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PLANE DOMAINS WITH SPECIAL CONE CONDITION

Abstract. The paper considers the domains with cone condition in $\mathbb{C}$. We say that domain G satisfies the (weak) *cone condition*, if $p + V(e(p), H) \subset G$ for all $p \in G$, where $V(e(p), H)$ denotes right-angled circular cone with vertex at the origin, a fixed solution ε and a height H, $0 < H \leq \infty$, and depending on the p vector $e(p)$ axis direction. Domains satisfying cone condition play an important role in various branches of mathematic (e. g. [1], [2], [3] (p. 1076), [4]). In the paper of P. Liczberski and V. V. Starkov, α–accessible domains were considered, $\alpha \in [0, 1)$, — the domains, accessible at every boundary point by the cone with symmetry axis on $\{pt : t > 1\}$. Unlike the paper of P. Liczberski and V. V. Starkov, here we consider domains, accessible outside by the cone, which symmetry axis inclined on fixed angle ϕ to the $\{pt : t > 1\}$, $0 < \|\phi\| < < \pi/2$. In this paper we give criteria for this class of domains when the boundaries of domains are smooth, and also give a sufficient condition when boundary is arbitrary. This article is the full variant of [5], published without proofs.

Key words: (α, β)*–accessible domain, cone condition*

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1. Introduction. In [6] (see also [7]) α–accessible domain, $\alpha \in [0, 1)$, were introduced and studied. *A domain $\Omega \subset \mathbb{R}^n$, $0 \in \Omega$, is called α–accessible, if for every point $p \in \partial\Omega$ there exists a number $r = r(p) > 0$ such that the cone*

$$K_+(p, \alpha, r) = \left\{ x \in \mathbb{R}^n : \left(x - p, \frac{p}{\|p\|} \right) \geq \|x - p\| \cos(\frac{\alpha\pi}{2}), \|x - p\| \leq r \right\}$$

is included in $\Omega' = \mathbb{R}^n \setminus \Omega$.

In particular, in [6] the authors proved that α–accessible domains are bounded and satisfy cone condition when $\alpha \in (0,1)$ and $e(p)=-p$. This condition of radiality axis of symmetry applies significant limitation on Ω.

The paper consider the case, when the axis of cone symmetry is lies on ray, containing 0 and p, and crosses the cone.

Definition 1. *A domain $\Omega \subset \mathbb{C}$, $0 \in \Omega$, is called (α,β)–accessible, $\alpha,\beta \in \in [0,1)$, if for every point $p \in \partial\Omega$ there exists a number $r = r(p) > 0$ so that the cone*

$$K_+(p,\alpha,\beta,r) = \left\{ z \in \mathbb{C} : -\frac{\beta\pi}{2} \leq \arg(z-p) - \arg(p) \leq \frac{\alpha\pi}{2}, |z-p| \leq r \right\}$$

is included in $\Omega' = \mathbb{C} \setminus \Omega$.

Let us denote $\alpha_0 = \min(\alpha,\beta)$, $\beta_0 = \max(\alpha,\beta)$. Note that the class of (α,β)–accessible domains is intermediate between α_0– and β_0–accessible classes.

The purpose of this paper is to discuss the failure of condition $e(p) = = -p$, when the angle (let us denote it by ϕ) of inclination axis of symmetry to the ray $\{pt : t > 0\}$ is a constant.

It is interesting to figure out how the properties of domains with this inclination will be changed. This problem is very difficult for large values of ϕ ($\phi > \frac{\pi}{2}$) even in the case of permanent angle ϕ. In this case, the methods by which the results were obtained in [6] are no longer applicable.

This work does not provide a complete description of these areas – this task is too complex, but at this stage it's unable to get rid of condition $e(p) = -p$ and replace it by the condition of the Def. 1, when ϕ is constant.

Let's introduce some other definitions.

Definition 2. *We call a domain Ω starlike with respect to 0 if for every point $z \in \Omega$ segment $[0,z]$ is contained in Ω.*

Definition 3. *We call a domain Ω a strong-starlike with respect to 0 if $[0,p] \cap \partial\Omega = p$ for every point $p \in \partial\Omega$.*

2. Case of arbitrary boundary.

Theorem 1. *If the domain Ω is (α,β)–accessible, $\alpha,\beta \in (0,1)$, then for each point $p \in \partial\Omega$ and for every $\varepsilon \in (0, \min(\alpha,\beta))$ there exists a number ρ such that $\rho(p) > 0$ and the cone $K_-(p,\alpha-\varepsilon,\beta-\varepsilon,\rho) \subset \Omega$, where*

$$K_-(p,\alpha-\varepsilon,\beta-\varepsilon,\rho) =$$

$$= \left\{ z \in \mathbb{C} : -\frac{(\beta-\varepsilon)\pi}{2} < \arg(z-p) - \arg(-p) < \frac{(\alpha-\varepsilon)\pi}{2}, |z-p| < \rho \right\}.$$

Proof. Suppose not. Then there exists a point $p \in \partial\Omega$ such that

$$K_-(p, \alpha-\varepsilon, \beta-\varepsilon, \rho) \cap \Omega' \neq \varnothing$$

for $\rho > 0$ and $\varepsilon \in (0, \min(\alpha, \beta))$. This shows that there exists a sequence of points such that $\{w_m\} \in K_-(p, \alpha-\varepsilon, \beta-\varepsilon, \rho) \cap \Omega'$, and $z_m \to p$ as $m \to \infty$. Consider $C(p, |w_m|)$ – circle with center of p and radius $|w_m|$. This circle intersects the segment $[0, \mathrm{p})$. Associate point w_m with those, which are obtained as a result of intersection $C(p, |z_m|) \cap [0, p)$ with arc of circle, are placed in $Int(K_-(p, \alpha-\varepsilon, \beta-\varepsilon, \rho))$. As $\partial\Omega$ is connected, this arc of circle intersects the bound of Ω. Thus we get sequence of points lying on bound of Ω, which converges to p. Let us denote this sequence $\{w_m\}$.

