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INEQUALITIES FOR THE RIEMANN–STIELTJES INTEGRAL OF S-DOMINATED INTEGRATORS WITH APPLICATIONS. I

Abstract. Assume that $u, v : [a, b] \to \mathbb{R}$ are monotonic nondecreasing on the interval $[a, b]$. We say that the complex-valued function $h : [a, b] \to \mathbb{C}$ is *S-dominated* by the pair (u, v) if

$$|h(y) - h(x)|^2 \leq [u(y) - u(x)][v(y) - v(x)]$$

for any $x, y \in [a, b]$. In this paper we show amongst other that

$$\left|\int_a^b f(t)\, dh(t)\right|^2 \leq \int_a^b |f(t)|\, du(t) \int_a^b |f(t)|\, dv(t),$$

for any continuous function $f : [a, b] \to \mathbb{C}$. Applications for the trapezoidal and midpoint inequalities are given. New inequalities for some Čebyšev and (CBS)-type functionals are presented. Natural applications for continuous functions of selfadjoint and unitary operators on Hilbert spaces are provided as well.

Key words: *Riemann–Stieltjes integral, functions of bounded variation, cumulative variation, selfadjoint operators, unitary operators, trapezoid and midpoint inequalities, Čebyšev and (CBS)-type functionals*

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1. Introduction.

One of the most important properties of the *Riemann–Stieltjes integral* $\int_a^b f(t)\, dg(t)$ is the fact that this integral exists if one of the functions is of *bounded variation* while the other is *continuous*. The following sharp inequality holds

$$\left|\int_a^b f(t)\, dg(t)\right| \leq \max_{t \in [a,b]} |f(t)| \bigvee_a^b (g),$$

provided that $f:[a,b]\to\mathbb{C}$ is continuous on $[a,b]$ and $g:[a,b]\to\mathbb{C}$ is of bounded variation on this interval. Here $\bigvee_a^b(g)$ denotes the *total variation* of g on $[a,b]$.

When g is *Lipschitzian* with the constant $L>0$, i. e.,

$$|g(t)-g(s)|\leq L|t-s|$$

for any $t,s\in[a,b]$, then we have

$$\left|\int_a^b f(t)\,dg(t)\right|\leq L\int_a^b|f(t)|\,dt$$

for any *Riemann integrable* function $f:[a,b]\to\mathbb{C}$.

Moreover, if the integrator g is *monotonic nondecreasing* on the interval $[a,b]$ and $f:[a,b]\to\mathbb{C}$ is continuous, then we have the *modulus inequality*

$$\left|\int_a^b f(t)\,dg(t)\right|\leq\int_a^b|f(t)|\,dg(t).$$

The above inequalities have been used by many authors to derive various integral inequalities. We provide here some simple examples.

The following *generalized trapezoidal inequality* for the function of bounded variation $f:[a,b]\to\mathbb{C}$ was obtained in 1999 by the author [1, Proposition 1]

$$\left|\int_a^b f(t)\,dt-(x-a)f(a)-(b-x)f(b)\right|\leq \tag{1}$$
$$\leq\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f),$$

where $x\in[a,b]$. The constant $\frac{1}{2}$ cannot be replaced by a smaller quantity. See also [2] for a different proof and other details.

The best inequality one can derive from (1) is the *trapezoid inequality*

$$\left|\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}(b-a)\right|\leq\frac{1}{2}(b-a)\bigvee_a^b(f).$$

Here the constant $\frac{1}{2}$ is also the best possible.

For related results, see [3]–[27].

In order to extend the classical *Ostrowski's inequality* for differentiable functions with bounded derivatives to the larger class of functions of bounded variation, the author obtained in 1999 (see [1] or the RGMIA preprint version of [28]) the following result

$$\left|\int_a^b f(t)\,dt - f(x)(b-a)\right| \le \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f), \tag{2}$$

for any $x \in [a,b]$ and $f : [a,b] \to \mathbb{C}$ is a function of bounded variation on $[a,b]$. Here $\bigvee_a^b (f)$ denotes the *total variation* of f on $[a,b]$ and the constant $\frac{1}{2}$ is the best possible in (2). The best inequality one can obtain from (2) is the *midpoint inequality*, namely

$$\left|\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a)\right| \le \frac{1}{2}(b-a) \bigvee_a^b (f),$$

for which the constant $\frac{1}{2}$ is also sharp.

