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REMARKS ON COMPLETE MONOTONICITY OF A FUNCTION INVOLVING THE GAMMA FUNCTION

Abstract. In the note, the authors give several remarks on the paper in "Chen and Haigang Zhou *On completely monotone of an arbitrary real parameter function involving the gamma function.* Applied Mathematics and Computation, 2014, vol. 242, pp. 658–663; DOI: 10.1016/j.amc.2014.05.034." By virtue of these, the authors point out several trivial extensions and generalizations and establish some new results on the complete monotonicity of a function involving the classical Euler gamma function.

Key words: *remark, parameter, gamma function, completely monotonic function, logarithmically completely monotonic function, inequality*

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Recall from [1, Chapter XIII], [2, Chapter 1], or [3, Chapter IV] that a function f is said to be completely monotonic on an interval I if it has derivatives of all orders on I and satisfies

$$0 \le (-1)^{k-1} f^{(k-1)}(x) < \infty$$

for $x \in I$ and $k \in \mathbb{N}$, where $f^{(0)}(x)$ means $f(x)$ and $\mathbb{N}$ stands for the set of all positive integers. The class of completely monotonic functions may be characterized by the famous Hausdorff–Bernstein–Widder theorem [3, p. 161, Theorem 12b]: A necessary and sufficient condition that $f(x)$ should be completely monotonic for $0 < x < \infty$ is that

$$f(x) = \int_0^\infty e^{-xt} \, \mathrm{d}\, \alpha(t),$$

where $\alpha(t)$ is non-decreasing and the above integral converges on $(0, \infty)$.

Recall from [2, Chapter 5] and [4]–[8] that an infinitely differentiable and positive function f is said to be logarithmically completely monotonic on an interval I if

$$(-1)^k[\ln f(x)]^{(k)} \ge 0$$

on I for all $k \ge 1$.

It is well known [9, p. 31, (1.01)] that the classical Euler's gamma function may be defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}\,t$$

for $\operatorname{Re}(z) > 0$. The logarithmic derivative of $\Gamma(z)$, denoted by

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)},$$

is called the psi function or digamma function, and the derivatives $\psi'(z)$ and $\psi''(z)$ are respectively called the trigamma and tetragamma functions. As a whole, the derivatives $\psi^{(m)}(z)$ for $m \in \{0\} \cup \mathbb{N}$ are called the polygamma functions. See [10, pp. 255–260].

In this note, we will give several remarks on the paper [11].

Remark 1. *Let*

$$q(t) = \begin{cases} \dfrac{1}{t}\ln\Gamma(t+1), & t \ne 0 \\ -\gamma, & t = 0 \end{cases}$$

on $(-1,\infty)$*, where* $\gamma = 0.57721\dots$ *denotes Euler-Mascheroni's constant. Comparing the function* $f(t)$ *defined by* (3) *below with*

$$h(t) = 1 - \ln(t+1) + q(t), \quad t > -1$$

in [11, p. 659, (8)], whose complete monotonicity on $(-1,\infty)$ *was established in [12], one may pose the following problem: What is the range of* β *such that the function*

$$h_\beta(t) = 1 - \ln(t+\beta) + q(t) \tag{1}$$

is completely monotonic, or equivalently, such that the function

$$H_\beta(t) = \frac{1}{t+\beta}e^{q(t)} \tag{2}$$

is logarithmically completely monotonic, on $(\max\{-1,-\beta\},\infty)$*?*

If the function $h_\beta(t)$ is completely monotonic on $(\max\{-1,-\beta\},\infty)$, *then its first derivative satisfies*

$$h'_\beta(t)=\frac{e^{q(t)}[(\beta+t)q'(t)-1]}{(\beta+t)^2}\le 0$$

which is equivalent to

$$\beta\le\frac{1}{q'(t)}-t.$$

When $\beta>1$, the function $h_\beta(t)$ is defined on $(-1,\infty)$ and, by the L'Hôspital rule,

$$\lim_{t\to-1^+}\left[\frac{1}{q'(t)}-t\right]=\lim_{t\to-1^+}\left[\frac{t^2}{t\psi(t+1)-\ln\Gamma(t+1)}-t\right]=1.$$

This means that, when $\beta>1$, the function $h_\beta(t)$ is not completely monotonic on $(-1,\infty)$.