Denote by $l(\theta)$ the ray, starting from 0 and passing through the segment $[0, \mathrm{p}]$ with angle θ. In [6, proof of Theorem 1] was proved existence of $l_\theta \cap \partial\Omega$ and a unique. Thus, Ω is a strong-starlike domain.

Introduce a function $r = r(\theta)$, the distance from 0 to the point of intersection of the ray $l(\theta)$ with $\partial\Omega$. From [6, proof of Theorem 1] it follows that $r(\theta)$ is continuous.

There exists $n \in \mathbb{N}$ such that for all $m > n$

$$|\arg(w_m) - \arg(p)| < \frac{\varepsilon\pi}{2}. \tag{$*$}$$

Denote by $\phi_m = \arg(w_m) - \arg(p)$, $\phi_m \in (-\pi; \pi]$.

Now let us consider that L is part of $\partial\Omega$, lying between $l(0)$ and $l(\phi_m)$.

As $w_m \in \partial\Omega$, then for it exists cone $K_+(w_m, \alpha, \beta, r_m) \subset \Omega$, $r_m > 0$.

Consider two ways:

1) Let $\phi_m > 0$. Draw a line through w_m parallel those sides of cone $K_-(p, \alpha-\varepsilon, \beta-\varepsilon, \rho)$, which intersect $l(\phi_m)$. This line intersects segment $[0, p]$ at the point $A : |A| < |p|$. The same side of cone $K_+(w_m, \alpha, \beta)$ intersects the segment $[0, p]$ at the point $B : |B| < |A|$. This is true, when:

$$\frac{\beta\pi}{2} > \frac{(\beta-\varepsilon)\pi}{2} + |\phi_m|. \tag{1}$$

2) Let now $\phi_m \leq 0$. By similar reasoning, we obtain:

$$\frac{\alpha\pi}{2} > \frac{(\alpha-\varepsilon)\pi}{2} + |\phi_m|. \tag{2}$$

From $(*)$ it follows that for sufficiently large number m the inequalities (1) and (2) hold. By the fact, that Ω is (α, β)–accessible and

$$K_+(w_m, \alpha, \beta, r_m) \cap \Omega = \varnothing$$

we have $L \cap K_+(w_m, \alpha, \beta, r_m) = \varnothing$.

Consider $L \cap [w_m, B]$. Let w_0 is closest to B point of intersection $L \cap [w_m, B]$. Denote by $\theta_0 = \arg(w_0) - \arg(p)$, the angle between the ray $l(\theta_0)$, going from 0 through point w_0, and the segment [0,p].

As $w_0 \in \partial\Omega$, for it exists cone $K_+(w_0, \alpha, \beta, \delta)$ such that

$$K_+(w_0, \alpha, \beta, \delta) \cap \Omega = \varnothing$$

for sufficiently small $\delta > 0$. The side of cone $K_+(w_0, \alpha, \beta)$ intersects the segment $[0, p]$ in point C in the way that $|C| < |B|$. It follows from the fact that cone $K_+(w_0, \alpha, \beta)$ obtains from cone $K_+(w_m, \alpha, \beta)$ by turning an angle $(\theta_0 - \phi_m)$.

For L to connect w_0 and p, it must either intersect (w_0, B), or intersect segment $[0, p]$. None of both is possible. Indeed, by the definition of w_0, L can't intersect the segment (w_0, B). On the other hand, by virtue of an unambiguous definition $r(\theta)$, $\partial\Omega$ can't contain the radial segments [6, Theorem 1], so it doesn't contain the points from $[0, p)$. Hence we get a contradiction with the fact, that theorem is wrong. The proof is complete now. □

Theorem 2. *If Ω is (α, β)–accessible, then for every point $p \in \partial\Omega$ and for every fixed $\alpha, \beta \in [0, 1)$ unbounded cone $K_+(p, \alpha, \beta, \infty) := K_+(p, \alpha, \beta)$ belongs to $\mathbb{C} \setminus \Omega = \Omega'$.*

Proof. Suppose that the theorem is wrong. Then there is a point $p \in \partial\Omega$ such, that $z \in K_+(p, \alpha, \beta) \cap \Omega$, $z \in \Omega$. Consequently there exists $w \in \partial\Omega$ such, that for every fixed $R > 0$, $w \in \partial K_+(p, \alpha, \beta, R)$. Let us suppose, that point w is first, except p, contained in $\partial K_+(p, \alpha, \beta, R)$, which means, that there were no other points from $\partial\Omega$ on $\partial K_+(p, \alpha, \beta, R)$.

Suppose that $w \notin \partial K_+(p, \alpha, \beta)$. Then $w \in \partial\mathbb{B}(p, R)$ and thus, there exists vicinity $U_w \subset K_+(p, \alpha, \beta)$. So, there is a point $v \in \Omega$ such that $v \in$ $\in U_w$. As Ω is starlike, $[0, v]$ is contained in Ω. From the other hand

$$[0, v] \cap K_+(p, \alpha, \beta, R) \neq \varnothing,$$

which contradicts the fact that $K_+(p, \alpha, \beta, R) \subset \Omega'$.