For related results, see [29]–[57].

Motivated by the above results, we establish in this paper a bound for the quantity

$$\left|\int_a^b f(t)\,dg(t)\right|$$

in the case when the integrand f is continuous while the function of bounded variation g is S-dominated by a pair of monotonic functions in the sense presented at the beginning of the next section. The applications for the trapezoidal and midpoint inequalities are given. New inequalities for some Čebyšev and (CBS)-type functionals are presented. Natural applications for continuous functions of selfadjoint and unitary operators on Hilbert spaces are provided as well.

2. Some General Inequalities. Assume that $u, v : [a,b] \to \mathbb{R}$ are *monotonic nondecreasing* on the interval $[a,b]$. We say that the complex-valued function $h : [a,b] \to \mathbb{C}$ is S-*dominated* by the pair (u,v) if

$$|h(y) - h(x)|^2 \le [u(y) - u(x)]\,[v(y) - v(x)] \tag{S}$$

for any $x, y \in [a,b]$.

We observe that by the monotonicity of the functions u and v and by the symmetry of the inequality (S) over x and y we can assume that (S) is satisfied only for $y > x$ with $x, y \in [a, b]$.

We can give numerous examples of such functions.

For instance, if we take $f, g \in L_2[a, b]$, where $L_2[a, b]$ is the Hilbert space of all complex-valued functions that are square-Lebesgue integrable, and denote

$$h(x) := \int_a^x f(t) g(t) \, dt, \ u(x) := \int_a^x |f(t)|^2 \, dt \text{ and } v(x) := \int_a^x |g(t)|^2 \, dt,$$

then we observe that u and v are monotonic nondecreasing on $[a, b]$ and by Cauchy–Bunyakovsky–Schwarz integral inequality we have

$$|h(y) - h(x)|^2 = \left| \int_x^y f(t) g(t) \, dt \right|^2 \leq \int_x^y |f(t)|^2 \, dt \int_x^y |g(t)|^2 \, dt \leq \\ \leq [u(y) - u(x)] [v(y) - v(x)].$$

for any $y > x$ with $x, y \in [a, b]$.

Now, for $p, q > 0$ if we consider $f(t) := t^p$ and $g(t) := t^q$ for $t \geq 0$, then

$$h_{p,q}(x) := \int_0^x t^{p+q} dt = \frac{1}{p+q+1} x^{p+q+1}$$

and

$$u_p(x) := \int_0^x t^{2p} dt = \frac{1}{2p+1} x^{2p+1}, \ v_q(x) := \int_0^x t^{2q} dt = \frac{1}{2q+1} x^{2q+1}.$$

Taking into account the above comments we observe that the function $h_{p,q}$ is *S-dominated* by the pair (u_p, v_q) on any subinterval of $[0, \infty)$.

Proposition 1. *If $h : [a, b] \to \mathbb{C}$ is S-dominated by the pair (u, v), then h is of bounded variation on any subinterval $[c, d] \subset [a, b]$ and*

$$\left[\bigvee_c^d (h) \right]^2 \leq [u(d) - u(c)] [v(d) - v(c)]. \tag{3}$$

Proof. Consider a division δ of the interval $[c, d]$ given by

$$\delta : c = x_0 < x_1 < ... < x_{n-1} < x_n = b.$$

Since $h:[a,b]\to\mathbb{C}$ is S-dominated by the pair (u,v) then we have

$$|h(x_{i+1})-h(x_i)|\le [u(x_{i+1})-u(x_i)]^{1/2}[v(x_{i+1})-v(x_i)]^{1/2}$$

for any $i\in\{0,...,n-1\}$.