When $0<\beta\le 1$, the function $h_\beta(t)$ is defined on $(-\beta,\infty)\supset(0,\infty)$ and

$$h_\beta(t)=h(t)+\ln\frac{t+1}{t+\beta}$$

for $t>-\beta$. When $\beta\le 0$, the function $h_\beta(t)$ is defined on $(-\beta,\infty)\subseteq$ $\subseteq(0,\infty)$ and

$$h_\beta(t)=f(t)+\ln\frac{t}{t+\beta}$$

for $t>-\beta$. By virtue of

$$\ln\frac{b}{a}=\int_0^\infty\frac{e^{-au}-e^{-bu}}{u}\,\mathrm{d}\,u$$

in [10, p. 230, 5.1.32], it follows that the functions

$$\ln\frac{t+1}{t+\beta}=\int_0^\infty\frac{e^{-(t+\beta)u}-e^{-(t+1)u}}{u}\,\mathrm{d}\,u=\int_0^\infty\frac{e^{-\beta u}-e^{-u}}{u}e^{-tu}\,\mathrm{d}\,u$$

for $0<\beta\le 1$ and

$$\ln\frac{t}{t+\beta}=\int_0^\infty\frac{e^{-(t+\beta)u}-e^{-tu}}{u}\,\mathrm{d}\,u=\int_0^\infty\frac{e^{-\beta u}-1}{u}e^{-tu}\,\mathrm{d}\,u$$

for $\beta \le 0$ are both completely monotonic on $(-\beta, \infty)$. Since the sum of finitely many completely monotonic functions is still a completely monotonic function, the function $h_\beta(t)$ for $\beta \le 1$ is completely monotonic on $(-\beta, \infty)$.

In conclusion, if and only if $\beta \le 1$, the function $h_\beta(t)$ defined by (1) is completely monotonic, or equivalently, the function $H_\beta(t)$ defined by (2) is logarithmically completely monotonic, on the interval $(\max\{-1, -\beta\}, \infty)$.

Remark 2. *The first main result of the paper [11] is [11, p. 659, Theorem 1] which reads that the function*

$$f_\alpha(x) = 1 - \ln(x + \alpha) + \frac{1}{x + \alpha} \ln \Gamma(x + \alpha + 1)$$

is completely monotone on $x > -\alpha$. Moreover, the function $f_\alpha(x)$ is decreasing on $x > -\alpha$, tends to 0 for $x \to \infty$ and to ∞ for $x \to -\alpha$.

Replacing $x + \alpha$ by t we obtain that the function $f_\alpha(x)$ becomes

$$f(t) = 1 - \ln t + \frac{1}{t} \ln \Gamma(t + 1), \quad t > 0, \tag{3}$$

whose complete monotonicity has been proved in [6, p. 605, Theorem 2] which was cited in [11, p. 659, (5)]. Conversely, since $f(t)$ is completely monotonic on $(0, \infty)$, by definition, it is easy to see that $f(t + \alpha) = f_\alpha(x)$ is completely monotonic in $x \in (-\alpha, \infty)$.

Remark 3. *The second main result of the paper [11] is [11, p. 659, Theorem 2] which states that the function*

$$g_\alpha(x) = \frac{[\Gamma(x + \alpha + 1)]^{1/(x+\alpha)}}{x + \alpha}$$

is completely monotone on $x > -\alpha$.

Replacing $x + \alpha$ by t we obtain that the function $g_\alpha(x)$ becomes

$$g(t) = \frac{[\Gamma(t + 1)]^{1/t}}{t}, \quad t > 0,$$

whose logarithmically complete monotonicity has also been proved in [6, p. 605, Theorem 2] which was cited in [11, p. 659, (6)]. Conversely, since $g(t)$ is logarithmically completely monotonic on $(0, \infty)$, by definition, it is easy to see that $g(t + \alpha) = g_\alpha(x)$ is logarithmically completely monotonic *in $x \in (-\alpha, \infty)$. Furthermore, the inclusion $\mathcal{L}[I] \subset \mathcal{C}[I]$ was proved and verified in [5, 6, 7, 13] once again, where $\mathcal{L}[I]$ and $\mathcal{C}[I]$ denote respectively the set of all logarithmically completely monotonic functions on an interval I and the set of all completely monotonic functions on I. As a result, the complete monotonicity of $g(t)$ on $(0, \infty)$ and $g_\alpha(x)$ on $(-\alpha, \infty)$ are direct and simple consequences of the logarithmically complete monotonicity of $g(t)$ on $(0, \infty)$ and $g_\alpha(x)$ on $(-\alpha, \infty)$.*