So $w \in \partial K_+(p, \alpha, \beta)$. Through Theorem 1 there exists $\rho = \rho(p)$ such that cone $K_-(w, \alpha - \varepsilon, \beta - \varepsilon, \rho) \subset \Omega$ for every $\varepsilon \in (o, \min(\alpha, \beta))$.

In $\mathbb{C}$, we introduce polar coordinates 0 – pole, $\overrightarrow{0p}$ – polar.

Consider the points $a_\lambda = p + (w - p)\lambda$, $\lambda \in (0, 1)$. We show, that $a_\lambda \in K_-(w, \alpha - \varepsilon, \beta - \varepsilon, \rho)$ for sufficiently small ρ¿0, when λ close to 1 and ε close to 0. If this is true, then one the one hand $a_\lambda \in \Omega$, which follows from Theorem 1, and on the other hand $a_\lambda \in \partial K_+(p, \alpha, \beta)$ since $w \in \partial K_+(p, \alpha, \beta)$. This contradiction get us that the theorem is true.

To prove the inclusion $a_\lambda \in K_-(p, \alpha - \varepsilon, \beta - \varepsilon, \rho)$ it is enough to show that

$$-\frac{(\beta - \varepsilon)\pi}{2} < \arg(a_\lambda - w) - \arg(-w) < \frac{(\alpha - \varepsilon)\pi}{2}. \tag{3}$$

Since $a_\lambda - w = p + (w - p)\lambda - w = (p - w)(1 - \lambda)$, (3) can be rewritten as:

$$-\frac{(\beta - \varepsilon)\pi}{2} < \arg(p - w) - \arg(-w) < \frac{(\alpha - \varepsilon)\pi}{2}. \tag{4}$$

Here we get two ways:

1) Suppose that

$$-\frac{\beta\pi}{2} < \arg(w) - \arg(p) < 0;$$

this means that $\arg(w - p) - \arg(p) = -\frac{\beta\pi}{2}$.

We see that $\arg(p - w) - \arg(-w) = \arg(w - p) - \arg(w)$, so

$$\arg(p - w) - \arg(-w) = -\frac{\beta\pi}{2} + \arg(p) - arg(w) < 0.$$

As $\arg(w) - \arg(p) < 0$, for sufficiently small $\varepsilon > 0$

$$\arg(p) - \arg(w) > \frac{\varepsilon\pi}{2},$$

and thus,

$$\arg(p) - \frac{\beta\pi}{2} - \arg(w) > -\frac{\beta\pi}{2} + \frac{\varepsilon\pi}{2}. \tag{5}$$

From inequality (5), it follows, that

$$-\frac{(\beta - \varepsilon)\pi}{2} < \arg(p - w) - \arg(-w) < 0.$$

2) Now let us suppose that

$$0 < \arg(w) - \arg(p) < \frac{\alpha\pi}{2};$$

this means that $\arg(w-p) - \arg(p) = \frac{\alpha\pi}{2}$.

We see that $\arg(p-w) - \arg(-w) = \arg(w-p) - \arg(w)$ and so

$$0 < \arg(p-w) - \arg(-w) = \frac{\alpha\pi}{2} + \arg(p) - arg(w).$$

As $\arg(w) - \arg(p) > 0$, for sufficiently $\varepsilon > 0$, one has

$$\arg(w) - \arg(p) > \frac{\varepsilon\pi}{2},$$

so that

$$\frac{\alpha\pi}{2} + \arg(p) - \arg(w) < \frac{\alpha\pi}{2} - \frac{\varepsilon\pi}{2}. \tag{6}$$

From (6), it follows, that

$$0 < \arg(p-w) - \arg(-w) < \frac{(\alpha-\varepsilon)\pi}{2}.$$

Thus, from cases 1) and 2), it follows, that inequality (3) is true, and thus $a_\lambda \in K_-(w, \alpha-\varepsilon, \beta-\varepsilon, \rho)$ with λ close enough to 1. Hence we get a contradiction. The proof is completed. □

Remark 1. *Observe that (α,β)–accessible domains are bounded, if $\alpha, \beta \in$ $\in (0,1)$, since these domains are α_0–accessible, $\alpha_0 = (min(\alpha,\beta))$, and in [6] it was shown that α_0–accessible domains are bounded for $\alpha_0 > 0$.*

Theorem 3. *If $\Omega \subset \mathbb{C}$, $0 \in \Omega$, $\alpha, \beta \in (0,1)$, then the following assertions are equivalent:*

(i) Ω is (α,β)–accessible domain;

(ii) every unbounded cone $K_+(p,\alpha,\beta) \subset \Omega'$, $p \in \partial\Omega$;

(iii) every unbounded cone $K_+(p,\alpha,\beta) \subset \Omega'$, $p \in \Omega'$;

(iv) for every point $p \in \partial\Omega$ and for every $\varepsilon \in (0, \min(\alpha,\beta))$ there exists an $r = r(p) > 0$ such that the bounded cone $K_-(p, \alpha-\varepsilon, \beta-\varepsilon, r) \subset \Omega$.

Proof. In view of Theorems 1 and 2, it is sufficient to prove the implications $(iv) \Rightarrow (i)$ and $(ii) \Rightarrow (iii)$.

