tilde{}(\tau) \cdot \gamma(\tau)\, d\tau =$$

$$= \int\limits_{\mathbb{R}^n} K\left(\tau_1, \ldots, \tau_n, \frac{M\nu}{(M+1)\nu + 1}\,\Delta, \frac{\nu}{(M+1)\nu + 1}\,\Delta \right) \gamma(\tau)\, d\tau -$$

$$- \sum_{\alpha=1}^{n} \int\limits_{\mathbb{R}^n} K\left(\tau_1, \ldots, \tau_\alpha - t_\alpha, \ldots, \tau_n, \frac{M\nu}{(M+1)\nu + 1}\,\Delta, \right.$$

$$\left. \frac{\nu}{(M+1)\nu + 1}\,\Delta \right) \gamma(\tau)\, d\tau + \sum_{\substack{\alpha,\,\beta=1 \\ \alpha<\beta}}^{n} \int\limits_{\mathbb{R}^n} K\left(\tau_1, \ldots, \tau_\alpha - t_\alpha, \ldots, \tau_\beta - \right.$$

$$\left. - t_\beta, \ldots, \tau_n, \frac{M\nu}{(M+1)\nu + 1}\,\Delta, \frac{\nu}{(M+1)\nu + 1}\,\Delta \right) \gamma(\tau)\, d\tau +$$

$$+ \ldots + (-1)^n \int\limits_{\mathbb{R}^n} K\left(\tau_1 - t_1, \ldots, \tau_n - t_n, \frac{M\nu}{(M+1)\nu + 1}\,\Delta, \right.$$

$$\left. \frac{\nu}{(M+1)\nu + 1}\,\Delta \right) \gamma(\tau)\, d\tau. \tag{3.5}$$

We demonstrate that

$$\lim_{\|u\|\to 0} \int\limits_{\mathbb{R}^n} e^{-i(u,\tau)} \left\{ L(y,t) \right\}\tilde{}(\tau) \cdot \gamma(\tau)\, d\tau = 0. \tag{3.6}$$

If $n = 1$, then according to (3.3)

$$\int\limits_{\mathbb{R}^1} e^{-iu\tau} \left\{ L(y,t) \right\}\tilde{}(\tau) \cdot \gamma(\tau)\, d\tau =$$

$$= \int\limits_{\mathbb{R}^1} e^{-iu\tau} \left\{ \frac{1 - e^{iyt}}{iy}\, \widehat{\Omega}(y) \right\}\tilde{}(\tau)\, \gamma(\tau)\, d\tau. \tag{3.7}$$

If $n \geq 2$, then according to (3.3) integral standing after the sign of a limit in (3.6) equals to the sum of $2^n - 2$ integrals of the type

$$I(J_1, J_2, u, t) = \int\limits_{\mathbb{R}^n} e^{-i(u,\tau)} \left\{ \prod_{\alpha \in J_1} \frac{1 - e^{-iy_\alpha t_\alpha}}{iy_\alpha} \left[1 - \widehat{\Omega}(y_\alpha)\right] \times \right.$$

$$\left. \times \prod_{\beta \in J_2} \frac{1 - e^{-iy_\beta t_\beta}}{iy_\beta} \widehat{\Omega}(y_\beta) \right\}^{\tilde{}}(\tau) \cdot \gamma(\tau)\, d\tau, \tag{3.8}$$

where $J_1 \cup J_2 = \{1, 2, \ldots, n\}$, $J_1 \cap J_2 = \varnothing$, $J_1 \neq \varnothing$, and of the integral

$$I(u, t) = \int\limits_{\mathbb{R}^n} e^{-i(u,\tau)} \left\{ \prod_{\beta=1}^{n} \frac{1 - e^{-iy_\beta t_\beta}}{iy_\beta} \widehat{\Omega}(y_\beta) \right\}^{\tilde{}}(\tau) \cdot \gamma(\tau)\, d\tau. \tag{3.9}$$

Note that integral (3.7) results from (3.9) at $n = 1$. We check now that each integral (3.8) and (3.9) tends to zero as $\|u\| \to 0$, from where the result (3.6) follows.