Summing this inequality over i from 0 to $n-1$ and utilizing the Cauchy–Bunyakovsky–Schwarz discrete inequality we have

$$\begin{aligned}
&\sum_{i=1}^{n-1}|h(x_{i+1})-h(x_i)|\le \qquad (4)\\
&\le\sum_{i=1}^{n-1}[u(x_{i+1})-u(x_i)]^{1/2}[v(x_{i+1})-v(x_i)]^{1/2}\le\\
&\le\left(\sum_{i=1}^{n-1}[u(x_{i+1})-u(x_i)]\right)^{1/2}\left(\sum_{i=1}^{n-1}[v(x_{i+1})-v(x_i)]\right)^{1/2}=\\
&=[u(d)-u(c)]^{1/2}[v(d)-v(c)]^{1/2}.
\end{aligned}$$

Taking the supremum over δ we deduce the desired result (3). □

Corollary 1. *If $h:[a,b]\to\mathbb{C}$ is S-dominated by the pair (u,v), then the cumulative variation function $V:[a,b]\to[0,\infty)$ defined by*

$$V(x):=\bigvee_a^x(h)$$

is also S-dominated by the pair (u,v).

Theorem 1. *Assume that $u,v:[a,b]\to\mathbb{R}$ are monotonic nondecreasing on the interval $[a,b]$. If $h:[a,b]\to\mathbb{C}$ is S-dominated by the pair (u,v) and $f:[a,b]\to\mathbb{C}$ is a continuous function on $[a,b]$, then the Riemann–Stieltjes integral $\int_a^b f(t)\,dh(t)$ exists and*

$$\left|\int_a^b f(t)\,dh(t)\right|^2\le\int_a^b|f(t)|\,du(t)\int_a^b|f(t)|\,dv(t). \qquad (5)$$

Proof. Since the Riemann–Stieltjes integral $\int_a^b f(t)\,dh(t)$ exists, then for any sequence of partitions

$$I_n^{(n)}:a=t_0^{(n)}<t_1^{(n)}<\cdots<t_{n-1}^{(n)}<t_n^{(n)}=b$$

with the norm

$$v\left(I_n^{(n)}\right) := \max_{i\in\{0,\dots,n-1\}}\left(t_{i+1}^{(n)} - t_i^{(n)}\right) \to 0 \quad \text{as} \quad n\to\infty,$$

and for any intermediate points $\xi_i^{(n)} \in [t_i^{(n)}, t_{i+1}^{(n)}]$, $i\in\{0,\dots,n-1\}$ we have:

$$\left|\int_a^b f(t)\,dh(t)\right| = \tag{6}$$

$$= \left|\lim_{v\left(I_n^{(n)}\right)\to 0}\sum_{i=0}^{n-1} f\left(\xi_i^{(n)}\right)\left[h\left(t_{i+1}^{(n)}\right) - h\left(t_i^{(n)}\right)\right]\right| \le$$

$$\le \lim_{v\left(I_n^{(n)}\right)\to 0}\sum_{i=0}^{n-1}\left|f\left(\xi_i^{(n)}\right)\right|\left|h\left(t_{i+1}^{(n)}\right) - h\left(t_i^{(n)}\right)\right| \le$$

$$\le \lim_{v\left(I_n^{(n)}\right)\to 0}\sum_{i=0}^{n-1}\left|f\left(\xi_i^{(n)}\right)\right|\left[u\left(t_{i+1}^{(n)}\right) - u\left(t_i^{(n)}\right)\right]^{\frac{1}{2}}\left[v\left(t_{i+1}^{(n)}\right) - v\left(t_i^{(n)}\right)\right]^{\frac{1}{2}} \le$$

$$\le \lim_{v\left(I_n^{(n)}\right)\to 0}\left(\sum_{i=0}^{n-1}\left|f\left(\xi_i^{(n)}\right)\right|\left[u\left(t_{i+1}^{(n)}\right) - u\left(t_i^{(n)}\right)\right]\right)^{1/2} \times$$

$$\times \lim_{v\left(I_n^{(n)}\right)\to 0}\left(\sum_{i=0}^{n-1}\left|f\left(\xi_i^{(n)}\right)\right|\left[v\left(t_{i+1}^{(n)}\right) - v\left(t_i^{(n)}\right)\right]\right)^{1/2} =$$

$$= \left(\int_a^b |f(t)|\,du(t)\right)^{1/2}\left(\int_a^b |f(t)|\,dv(t)\right)^{1/2},$$

where for the last inequality we employed the Cauchy–Bunyakovsky–Schwarz weighted discrete inequality

$$\sum_{k=1}^n m_k a_k b_k \le \left(\sum_{k=1}^n m_k a_k^2\right)^{1/2}\left(\sum_{k=1}^n m_k b_k^2\right)^{1/2},$$

where m_k, a_k, $b_k \ge 0$ for $k\in\{1,...,n\}$. $\square$

3. Trapezoid and Midpoint Inequalities. We can use the inequality (5) to derive various inequalities of trapezoidal and midpoint type as follows.