Remark 4. *The function $f(t)$ defined by (3) above may be rearranged as*

$$f(t) = \ln \frac{e[\Gamma(t+1)]^{1/t}}{t} = \ln[eg(t)], \quad t > 0.$$

By definition, it is clear that the product between a positive scalar and a logarithmically completely monotonic function is still a logarithmically completely monotonic function and that a logarithm of a logarithmically completely monotonic function whose values are not less than 1 is surely a completely monotonic function. Consequently, the complete monotonicity of $f(t)$ may be derived from the logarithmically complete monotonicity of $g(t)$, it also may be conversely. In conclusion, the complete monotonicity of $f(t)$ and the logarithmically complete monotonicity of $g(t)$ are the same one.

Remark 5. *In a word, for any function $f(x)$ on an interval I and any real scalar α, one should not regard the functions $f(x + \alpha)$ and $f(x)$ as different ones, and then one should not regard all the conclusions on the function $f(x + \alpha)$ as generalizations of those on $f(x)$.*

Remark 6. *This paper is a shortened and simplified version of the preprint [14].*

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References

[1] Mitrinović D. S., Pečarić J. E., and Fink A. M. *Classical and New Inequalities in Analysis.* Dordrecht; Boston; London: Kluwer Academic Publishers, 1993. DOI: 10.1007/978-94-017-1043-5.

[2] Schilling R. L., Song R., and Vondraček Z. *Bernstein Functions—Theory and Applications.* 2nd ed., de Gruyter Studies in Mathematics, 2012, vol. 37. DOI: 10.1515/9783110269338.

[3] Widder D. V. *The Laplace Transform.* Princeton: Princeton University Press, 1946.

[4] Atanassov R. D. and Tsoukrovski U. V. *Some properties of a class of logarithmically completely monotonic functions.* C. R. Acad. Bulgare Sci., 1988, vol. 41, no. 2, pp. 21–23.

[5] Berg C. *Integral representation of some functions related to the gamma function.* Mediterr. J. Math., 2004, vol. 1, pp. 433–439. DOI: 10.1007/s00009-004-0022-6.

[6] Qi F. and Chen C. P. *A complete monotonicity property of the gamma function.* J. Math. Anal. Appl., 2004, vol. 296, pp. 603–607. DOI: 10.1016/j.jmaa.2004.04.026.

[7] Qi F. and Guo B. N. *Complete monotonicities of functions involving the gamma and digamma functions.* RGMIA Res. Rep. Coll., 2004, vol. 7, no. 1. Art. 8, pp. 63–72. URL: http: rgmia.org/v7n1.php.

[8] Qi F., Guo B. N., and Chen C. P. *Some completely monotonic functions involving the gamma and polygamma functions.* RGMIA Res. Rep. Coll., 2004, vol. 7, no. 1. Art. 5, pp. 31–36. URL: http://rgmia.org/v7n1.php.

[9] Olver F. W. J. *Asymptotics and Special Functions.* New York; San Francisco; London: Academic Press., 1974.

[10] Abramowitz M. and Stegun I. A. (Eds) *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* Washington: National Bureau of Standards, Applied Mathematics Series, 1972, vol. 55, 10th printing.

[11] Chen B. and Zhou H.-G. *On completely monotone of an arbitrary real parameter function involving the gamma function.* Appl. Math. Comput., 2014, vol. 242, pp. 658–663. DOI: 10.1016/j.amc.2014.05.034.

[12] Vogt H. and Voigt J. *A monotonicity property of the* Γ*-function.* J. Inequal. Pure Appl. Math., 2002, vol. 3, no. 5. Art. 73. URL: http://www.emis.de/journals/JIPAM/article225.html.

[13] Guo B. N. and Qi F. *A property of logarithmically absolutely monotonic functions and the logarithmically complete monotonicity of a power-exponential function.* Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 2010, vol. 72, no. 2, pp. 21–30.

[14] Guo B. N. and Qi F. *Remarks on complete monotonicity of a function involving the gamma function.* ResearchGate Dataset, available online at URL: http://dx.doi.org/10.13140/RG.2.1.2378.2249.

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