Consider any integral $I(J_1, J_2, u, t)$. In that integral:

$$\left\{ \prod_{\alpha \in J_1} \frac{1 - e^{-iy_\alpha t_\alpha}}{iy_\alpha} \left[1 - \widehat{\Omega}(y_\alpha)\right] \cdot \prod_{\beta \in J_2} \frac{1 - e^{-iy_\beta t_\beta}}{iy_\beta} \widehat{\Omega}(y_\beta) \right\}^{\tilde{}}(\tau) =$$

$$\left(\frac{1}{2\pi}\right)^n \int\limits_{\mathbb{R}^n} e^{i(y,\tau)} \prod_{\alpha \in J_1} \frac{1 - e^{-iy_\alpha t_\alpha}}{iy_\alpha} \left[1 - \widehat{\Omega}(y_\alpha)\right] \prod_{\beta \in J_2} \frac{1 - e^{-iy_\beta t_\beta}}{iy_\beta} \widehat{\Omega}(y_\beta)\, dy =$$

$$= \prod_{\alpha \in J_1} \frac{1}{2\pi} \int\limits_{\mathbb{R}^1} e^{iy_\alpha \tau_\alpha} \frac{1 - e^{-iy_\alpha t_\alpha}}{iy_\alpha} \left[1 - \widehat{\Omega}(y_\alpha)\right] dy_\alpha \times$$

$$\times \prod_{\beta \in J_2} \frac{1}{2\pi} \int\limits_{\mathbb{R}^1} e^{iy_\beta \tau_\beta} \frac{1 - e^{-iy_\beta t_\beta}}{iy_\beta} \widehat{\Omega}(y_\beta)\, dy_\beta. \tag{3.10}$$

Introduce the following notions for (3.10):

$$f(\tau_\alpha, t_\alpha) = \frac{1}{2\pi} \int\limits_{\mathbb{R}^1} e^{iy_\alpha \tau_\alpha} \frac{1 - e^{-iy_\alpha t_\alpha}}{iy_\alpha} \left[1 - \widehat{\Omega}(y_\alpha)\right] dy_\alpha, \quad \alpha \in J_1,$$

$$l(\tau_\beta, t_\beta) = \frac{1}{2\pi} \int\limits_{\mathbb{R}^1} e^{iy_\beta \tau_\beta} \frac{1 - e^{-iy_\beta t_\beta}}{iy_\beta} \widehat{\Omega}(y_\beta)\, dy_\beta, \quad \beta \in J_2.$$

Proposition 3.1.1. *Let* $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, $\Delta = (\Delta_1, \ldots, \Delta_n)$, $M \in (1, +\infty)$, $\nu \in (1, +\infty)$ *and* $r \in [1, +\infty)$. *Then:*

1) $$f(\tau_\alpha, t_\alpha) = K\left(\tau_\alpha, \frac{M\nu}{(M+1)\nu+1}\,\Delta_\alpha, \frac{\nu}{(M+1)\nu+1}\,\Delta_\alpha\right) - \\ -K\left(\tau_\alpha - t_\alpha, \frac{M\nu}{(M+1)\nu+1}\,\Delta_\alpha, \frac{\nu}{(M+1)\nu+1}\,\Delta_\alpha\right),\ 1 \le \alpha \le n,$$

$f(\,\cdot\,, t_\alpha) \in L^r(\mathbb{R}^1)$, $1 \le \alpha \le n$;

2) $$\operatorname{supp} \widehat{l}\,(y_\beta, t_\beta) = \left[-\frac{(M+1)\nu}{(M+1)\nu+1}\,;\, \frac{(M+1)\nu}{(M+1)\nu+1}\right],\ 1 \le \beta \le n; \quad (3.11)$$

3) $l(\,\cdot\,, t_\beta) \in L^r(\mathbb{R}^1)$, $t_\beta \in \mathbb{R}^1$, $1 \le \beta \le n$.

Proof of proposition 3.1.1. 1) Taken $n = 1$, in (2.5) and (2.6) we deduce from definition of function $f(\tau_\alpha, t_\alpha)$ that for any $1 \le \alpha \le n :$

$$f(\tau_\alpha, t_\alpha) = K\left(\tau_\alpha, \frac{M\nu}{(M+1)\nu+1}\,\Delta_\alpha, \frac{\nu}{(M+1)\nu+1}\,\Delta_\alpha\right) - \\ -K\left(\tau_\alpha - t_\alpha, \frac{M\nu}{(M+1)\nu+1}\,\Delta_\alpha, \frac{\nu}{(M+1)\nu+1}\,\Delta_\alpha\right),$$

and as owing to Lemma 2.2

$$K\left(\tau_\alpha, \frac{M\nu}{(M+1)\nu+1}\,\Delta_\alpha, \frac{\nu}{(M+1)\nu+1}\,\Delta_\alpha\right) \in L^r(\mathbb{R}^1)$$

for any $r \in [1, +\infty)$, then $f(\,\cdot\,, t_\alpha) \in L^r(\mathbb{R}^1)$.