Theorem 2. *Assume that $u, v : [a,b] \to \mathbb{R}$ are monotonic nondecreasing on the interval $[a,b]$. If $h : [a,b] \to \mathbb{C}$ is S-dominated by the pair (u,v), then*

$$\left|\frac{h(a)+h(b)}{2}(b-a)-\int_a^b h(t)\,dt\right|^2 \leq \tag{7}$$
$$\leq \left[\frac{1}{2}(b-a)[u(b)-u(a)]-\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)u(t)\,dt\right]\times$$
$$\times\left[\frac{1}{2}(b-a)[v(b)-v(a)]-\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)v(t)\,dt\right]\leq$$
$$\leq \frac{1}{4}(b-a)^2[u(b)-u(a)][v(b)-v(a)].$$

Proof. Integrating by parts in the Riemann–Stieltjes integral, we have

$$\frac{h(a)+h(b)}{2}(b-a)-\int_a^b h(t)\,dt=\int_a^b\left(t-\frac{a+b}{2}\right)dh(t). \tag{8}$$

Applying the inequality (5) we have

$$\left|\int_a^b\left(t-\frac{a+b}{2}\right)dh(t)\right|^2\leq\int_a^b\left|t-\frac{a+b}{2}\right|du(t)\int_a^b\left|t-\frac{a+b}{2}\right|dv(t). \tag{9}$$

Integrating by parts in the Riemann–Stieltjes integral we also have

$$\int_a^b\left|t-\frac{a+b}{2}\right|du(t)= \tag{10}$$
$$=\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)du(t)+\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)du(t)=$$
$$=\left(\frac{a+b}{2}-t\right)u(t)\Big|_a^{\frac{a+b}{2}}+\int_a^{\frac{a+b}{2}}u(t)\,dt+$$
$$+\left(t-\frac{a+b}{2}\right)u(t)\Big|_{\frac{a+b}{2}}^b-\int_{\frac{a+b}{2}}^b u(t)\,dt=$$
$$=-\frac{b-a}{2}u(a)+\int_a^{\frac{a+b}{2}}u(t)\,dt+\frac{b-a}{2}u(b)-\int_{\frac{a+b}{2}}^b u(t)\,dt=$$

$$= \frac{1}{2}(b-a)[u(b)-u(a)] - \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) u(t)\, dt$$

and a similar relation for v.

By the Čebyšev inequality for monotonic nondecreasing functions F, G that states that

$$\frac{1}{b-a}\int_a^b F(t)\,G(t)\,dt \geq \frac{1}{b-a}\int_a^b F(t)\,dt \cdot \frac{1}{b-a}\int_a^b G(t)\,dt$$

we also have

$$\int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) u(t)\, dt \geq \tag{11}$$
$$\geq \frac{1}{b-a}\int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt \int_a^b u(t)\, dt = 0$$

and a similar result for v.

Utilizing (8)–(11) we deduce the desired result (7). □

Theorem 3. *Assume that $u, v : [a,b] \to \mathbb{R}$ are monotonic nondecreasing on the interval $[a,b]$. If $h : [a,b] \to \mathbb{C}$ is S-dominated by the pair (u,v), then*

$$\left| h\left(\frac{a+b}{2}\right)(b-a) - \int_a^b h(t)\, dt \right|^2 \leq \tag{12}$$
$$\leq \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) u(t)\, dt \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) v(t)\, dt \leq$$
$$\leq \frac{1}{4}(b-a)^2 [u(b)-u(a)][v(b)-v(a)].$$