Let $w = I(z)$ be the mapping inversion, defined as:

$$w = \frac{1}{\bar{z}}. \tag{7}$$

For the proof of $(iv) \Rightarrow (i)$, under this mapping, consider the image of the cone $K_+(p', \beta, \alpha) \setminus \{p'\}$ to $K_-(p, \alpha, \beta)$, where $p = 1/\overline{p'}$. Indeed, (7) is a bilinear mapping, having a circular feature and the property of preserving angles, so that the boundary of $K_+(p', \alpha, \beta)$ transfers into arcs, intersecting at points p and 0, and the angle of intersections of those circles at the point p is $(\alpha+\beta)\pi/2$, and the image will be lying inside intersection of these circles.

Now, let us consider the condition (iv). Denote $G \subset \mathbb{C}$ as the image $I(\Omega \setminus 0)$. We will show that domain $G' = \mathbb{C} \setminus G$ is (β, α)–accessible.

To show this, we note that $0 \in G'$, and for every point $p \in \partial\Omega$ there exists cone $K_-(p, \alpha - \varepsilon, \beta - \varepsilon, \rho) \subset \Omega$, for sufficient small $\rho = \rho(p, \varepsilon) > 0$. Obviously, there exists a number $r = r(p, \rho) > 0$ such that when $z = 1/\overline{w}$ follows the inclusion

$$I(K_+(p', \beta - \varepsilon, \alpha - \varepsilon, r) \setminus \{p'\}) \subset K_-(p, \alpha - \varepsilon, \beta - \varepsilon, \rho),$$

which means that $I(K_+(p', \beta - \varepsilon, \alpha - \varepsilon, r) \setminus \{p'\}) \subset \Omega$ for every point $p' \in \partial G'$. Thus G' is $(\beta - \varepsilon, \alpha - \varepsilon)$–accessible domain, $\varepsilon \in (0, \min(\alpha, \beta))$. From Theorem 2 it follows, that $K_+(p', \beta - \varepsilon, \alpha - \varepsilon) \subset G$. Passing to the limit $\varepsilon \to 0$ we get that $K_+(p', \beta, \alpha) \subset G'$, so G' is (β, α)–accessible. Hence, from Theorem 1, it follows that for every point $p \in \partial G'$ and for every $\varepsilon \in (0, \min(\alpha, \beta))$ there exists an $r = r(p', \varepsilon) > 0$ such that the cone $K_-(p', \alpha - \varepsilon, \beta - \varepsilon, r)$ belongs to G'.

Note that under the mapping (7) the image of cone $K_-(p', \alpha - \varepsilon, \beta - \varepsilon, r)$ belongs to $K_+(p, \alpha - \varepsilon, \beta - \varepsilon, R)$ for some $r = r(p, R) > 0$, so that $I(K_-(p', \alpha - \varepsilon, \beta - \varepsilon, r)) \subset \Omega'$. Hence and from definition we see that Ω is $(\alpha - \varepsilon, \beta - \varepsilon)$–accessible domain. Using Theorem 1 and allowing $\varepsilon \to 0$ we get, that Ω is (α, β)–accessible. This proves the implication $(iv) \Rightarrow (i)$.

We now show, that if Ω satisfies the condition (ii), then Ω satisfies the condition (iii). Take arbitrary point $p \in \Omega' \setminus \partial\Omega$. The segment $[0, p]$ intersects $\partial\Omega$. If this intersection has more than one point, then we take the closest to p and denote it as p', and the next one – as p''. Then the cone $K_+(p', \alpha, \beta)$ contains inside sufficient small surroundings of point p'' and therefore points from Ω. On the other hand, Theorem 2 says that $K_+(p', \alpha, \beta) \subset \Omega'$. This is a contradiction the fact that $[0, p] \cap \cap \partial\Omega = p'$. Hence, from Theorem 2, it follows that $K_+(p', \alpha, \beta) \subset \Omega'$. We will now show, that $K_+(p, \alpha, \beta) \subset \Omega'$. Indeed, since $|p| > |p'|$, we have $K_+(p, \alpha, \beta) = K_+(p', \alpha, \beta) + (p - p')$, so that $\arg(p) = \arg(p')$. Then for every point $z \in K_+(p', \alpha, \beta)$, one has $z + (p - p') \in K_+(p, \alpha, \beta)$. Let us show that $z+(p-p')$ belongs to $K_+(p',\alpha,\beta)$. Since $z+(p-p')\in K_+(p,\alpha,\beta)$, we see that

$$-\frac{\beta\pi}{2}\le \arg(z+(p-p')-p)-\arg(p)\le \frac{\alpha\pi}{2},$$

and so, as $\arg(p)=\arg(p')$,

$$-\frac{\beta\pi}{2}\le \arg(z-p')-\arg(p')\le \frac{\alpha\pi}{2}.$$

Hence, from definition of $K_+(p',\alpha,\beta)$, we obtain that $z+(p-p')\in$ $\in K_+(p',\alpha,\beta)$. Thus $K_+(p,\alpha,\beta)\subset\Omega'$. Since the point $p\in\Omega'\setminus\partial\Omega$ is arbitrary, we get the implication $(ii)\Rightarrow(iii)$. □

Remark 2. *If $\{\Omega_\gamma\}$ is a family of (α,β)–accessible domains, then the union $\Omega=\bigcup_\gamma\{\Omega_\gamma\}$ is also a (α,β)–accessible domain. Actually, from Theorem 3, it follows that Ω is (α,β)–accessible domain if and only if $K_+(p,\alpha,\beta)\cap\Omega=\varnothing$ for every point $p\in\Omega'$. If $p\notin\Omega$, then $p\notin\Omega_\gamma$ for every γ. In this situation, $K_+(p,\alpha,\beta)\cap\Omega_\gamma=\varnothing$ for every γ. Thus, $K_+(p,\alpha,\beta)\cap(\bigcup\Omega_\gamma)=\varnothing$.*