2) From definition of function $l(\tau_\beta, t_\beta)$ we obtain, that

$$\widehat{l}\,(y_\beta, t_\beta) = \frac{1 - e^{-iy_\beta t_\beta}}{iy_\beta}\,\widehat{\Omega}(y_\beta).$$

Hence, by (2.4)

$$\operatorname{supp} \widehat{l}\,(y_\beta, t_\beta) = \operatorname{supp} \widehat{\Omega}(y_\beta) = \\ = \left[-\frac{M\nu}{(M+1)\nu+1} - \frac{\nu}{(M+1)\nu+1};\, \frac{M\nu}{(M+1)\nu+1} + \frac{\nu}{(M+1)\nu+1}\right].$$

3) As $\dfrac{1 - e^{-iy_\beta t_\beta}}{iy_\beta}$ is bounded and owing to (2.4) $\widehat{\Omega}(y_\beta) \in L^1(\mathbb{R}^1)$, then $\widehat{l}\,(\,\cdot\,, t_\beta) \in L^1(\mathbb{R}^1)$. So with respect to the variable τ_β the function $l(\tau_\beta, t_\beta)$

is continuous and bounded. In addition, when $\tau_\beta \neq 0$ as a result of double integration by parts, we have according to (2.4) and (3.11)

$$l(\tau_\beta, t_\beta) = \frac{1}{2\pi} \int\limits_{-\frac{(M+1)\nu}{(M+1)\nu+1}\Delta_\beta}^{+\frac{(M+1)\nu}{(M+1)\nu+1}\Delta_\beta} e^{iy_\beta \tau_\beta} \left\{ \frac{1-e^{-iy_\beta t_\beta}}{iy_\beta}\,\widehat{\Omega}(y_\beta)\right\} dy_\beta =$$

$$= -\frac{1}{2\pi\tau_\beta^2} \int\limits_{-\frac{(M+1)\nu}{(M+1)\nu+1}\Delta_\beta}^{\frac{(M+1)\nu}{(M+1)\nu+1}\Delta_\beta} \left\{ \frac{1-e^{-iy_\beta t_\beta}}{iy_\beta}\,\widehat{\Omega}(y_\beta)\right\}''_{y_\beta^2} e^{iy_\beta \tau_\beta}\, dy_\beta = \frac{O(1)}{\tau_\beta^2}.$$

Therefore, $l(\,\cdot\,, t_\beta) \in L^r(\mathbb{R}^1)$ for any $r \in [1, +\infty)$. $\square$

Proposition 3.1.2. *Let* $n \geq 2$, $t = (t_1, \ldots, t_n)$, $\Delta = (\Delta_1, \ldots, \Delta_n)$, $\Delta_k > 0$, $1 \leq k \leq n$, $u = (u_1, \ldots, u_n)$ *and* $|u_k| < \dfrac{1}{(M+1)\nu+1}\Delta_k$, $1 \leq k \leq n$. *Then* $I(J_1, J_2, u, t) = 0$.

Proof of proposition 3.1.2. Changing, if necessary, numbering of variables $y_1, \ldots, y_n$ in (3.8), one can say without loss of generality that $J_1 = \{1, \ldots, k\}$, $J_2 = \{k+1, \ldots, n\}$, where $1 \leq k < n$. Then according to (3.10) and proposition 3.1.1

$$I(J_1, J_2, u, t) = \int\limits_{\mathbb{R}^n} e^{-i(u,\tau)} \left\{ \prod_{\alpha=1}^{k} \frac{1-e^{-iy_\alpha t_\alpha}}{iy_\alpha} \left[1 - \widehat{\Omega}(y_\alpha)\right] \times \right.$$

$$\left. \times \prod_{\beta=k+1}^{n} \frac{1-e^{-iy_\beta t_\beta}}{iy_\beta}\,\widehat{\Omega}(y_\beta) \right\}^{\sim}(\tau) \cdot \gamma(\tau)\, d\tau =$$

$$= \int\limits_{\mathbb{R}^n} e^{-i(u,\tau)} \left\{ \prod_{\alpha=1}^{k} \left[K\left(\tau_\alpha, \frac{M\nu}{(M+1)\nu+1}\Delta_\alpha, \frac{\nu}{(M+1)\nu+1}\Delta_\alpha\right) - \right.\right.$$