Theorem 4. *If Ω is (α,β)–accessible domain, $\alpha,\beta\in(0,1)$, then for every $\varepsilon\in(0,\min(\alpha,\beta))$ there exists an $R=R(\varepsilon)>0$ such that the cone $K_-(p,\alpha-\varepsilon,\beta-\varepsilon,R)\subset\Omega$ for every point $p\in\partial\Omega$.*

Proof. From the implication $(iv)\Rightarrow(i)$ in proof of Theorem 3, it follows that for (α,β)–accessible domains Ω, the interior of complement $I(\Omega')=$ $=G'$, using $z=I(w)=1/\overline{w}$, is (β,α)–accessible domain. Therefore it is enough to show that for every fixed $\varepsilon\in(0,\min(\alpha,\beta))$ there exists an $R=R(\varepsilon)>0$ such that for every point $p\in\partial\Omega$, the image of every $w\in$ $\in K_-(p,\alpha-\varepsilon,\beta-\varepsilon,R)$ using $z=I(w)$ considered inside $K_+(p',\beta,\alpha)$, $p'=1/\overline{p}$. Indeed, if it will be shown, then

$$I(K_-(p,\alpha-\varepsilon,\beta-\varepsilon,R))\subset G=I(\Omega).$$

Hence, as $I(w)$ is homeomorphism, we get $K_-(p,\alpha-\varepsilon,\beta-\varepsilon,R)\subset\Omega$.

Since $w\in K_-(p,\alpha-\varepsilon,\beta-\varepsilon,R)$, $w=p+re^{i(\phi+\arg(p))}$, $r\in(0,R]$,

$$\phi\in((2-\beta+\varepsilon)\pi/2,(2+\alpha-\varepsilon)\pi/2),$$

so that $\pi-\phi\in((\beta-\varepsilon)\pi/2,(\alpha+\varepsilon)\pi/2)$.

By definition of the $K_+(p,\beta,\alpha)$, we get $I(w) = 1/\overline{w} \in IntK_+(p',\beta,\alpha)$ if and only if

$$-\frac{\alpha\pi}{2} < \arg\left(\frac{1}{\overline{w}} - \frac{1}{\overline{p}}\right) - \arg\left(\frac{1}{\overline{p}}\right) < \frac{\beta\pi}{2}. \tag{8}$$

Now

$$\arg\left(\frac{1}{\overline{w}} - \frac{1}{\overline{p}}\right) - \arg\left(\frac{1}{\overline{p}}\right) = \arg\left(\frac{\overline{p-w}}{\overline{wp}}\right) - \arg(p) = \arg\left(\frac{\overline{p-w}}{\overline{w}}\right) =$$

$$= \arg\left(\frac{-re^{-i(\phi+\arg(p))}}{\overline{p}+re^{-i(\phi+\arg(p))}}\right) = \arg\left(e^{i(\pi-\phi-\arg(p))}\right) + \arg\left(p+re^{i(\phi+\arg(p))}\right) =$$

$$= \pi - \phi - arg(p) + \arg\left(p+re^{i(\phi+\arg(p))}\right).$$

Since $0 \in \Omega$, we have $p \neq 0$. Then there exists an $R \in (0, \min_{p\in\partial\Omega} |p|)$ such that

$$\left|\arg\left(p+Re^{i(\phi+\arg(p))}\right) - \arg(p)\right| < \frac{\varepsilon\pi}{2},$$

therefore, for every $r \in (0,R)$ and for every $p \in \partial\Omega$, the following inequality holds:

$$\left|\arg\left(p+re^{i(\phi+\arg(p))}\right) - \arg(p)\right| < \frac{\varepsilon\pi}{2},$$

thus the inequality (8) holds.

Hence we get that there exists an $R = R(\varepsilon) > 0$ such that for every $p \in \partial\Omega$, the image of the cone $K_-(p,\alpha-\varepsilon,\beta-\varepsilon,R)$ belongs to $IntK_+(p',\beta,\alpha)$. This proves the theorem. □

Theorem 5. *If a domain $\Omega \subset \mathbb{C}$ ($\Omega \neq \mathbb{C}$) is (α,β)–accessible, $\alpha,\beta \in (0,1)$, then for every $\varepsilon \in (0,\min(\alpha,\beta))$ there exists an $R = R(\varepsilon) > 0$ such that the cone $K_-(p,\alpha-\varepsilon,\beta-\varepsilon,R)$ belongs to Ω for every $p \in \overline{\Omega}$.*

Proof. Assume that theorem is wrong. Then for some $\varepsilon \in (0,\min(\alpha,\beta))$ there exists sequence of points $w_k \in \Omega$ and a sequence of numbers r_k such that the cone

$$K_-(w_k,\alpha-\varepsilon,\beta-\varepsilon,r_k) \cap \Omega' \neq \varnothing \tag{9}$$

for every number $k \in \mathbb{N}$, and $r_k \to 0$. Since $\overline{\Omega}$ is compact, there exists a convergent subsequence of sequence $\{w_k\}$, that $w'_k \to w'_0$. Denote this subsequence as $\{w'_k\}$. If $w'_0 \in \Omega$, then for sufficiently small $\rho > 0$ ball

$\mathbb{B}(w_0', \rho) \subset \Omega$. Starting from some number $k \geq N$, points $w_k' \in \mathbb{B}(w_0', \rho)$, we have $K_-(w_k', \alpha-\varepsilon, \beta-\varepsilon) \cap \mathbb{B}(w_0', \rho) \subset \Omega$. Since the last fact contradicts (9), we get that $w_0' \in \partial\Omega$.