$$\left.\left. -K\left(\tau_\alpha - t_\alpha, \frac{M\nu}{(M+1)\nu+1}\Delta_\alpha, \frac{\nu}{(M+1)\nu+1}\Delta_\alpha\right)\right] \cdot \prod_{\beta=k+1}^{n} l(\tau_\beta, t_\beta) \right\} \times$$

$$\times \gamma(\tau_1, \ldots, \tau_k, \tau_{k+1}, \ldots, \tau_n)\, d\tau_1 \ldots d\tau_k d\tau_{k+1} \ldots d\tau_n =$$

$$= \int\limits_{\mathbb{R}^n} e^{-i(u,\tau)} \mathcal{F}(\tau,t)\Phi(\tau,t)\,d\tau,$$

where

$$\mathcal{F}(\tau,t) = \gamma(\tau_1,\ldots,\tau_k,\tau_{k+1},\ldots,\tau_n)\times$$

$$\times \prod_{\alpha=1}^{k}\left[K\left(\tau_\alpha, \frac{M\nu}{(M+1)\nu+1}\,\Delta_\alpha, \frac{\nu}{(M+1)\nu+1}\,\Delta_\alpha\right)-\right.$$

$$\left.-K\left(\tau_\alpha - t_\alpha, \frac{M\nu}{(M+1)\nu+1}\,\Delta_\alpha, \frac{\nu}{(M+1)\nu+1}\,\Delta_\alpha\right)\right],$$

$$\Phi(\tau,t) = \prod_{\alpha=1}^{k} 1 \prod_{\beta=k+1}^{n} l(\tau_\beta, t_\beta).$$

Consider functions $\widehat{\mathcal{F}}(u,t)$ and $\widehat{\Phi}(u,t)$. Since, due to theorem conditions, $\operatorname{supp}\widehat{\gamma}(u)\cap Q(\Delta)=\varnothing$, then according to Lemma 1.1:

$$\widehat{\mathcal{F}}(u,t)\cap G(\Delta_{k+1},\ldots,\Delta_n)=\varnothing.$$

Function $\Phi(\tau,t)$ represents the direct product of functions 1, $1\le\alpha\le k$ and functions $l(\tau_\beta,t_\beta)$, $k+1\le\beta\le n$. It follows from definition $l(\tau_\beta,t_\beta)$, $k+1\le\beta\le n$ that:

$$\{l(\tau_\beta,t_\beta)\}\hat{\ }(u_\beta) = \frac{1-e^{-iu_\beta t_\beta}}{iu_\beta}\,\widehat{\Omega}(u_\beta), \quad u_\beta\in\mathbb{R}^1, \quad k+1\le\beta\le n.$$

Therefore according to [4, p. 240–241] and (1.1):

$$\widehat{\Phi}(u,t) = (2\pi)^k\left[\prod_{\alpha=1}^{k}\delta(u_\alpha)\right]\prod_{\beta=k+1}^{n}\frac{1-e^{-iu_\beta t_\beta}}{iu_\beta}\,\widehat{\Omega}(u_\beta),$$

where $u=(u_1,\ldots,u_n)$. In addition, considering (3.11), we obtain

$$\operatorname{supp}\,\widehat{\Phi}(u,t) = \{u \,|\, u=(u_1,\ldots,u_k,u_{k+1},\ldots,u_n),\ u_\alpha=0,\ 1\le\alpha\le k,$$

$$\left.|u_\beta|\le\frac{(M+1)\nu}{(M+1)\nu+1}\,\Delta_\beta,\ k+1\le\beta\le n\right\}.$$

So the support of function $\widehat{\Phi}(u,t)$ is bounded. Hence (see, for example, [4, p. 135]), the convolution of functions $\widehat{\mathcal{F}}(u,t)$, $\widehat{\Phi}(u,t)$ is defined and according to (1.1) we have:

$$I(J_1,J_2,u,t)=\left(\frac{1}{2\pi}\right)^n \widehat{\mathcal{F}}(u,t)*\widehat{\Phi}(u,t).$$