Consider a sequence of points $p_k \in \partial\Omega$, $p_k = \lambda_k w_k'$, $\lambda_k > 1$. Note that $p_k \to w_0'$ when $k \to \infty$ and $\lim\limits_{k\to\infty} p_k = p_0 = w_0'$. Indeed, if it is wrong, then $p_0 = \lambda_0 w_0'$, $\lambda_0 > 0$ and $\lambda \neq 1$. Since $p_0 \in \partial\Omega$, for every surroundings U_{p_0}: $U_{p_0} \cap \Omega \neq \varnothing$. On the one hand Ω is (α, β)–accessible domain and $w_0' \in \partial\Omega$, so the cone $K_+(w_0', \alpha, \beta)$ belongs to Ω'. On the other hand, since $|p_0| > |w_0'|$, the sufficient small surroundigs $U_{p_0} \subset K_+(w_0', \alpha, \beta)$, so that $K_+(w_0', \alpha, \beta) \cap \Omega \neq \varnothing$, but this can not be true (see Theorem 2). Hence we get that $p_0 = w_0'$.

Since $\lim\limits_{k\to\infty} w_k' = p_0 = \lim\limits_{k\to\infty} p_k$, $p_k = \lambda_k w_k'$, $\lambda_k \to 1^+$ as $k \to \infty$. Therefore for number R from Theorem 4 and for sufficient large number k, points $w_k' \in K_-(p_k, \alpha-\varepsilon, \beta-\varepsilon, R)$ and

$$K_-(w_k', \alpha-\varepsilon, \beta-\varepsilon, r_k') \subset K_-(p_k, \alpha-\varepsilon, \beta-\varepsilon, R).$$

By Theorem 4, the cone $K_-(p_k, \alpha-\varepsilon, \beta-\varepsilon, R) \subset \Omega$ for some fixed $R = R(\varepsilon) > 0$, so that $K_-(w_k', \alpha-\varepsilon, \beta-\varepsilon, r_k') \subset \Omega$. The last contradicts the relation (9). Theorem 5 is proved. $\Box$

3. Case of domains with smooth boundary. Here we assume that the domain $\Omega \subset \mathbb{R}^2$ has smooth boundary $\partial\Omega$ given by equation:

$$F(x, y) = 0,$$

and

$$F(x, y) < 0.$$

is Ω.

Smooth function $F(x, y)$ can be set locally which means that $F(x, y) = F_p(x, y)$ in the neighborhood of each point $p \in \partial\Omega$. Since $\partial\Omega$ in the neighborhood of each point $p \in \partial\Omega$ can be defined by the equation:

$$x = f(y) \quad or \quad y = f(x),$$

we can assume that $gradF(p) \neq 0$ for every point $p \in \partial\Omega$.

Denote by $n(p) = \dfrac{gradF(p)}{\|gradF(p)\|}$, the external unit normal vector at point $p \in \partial\Omega$.

The following lemma is a consequence of the lemma from [6].

Lemma 1. *Let $\Omega \subset \mathbb{C}$ with smooth boundary $\partial\Omega$, and $n(p)$ is external normal vector at point $p \in \partial\Omega$. Then for every fixed $\alpha, \beta \in (0,1)$ there exists $r > 0$ such that*

$$K^{+}(p,\alpha,\beta,r) =$$

$$= \left\{ z \in \mathbb{C} : -\frac{\beta\pi}{2} < \arg(z-p) - \arg(n(p)) < \frac{\alpha\pi}{2}, \|z-p\| < r \right\} \subset \Omega',$$

$$K^{-}(p,\alpha,\beta,r) =$$

$$= \left\{ z \in \mathbb{C} : -\frac{\beta\pi}{2} < \arg(z-p) - \arg(-n(p)) < \frac{\alpha\pi}{2}, \|z-p\| < r \right\} \subset \Omega.$$

Theorem 6. *Let $\Omega \in \mathbb{C}$, $\partial\Omega$ be smooth boundary. Then for every fixed $\alpha, \beta \in (0,1)$ domain Ω is (α,β)–accessible if and only if*

$$-\frac{(1-\beta)\pi}{2} \le \arg(p) - \arg(n(p)) \le \frac{(1-\alpha)}{2} \tag{10}$$

for every point $p \in \partial\Omega$.

Proof. Suppose that Ω is (α,β)–accessible. We will show that the inequality (10) holds. As Ω is (α,β)–accessible, it is starlike with respect to 0, and under our assumptions about $F(z)$ it follows from [7] that Ω starlike if and only if $\left(\frac{p}{\|p\|}, \frac{grad(F(p))}{\|gradF(p)\|}\right) \ge 0$ for every $p \in \partial\Omega$. Indeed, $\frac{gradF(p)}{\|gradF(p)\|} = n(p)$ is external normal vector at point p and

$$\left(\frac{p}{\|p\|}, \frac{n(p)}{\|n(p)\|}\right) \ge 0 \Leftrightarrow \cos\phi_p \ge 0,$$