For $A,B\subseteq\mathbb{R}^n$ denote:

$$A+B=\{x\,|\,x=a+b,\ a\in A,\ b\in B\}.$$

Sinse (see [5, p. 69]) the support of convolution of two functions is contained in closure of the sum of supports of these functions, then

$$\operatorname{supp} I(J_1,J_2,u,t)\subseteq\overline{\operatorname{supp}\widehat{\mathcal{F}}(u,t)+\operatorname{supp}\widehat{\Phi}(u,t)}\subseteq$$

$$\subseteq\{x\,|\,x=(x_1,\dots,x_k,x_{k+1},\dots,x_n),\ x_\alpha\in\mathbb{R}^1,\ 1\le\alpha\le k,$$

$$x_\beta\in\mathbb{R}^1\setminus(-\Delta_\beta,\Delta_\beta),\ k+1\le\beta\le n\}+\{y\,|\,y=(y_1,\dots,y_k,y_{k+1},\dots,y_n),$$

$$y_\alpha=0,\ 1\le\alpha\le k,\ |y_\beta|\le\frac{(M+1)\nu}{(M+1)\nu+1}\,\Delta_\beta,\ k+1\le\beta\le n\}=$$

$$=\{u\,|\,u=(u_1,\dots,u_k,u_{k+1},\dots,u_n),\ u_\alpha=x_\alpha+y_\alpha,\ x_\alpha\in\mathbb{R}^1,\ y_\alpha=0,$$

$$1\le\alpha\le k,\ u_\beta=x_\beta+y_\beta,\ x_\beta\in\mathbb{R}^1\setminus(-\Delta_\beta,\Delta_\beta),\ |y_\beta|\le\frac{(M+1)\nu}{(M+1)\nu+1}\,\Delta_\beta,$$

$$k+1\le\beta\le n\}=\{u\,|\,u=(u_1,\dots,u_k,u_{k+1},\dots,u_n),\ u_\alpha\in\mathbb{R}^1,\ 1\le\alpha\le k,$$

$$|u_\beta|\ge\frac{1}{(M+1)\nu+1}\,\Delta_\beta,\ k+1\le\beta\le n\}.$$

Therefore, if $u\in\{u\,|\,u=(u_1,\dots,u_n),\ |u_k|<\dfrac{1}{(M+1)\nu+1}\,\Delta_k,\ 1\le$ $\le k\le n\}$, then $I(J_1,J_2,u,t)=0$. $\square$

Proposition 3.1.3. *Let* $n\ge1$, $t=(t_1,\dots,t_n)$, $\Delta=(\Delta_1,\dots,\Delta_n)$, $\Delta_k>0$, $1\le k\le n$, $u=(u_1,\dots,u_n)$ *and* $|u_k|<\dfrac{1}{(M+1)\nu+1}\,\Delta_k$, $1\le$ $\le k\le n$. *Then* $I(u,t)=0$.

Proof of proposition 3.1.3. Since the set

$$\operatorname{supp}\left[\prod_{\beta=1}^{n}\frac{1-e^{-iu_\beta t_\beta}}{iu_\beta}\,\widehat{\Omega}(u_\beta)\right]=$$

$$=\left\{u\,|\,u=(u_1,\dots,u_n),\ |u_\beta|\le\frac{(M+1)\nu}{(M+1)\nu+1}\,\Delta_\beta,\ 1\le\beta\le n\right\}$$

is bounded in $\mathbb{R}^n$, then the convolution of functions

$$\prod_{\beta=1}^{n}\frac{1-e^{-iu_\beta t_\beta}}{iu_\beta}\,\widehat{\Omega}(u_\beta),\quad \widehat{\gamma}(u_1,\dots,u_n)$$

is well-defined. Consequently

$$I(u,t)=\left\{\prod_{\beta=1}^{n}\frac{1-e^{-iu_\beta t_\beta}}{iu_\beta}\,\widehat{\Omega}(u_\beta)\right\}*\widehat{\gamma}(u_1,\dots,u_n).$$

Thus taking into account (3.11) we obtain:

$$\operatorname{supp}I(u,t)\subseteq\overline{\operatorname{supp}\left\{\prod_{\beta=1}^{n}\frac{1-e^{-iu_\beta t_\beta}}{iu_\beta}\,\widehat{\Omega}(u_\beta)\right\}+\operatorname{supp}\widehat{\gamma}(u)}=$$

$$=\left\{u\,|\,u=(u_1,\dots,u_n),\ |u_\beta|\le\frac{(M+1)\nu}{(M+1)\nu+1}\,\Delta_\beta,\ 1\le\beta\le n\right\}+$$

$$+\{u\,|\,u=(u_1,\dots,u_n),\ |u_\beta|\ge\Delta_\beta,\ 1\le\beta\le n\}=$$

$$=\left\{u\,|\,u=(u_1,\dots,u_n),\ |u_\beta|\ge\frac{1}{(M+1)\nu+1}\,\Delta_\beta,\ 1\le\beta\le n\right\}.$$