which means that $|\phi_p| \le \pi/2$. Let $\phi_p = \arg(p) - \arg(n(p))$, $\arg(p) \in$ $\in [0,2\pi]$. $\arg(p)$ increases when crawling $\partial\Omega$ in positive direction, and $arg(n(p))$ changes continuously with a continuous changing of $p \in \partial\Omega$. Suppose that at point p the inequality (10) doesn't hold, then we get:

$$\frac{(1-\alpha)\pi}{2} < \arg(p) - \arg(n(p)) \le \frac{\pi}{2}, \tag{11}$$

or

$$-\frac{\pi}{2} \le \arg(p) - \arg(n(p)) < -\frac{(1-\beta)\pi}{2}. \tag{12}$$

For simplicity, we assume that $arg(p) = 0$, $p \in \mathbb{R}$ (this could be achieved by converting the rotation on which the domain Ω is not sensitive). Thus

$$-\frac{\pi}{2} \leq arg(n(p)) < \frac{(1-\alpha)\pi}{2} \tag{13*}$$

or

$$\frac{(1-\beta)\pi}{2} < arg(n(p)) \leq \frac{\pi}{2}. \tag{14*}$$

As Ω is (α, β)–accessible, the cone $K_+(p, \alpha, \beta) \subset \Omega'$. Let

$$K^-(p, \gamma, m) =$$

$$= \left\{ z \in \mathbb{C} : -\frac{\gamma\pi}{2} \leq \arg(z-p) - \arg(-n(p)) \leq \frac{\gamma\pi}{2}, |z-p| < m \right\}.$$

From a lemma proved in [6], it follows that for every fixed $\gamma \in (0, 1)$ there exists an $m > 0$ such that $K^-(p, \gamma, m) \subset \Omega$. Take a point $z \in$ $\in \partial K_+(p, \alpha, \beta, r)$, $z = p + \rho e^{i\phi}$, $\phi = \{\alpha\pi/2, -\beta\pi/2\}$, $0 < \rho < r$.

Separately consider the cases (13*), (14*).

1) Case (13*). Let $z^+ = p + \rho e^{i\alpha\pi/2}$. We will show that z^+ belongs to $K^-(p, \gamma, m)$ if $\rho < m$. Choose $\arg(-n(p))$ such that

$$\arg(-n(p)) = \pi + \arg(n(p)).$$

Then

$$\frac{\pi}{2} \leq \arg(-n(p)) < \frac{(1+\alpha)\pi}{2}. \tag{15}$$

Since $\arg(z^+ - p) = \alpha\pi/2$, one has

$$-\frac{\pi}{2} < \arg(z^+ - p) - \arg(-n(p)) \leq -\frac{(1-\alpha)\pi}{2}. \tag{16}$$

From (16) it follows that for sufficiently small $m > 0$ there exists $\gamma \in (0, 1)$ with

$$-\frac{\gamma\pi}{2} < \arg(z^+ - p) - \arg(-n(p)) \leq \frac{\gamma\pi}{2}.$$

Last inequality means that $z^+ \in K^-(p, \gamma, m)$ with $\rho < m$. A lemma from [6] guarantees that $K^-(p, \gamma, m) \subset \Omega$ and thus z^+ also belongs Ω, which contradicts the fact that $z^+ \in \partial K_+(p, \alpha, \beta) \subset \Omega^{'}$.

2) Case (14*). Now let $z^- = p + \rho e^{-i\frac{\beta\pi}{2}}$. We will show that z^- belongs to $K^-(p, \gamma, m)$ if $\rho < m$. Choose $\arg(-n(p))$ such that

$$\arg(-n(p)) = \arg(n(p)) - \pi.$$

Then
$$-\frac{(1+\beta)\pi}{2} < \arg(-n(p)) \le -\frac{\pi}{2}. \quad (17)$$
Since $\arg(z^- - p) = -\beta\pi/2$, we have
$$\frac{(1-\beta)\pi}{2} \le \arg(z^- - p) - \arg(-n(p)) < \frac{\pi}{2}. \quad (18)$$
From (18) it follows that for sufficiently small $m > 0$ there exists $\gamma \in (0,1)$ with
$$-\frac{\gamma\pi}{2} < \arg(z^- - p) - \arg(-n(p)) \le \frac{\gamma\pi}{2}.$$
Last inequality means, that $z^- \in K^-(p,\gamma,m)$ with $\rho < m$. A lemma from [6] guarantees that $K^-(p,\gamma,m) \subset \Omega$. Thus z^- also belongs to Ω, which contradicts the fact that $z^- \in \partial K_+(p,\alpha,\beta) \subset \Omega^{'}$.

Contradictions in cases 1) and 2) mean that if Ω is (α,β)–accessible domain then inequality (10) holds.

Now let for every point $p \in \partial\Omega$ the inequality (10) hold. We show that Ω is (α,β)–accessible domain. At first show that Ω is (η,θ)–accessible domain for $\eta \in (0,\alpha), \theta \in (0,\beta)$, i.e. for every point $p \in \partial\Omega$ the cone $K_+(p,\eta,\theta,r) \subset \Omega^{'}$, $r = r(p) > 0$. Fix p and take a point $z \in K_+(p,\eta,\theta,r)$ with sufficiently small r then
$$-\frac{\theta\pi}{2} \le \arg(z-p) - \arg(p) \le \frac{\eta\pi}{2}. \quad (19)$$
Compose (10) and (19):
$$-\frac{(1-\beta+\theta)\pi}{2} \le \arg(z-p) - \arg(n(p)) \le \frac{(1-\alpha+\eta)\pi}{2}.$$
Last inequality means that z belongs to $K^+(p, 1-\alpha+\eta, 1-\beta+\theta, r)$. Denote $\psi = \max(1-\alpha+\eta, 1-\beta+\theta)$, $\psi \in (0,1)$. Then
$$K^+(p, 1-\alpha+\eta, 1-\beta+\theta, r) \subset K^+(p,\psi,r).$$
A lemma from [6] guarantees that $K^+(p,\psi,r) \subset \Omega'$ for sufficiently small $r > 0$, and thus $K^+(p, 1-\alpha+\eta, 1-\beta+\theta, r) \subset \Omega'$.