Thus, if $u\in\{u\,|\,u=(u_1,\dots,u_n),\ |u_k|<\dfrac{1}{(M+1)\nu+1}\,\Delta_k,\ 1\le k\le$ $\le n\}$, then $I(u,t)=0$. Proposition 3.1.3 is proved. □

Continue the proof of Theorem 3.1. From (3.3), (3.5) and (3.6) we have:

$$(-1)^{n(t)}\int\limits_{E_t}\gamma(\tau)\,d\tau=\lim_{\|u\|\to 0}(-1)^{n(t)}\int\limits_{E_t}e^{-i(u,\tau)}\gamma(\tau)\,d\tau=$$

$$= \int\limits_{\mathbb{R}^n} K\left(\tau_1, \ldots, \tau_n, \frac{M\nu}{(M+1)\nu+1}\,\Delta, \frac{1}{(M+1)\nu+1}\,\Delta\right)\gamma(\tau)\,d\tau -$$

$$-\sum_{\alpha=1}^{n}\int\limits_{\mathbb{R}^n} K\left(\tau_1, \ldots, \tau_\alpha - t_\alpha, \ldots, \tau_n, \frac{M\nu}{(M+1)\nu+1}\,\Delta, \frac{1}{(M+1)\nu+1}\,\Delta\right)\times$$

$$\times\gamma(\tau)\,d\tau + \sum_{\substack{\alpha,\,\beta=1\\ \alpha<\beta}}^{n}\int\limits_{\mathbb{R}^n} K\,(\tau_1, \ldots, \tau_\alpha - t_\alpha, \ldots, \tau_\beta -$$

$$-t_\beta, \ldots, \tau_n, \frac{M\nu}{(M+1)\nu+1}\,\Delta, \frac{1}{(M+1)\nu+1}\,\Delta\Big)\,\gamma(\tau)\,d\tau +$$

$$+\ldots+(-1)^n\int\limits_{\mathbb{R}^n} K\left(\tau_1 - t_1, \ldots, \tau_n - t_n, \frac{M\nu}{(M+1)\nu+1}\,\Delta,\right.$$

$$\left.\frac{1}{(M+1)\nu+1}\,\Delta\right)\gamma(\tau)\,d\tau. \tag{3.12}$$

□

Proposition 3.1.4. *In the assumptions, formulated above, we have:*

$$\lim_{\nu\to+\infty}\left\|K\left(\tau_1, \ldots, \tau_n, \frac{M\nu}{(M+1)\nu+1}\,\Delta, \frac{1}{(M+1)\nu+1}\,\Delta\right) -\right.$$

$$\left.-K\left(\tau_1, \ldots, \tau_n, \frac{M}{M+1}\,\Delta, \frac{1}{M+1}\,\Delta\right)\right\|_{L^q(\mathbb{R}^n)} = 0.$$

Proof of proposition 3.1.4. We use the induction method on number of variables n.

Let at first $n = 1$. According to (2.6)

$$K\left(\tau, a, \frac{1}{M}\,a\right) = \frac{1}{\pi}\int\limits_{\frac{1}{2M}\,a\tau}^{sign\,\tau\cdot\infty}\frac{\sin 2M\theta\cdot\sin^2\theta}{\theta^3}\,d\theta, \quad \tau\in\mathbb{R}^1,$$

then, taking $K(\tau) =$

$$= K\left(\tau, \frac{M\nu}{(M+1)\nu+1}\,\Delta, \frac{1}{(M+1)\nu+1}\,\Delta\right) - K\left(\tau, \frac{M}{M+1}\,\Delta, \frac{1}{M+1}\,\Delta\right),$$

we obtain:

$$\|K(\tau)\|_{L^q(\mathbb{R}^1)} = \left\{ \int\limits_{\mathbb{R}^1} \left| \frac{1}{\pi} \int\limits_{\frac{\nu/2}{(M+1)\nu+1}\Delta\tau}^{\frac{1/2}{M+1}\Delta\tau} \frac{\sin 2M\theta \cdot \sin^2\theta}{\theta^3}\, d\theta \right|^q dt \right\}^{1/q} =$$

$$= \left\{ \frac{4(M+1)}{\Delta \pi^q} \int\limits_0^\infty \left| \int\limits_{\frac{(M+1)\nu}{(M+1)\nu+1}x}^{x} \frac{\sin 2M\theta \cdot \sin^2\theta}{\theta^3}\, d\theta \right|^q dx \right\}^{1/q} \le$$

$$\le \frac{2M}{\pi} \left\{ \frac{4(M+1)}{\Delta} \int\limits_0^\infty \left| \int\limits_{\frac{(M+1)\nu}{(M+1)\nu+1}x}^{x} \frac{\sin 2M\theta \cdot \sin^2\theta}{2M\theta \cdot \theta^2}\, d\theta \right|^q dx \right\}^{1/q}.$$