We get that $z \in \Omega'$ for every point $z \in K_+(p,\eta,\theta,r)$ with sufficiently small $r > 0$. Thus Ω is (η,θ)–accessible domain. Now, applying Theorem

2 and passing to a limit $\eta \to \alpha$, $\theta \to \beta$ we get that Ω is a (α, β)–accessible domain. Theorem 6 proved.
□

Corollary 1. *Denote by e the symmetry axis of the cone $K_+(p, \alpha, \beta)$. Then a condition (10) is equivalent to*

$$\left(\frac{e}{\|e\|}, n(p)\right) \geq \sin\left(\frac{(\alpha+\beta)\pi}{4}\right).$$

Proof. Fix $p \in \partial\Omega$. With rotation transformation, assume that arg(p)=0. The solution of the cone $K_+(p, \alpha, \beta, r)$ is $\frac{(\alpha+\beta)\pi}{2}$. Note that $\arg(e) \in$ $\in (-\frac{\pi}{4}, \frac{\pi}{4})$. Then

$$\frac{\alpha\pi}{2} - \arg(e) = \frac{(\alpha+\beta)\pi}{4} if \arg(e) \geq 0,$$

or

$$\frac{\beta\pi}{2} + \arg(e) = \frac{(\alpha+\beta)\pi}{4} if \arg(e) < 0.$$

From the last inequalities we get $\arg(e) = \frac{(\alpha-\beta)\pi}{4}$. Thus from (10) we get:

$$-\frac{(2-\alpha-\beta)\pi}{4} \leq \arg(e) - \arg(n(p)) \leq \frac{(2-\alpha-\beta)\pi}{4},$$

and this is equivalent:

$$\left(\frac{e}{\|e\|}, n(p)\right) \geq \sin\left(\frac{(\alpha+\beta)\pi}{4}\right).$$

□

The following theorem gives a sufficient condition for (α, β)–accessible domains. Here A^* denotes a matrix, conjugate to a matrix A. Let e, as in corollary to Theorem 6, be a vector lying on the symmetry axis of the cone $K_+(p, \alpha, \beta)$.

Theorem 7. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $0 \in \Omega$, $\alpha, \beta \in (0, 1)$. Let $f = \begin{pmatrix} u \\ v \end{pmatrix}$ be a diffeomorphism of a domain Ω at the unit circle centered at the point 0, $f(0) = 0$, and $Df(x)$ is nonsingular differential in*

every point $x \in \Omega$.
If for a number $\delta > 0$ *the inequality:*

$$\frac{f^*(x)Df(x)e}{\|f^*(x)Df(x)\|\|e\|} \geq \sin\left(\frac{(\alpha+\beta)\pi}{4}\right)$$

holds in $\Omega(\delta) = \{x \in \Omega : dist(x, \partial\Omega) < \delta\}$, *then* Ω *is* (α, β)*–accessible domain.*

Proof. Denote by $\Omega_r = \left\{x \in \Omega : u^2 + v^2 < r^2\right\}$ with $r \in (0,1)$. $\Omega_r \subset \Omega$ and $\partial\Omega_r$ – smooth boundary, given by equation:

$$F(x) = u^2 + v^2 - r^2 = 0.$$

Since $Df(x)$ is nonsingular for every $x \in \Omega$, then $f^*(p)Df(p) \neq 0$ for every point $p \in \partial\Omega_r$. Note that

$$f^*(x)Df(x) = (u, v)\begin{pmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial x_2} \end{pmatrix} =$$

$$= (u\frac{\partial u}{\partial x_1} + v\frac{\partial v}{\partial x_1}, u\frac{\partial u}{\partial x_2} + v\frac{\partial v}{\partial x_2}) = \left(\frac{1}{2}\frac{\partial}{\partial x_1}(u^2+v^2), \frac{1}{2}\frac{\partial}{\partial x_2}(u^2+v^2)\right) =$$

$$= \frac{1}{2} gradF(x).$$

Then $gradF(p) = 2f^*(p)Df(p) \neq 0$ for every point $p \in \partial\Omega$.
As Ω is bounded, for fixed $\delta > 0$ $\partial\Omega_r \subset \Omega(\delta)$ for $r \in (r_0, 1)$, with r_0 sufficiently close to 1.
By the condition in Theorem 7 we get,

$$\left(\frac{e}{\|e\|}, \frac{gradF^*(x)}{\|gradF^*(x)\|}\right) = \frac{gradF(x)e}{\|gradF(x)\|\|e\|} = \frac{f^*(x)Df(x)e}{\|f^*(x)Df(x)\|\|e\|} \geq$$

$$\geq \sin\left(\frac{(\alpha+\beta)\pi}{4}\right).$$

Now, from the corollary after Theorem 6 we get that Ω_r is (α, β)–accessible domain, and from remark after Theorem 3 it follows that $\Omega = \bigcup_{r \in (r_0,1)} \Omega_r$ is (α, β)–accessible. Thus, Theorem 7 is proved. $\square$

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