For $\theta > 0$ and $M \ge 1$

$$\frac{\sin\theta}{\theta} < \frac{2}{1+\theta}, \quad \frac{\sin 2M\theta}{2M\theta} < \frac{2}{1+\theta},$$

then

$$\|K(\tau)\|_{L^q(\mathbb{R}^1)} \le \frac{16M}{\pi} \left\{ \frac{4(M+1)}{\Delta} \int\limits_0^\infty \left| \int\limits_{\frac{(M+1)\nu}{(M+1)\nu+1}x}^{x} \frac{d\theta}{(1+\theta)^3} \right|^q dx \right\}^{1/q} <$$

$$< \frac{16M}{\pi} \left\{ \frac{4(M+1)}{\Delta} \int\limits_0^\infty \left| \frac{x}{(M+1)\nu+1} \cdot \frac{1}{\left[1 + \frac{(M+1)\nu}{(M+1)\nu+1}x\right]^3} \right|^q dx \right\}^{1/q} <$$

$$< \frac{1}{(M+1)\nu+1} \cdot \frac{16M}{\pi} \left\{ \frac{4(M+1)}{\Delta} \int\limits_0^\infty \left| \frac{x}{(1+x)^3} \right|^q dx \right\}^{1/q} \to 0$$

while $\nu \to +\infty$.

Let the statement of proposition 3.1.4 be true for $n = k \geq 1$. Then it is easy to prove that the statement of proposition 3.1.4 is true for $n = k+1$:

$$\left\| K\left(\tau_1, \dots, \tau_k, \tau_{k+1}, \frac{M\nu}{(M+1)\nu+1}\Delta, \frac{1}{(M+1)\nu+1}\Delta\right) - \right.$$

$$\left. -K\left(\tau_1, \dots, \tau_k, \tau_{k+1}, \frac{M}{M+1}\Delta, \frac{1}{M+1}\Delta\right) \right\|_{L^q(\mathbb{R}^{k+1})} =$$

$$= \left\| \prod_{j=1}^{k+1} K\left(\tau_j, \frac{M\nu}{(M+1)\nu+1}\Delta_j, \frac{1}{(M+1)\nu+1}\Delta_j\right) - \right.$$

$$- \left[\prod_{j=1}^{k} K\left(\tau_j, \frac{M}{M+1}\Delta_j, \frac{1}{M+1}\Delta_j\right)\right] \times$$

$$\times K\left(\tau_{k+1}, \frac{M\nu}{(M+1)\nu+1}\Delta_{k+1}, \frac{1}{(M+1)\nu+1}\Delta_{k+1}\right) +$$

$$+ \left[\prod_{j=1}^{k} K\left(\tau_j, \frac{M}{M+1}\Delta_j, \frac{1}{M+1}\Delta_j\right)\right] \times$$

$$\times K\left(\tau_{k+1}, \frac{M\nu}{(M+1)\nu+1}\Delta_{k+1}, \frac{1}{(M+1)\nu+1}\Delta_{k+1}\right) -$$

$$\left. - \prod_{j=1}^{k+1} K\left(\tau_j, \frac{M}{M+1}\Delta_j, \frac{1}{M+1}\Delta_j\right) \right\|_{L^q(\mathbb{R}^{k+1})} \leq$$

$$\leq \left\| \prod_{j=1}^{k} K\left(\tau_j, \frac{M\nu}{(M+1)\nu+1}\Delta_j, \frac{1}{(M+1)\nu+1}\Delta_j\right) - \right.$$

$$\left. - \prod_{j=1}^{k} K\left(\tau_j, \frac{M}{M+1}\Delta_j, \frac{1}{M+1}\Delta_j\right) \right\|_{L^q(\mathbb{R}^{k})} \times$$

$$\times \left\| K\left(\tau_{k+1}, \frac{M\nu}{(M+1)\nu+1}\Delta_{k+1}, \frac{1}{(M+1)\nu+1}\Delta_{k+1}\right) \right\|_{L^q(\mathbb{R}^{1})} +$$

$$+\left\|\prod_{j=1}^{k} K\left(\tau_j, \frac{M}{M+1}\,\Delta_j, \frac{1}{M+1}\,\Delta_j\right)\right\|_{L^q(\mathbb{R}^k)} \times$$

$$\times\left\|K\left(\tau_{k+1}, \frac{M\nu}{(M+1)\nu+1}\,\Delta_{k+1}, \frac{1}{(M+1)\nu+1}\,\Delta_{k+1}\right) -\right.$$

$$\left.-K\left(\tau_{k+1}, \frac{M}{M+1}\,\Delta_{k+1}, \frac{1}{M+1}\,\Delta_{k+1}\right)\right\|_{L^q(\mathbb{R}^1)} \to 0$$

while $\nu \to +\infty$. □

Continue the proof of Theorem 3.1. Proceeding in (3.12) to the limit as $\nu \to +\infty$, we obtain the statement (3.1). □

Theorem 3.2. *Let $n \geq 1$, $p \in (2, +\infty]$ and $\Delta = (\Delta_1, \ldots, \Delta_n)$, $\Delta_k > 0$, $1 \leq k \leq n$. Then for any function $\gamma(\tau) \in \Gamma(\mathbb{R}^n \setminus Q(\Delta), p)$ the next inequality is fulfilled:*

$$\left\|\int\limits_{E_t} \gamma(\tau)d\tau\right\|_{L^\infty(\mathbb{R}^n)} \leq C^n(q)\left[\prod_{k=1}^{n} \frac{1}{\Delta_k^{1/q}}\right] \|\gamma(\tau)\|_{L^p(\mathbb{R}^n)},$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, and the constant $C(q) > 0$ does not depend on $\gamma(\tau)$ and the vector Δ.

Proof. From (3.1) we obtain:

$$\left\|\int\limits_{E_t} \gamma(\tau)d\tau\right\|_{L^\infty(\mathbb{R}^n)} \leq 2^n\left\|K\left(\tau, \frac{M}{M+1}\,\Delta, \frac{1}{M+1}\,\Delta\right)\right\|_{L^q(\mathbb{R}^n)} \cdot \|\gamma(\tau)\|_{L^p(\mathbb{R}^n)}.$$

Owing to (2.5), (2.6) and Lemma 2.2

$$\left\|K\left(\tau, \frac{M}{M+1}\,\Delta, \frac{1}{M+1}\,\Delta\right)\right\|_{L^q(\mathbb{R}^n)} =$$

$$= \prod_{k=1}^{n}\left\|K\left(\tau_k, \frac{M}{M+1}\,\Delta_k, \frac{1}{M+1}\,\Delta_k\right)\right\|_{L^q(\mathbb{R}^1)} \leq C_2^n(M,q)\left[\prod_{k=1}^{n} \frac{1}{\Delta_k^{1/q}}\right],$$

where $C_2(M,q) = \left\{4(M+1)\int\limits_0^\infty \left|\frac{1}{\pi}\int\limits_x^\infty \frac{\sin 2M\theta \cdot \sin^2\theta}{\theta^3}\,d\theta\right|^q dx\right\}^{1/q}$, $1 \leq q < +\infty$.

Hence,

$$\left\| \int\limits_{E_t} \gamma(\tau) d\tau \right\|_{L^\infty(\mathbb{R}^n)} \leq 2^n C_2^n(M,q) \left[\prod_{k=1}^{n} \frac{1}{\Delta_k^{1/q}} \right] \|\gamma(\tau)\|_{L^p(\mathbb{R}^n)} . \qquad (3.13)$$

The inequality (3.13) is correct for any $M > 1$, therefore taking

$$C(q) = 2 \inf_{M>1} C_2(M,q),$$

we obtain:

$$\left\| \int\limits_{E_t} \gamma(\tau) d\tau \right\|_{L^\infty(\mathbb{R}^n)} \leq 2^n \left[\inf_{M>1} C_2(M,q) \right]^n \left(\prod_{k=1}^{n} \frac{1}{\Delta_k^{1/q}} \right) \|\gamma(\tau)\|_{L^p(\mathbb{R}^n)} =$$

$$= [C(q)]^n \cdot \left(\prod_{k=1}^{n} \frac{1}{\Delta_k^{1/q}} \right) \|\gamma(\tau)\|_{L^p(\mathbb{R}^n)} .$$

Theorem 3.2 is proved. □

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Received July 14, 2014.
In revised form, September 24, 2014.

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