Proof. Integrating by parts in the Riemann–Stieltjes integral we have

$$h\left(\frac{a+b}{2}\right)(b-a) - \int_a^b h(t)\, dt = \tag{13}$$
$$= \int_a^{\frac{a+b}{2}} (t-a)\, dh(t) - \int_{\frac{a+b}{2}}^b (b-t)\, dh(t).$$

Taking the modulus in (13) we have

$$\left| h\left(\frac{a+b}{2}\right)(b-a) - \int_a^b h(t)\,dt \right| \leq \tag{14}$$
$$\leq \left| \int_a^{\frac{a+b}{2}} (t-a)\,dh(t) \right| + \left| \int_{\frac{a+b}{2}}^b (b-t)\,dh(t) \right|.$$

Applying the inequality (5) twice, we have

$$\left| \int_a^{\frac{a+b}{2}} (t-a)\,dh(t) \right| \leq \left(\int_a^{\frac{a+b}{2}} (t-a)\,du(t) \right)^{1/2} \left(\int_a^{\frac{a+b}{2}} (t-a)\,dv(t) \right)^{1/2}$$

and

$$\left| \int_{\frac{a+b}{2}}^b (b-t)\,dh(t) \right| \leq \left(\int_{\frac{a+b}{2}}^b (b-t)\,du(t) \right)^{1/2} \left(\int_{\frac{a+b}{2}}^b (b-t)\,dv(t) \right)^{1/2}.$$

Summing these inequalities and utilizing the elementary result

$$\alpha\beta + \lambda\delta \leq \left(\alpha^2 + \lambda^2\right)^{1/2} \left(\beta^2 + \delta^2\right)^{1/2}$$

where $\alpha, \beta, \lambda, \delta \geq 0$, we have

$$\left| \int_a^{\frac{a+b}{2}} (t-a)\,dh(t) \right| + \left| \int_{\frac{a+b}{2}}^b (b-t)\,dh(t) \right| \leq \tag{15}$$
$$\leq \left(\int_a^{\frac{a+b}{2}} (t-a)\,du(t) \right)^{1/2} \left(\int_a^{\frac{a+b}{2}} (t-a)\,dv(t) \right)^{1/2} +$$
$$+ \left(\int_{\frac{a+b}{2}}^b (b-t)\,du(t) \right)^{1/2} \left(\int_{\frac{a+b}{2}}^b (b-t)\,dv(t) \right)^{1/2} \leq$$
$$\leq \left(\int_a^{\frac{a+b}{2}} (t-a)\,du(t) + \int_{\frac{a+b}{2}}^b (b-t)\,du(t) \right)^{1/2} +$$
$$+ \left(\int_a^{\frac{a+b}{2}} (t-a)\,dv(t) + \int_{\frac{a+b}{2}}^b (b-t)\,dv(t) \right)^{1/2}.$$

Integrating by parts in the Riemann–Stieltjes integral we have

$$\int_a^{\frac{a+b}{2}} (t-a)\,du(t) + \int_{\frac{a+b}{2}}^b (b-t)\,du(t) = \tag{16}$$
$$= (t-a)\,u(t)\Big|_a^{\frac{a+b}{2}} - \int_a^{\frac{a+b}{2}} u(t)\,dt + (b-t)\,u(t)\Big|_{\frac{a+b}{2}}^b + \int_{\frac{a+b}{2}}^b u(t)\,dt =$$
$$= \frac{1}{2}(b-a)\,u\left(\frac{a+b}{2}\right) - \int_a^{\frac{a+b}{2}} u(t)\,dt -$$
$$- \frac{1}{2}(b-a)\,u\left(\frac{a+b}{2}\right) + \int_{\frac{a+b}{2}}^b u(t)\,dt =$$
$$= \int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right) u(t)\,dt$$

and the last integral is nonnegative as shown in the proof of Theorem 2.

The same equality holds for v as well.

Utilising the Grüss integral inequality

$$\left|\frac{1}{b-a}\int_a^b F(t)\,G(t)\,dt - \frac{1}{b-a}\int_a^b F(t)\,dt \cdot \frac{1}{b-a}\int_a^b G(t)\,dt\right| \le \tag{17}$$
$$\le \frac{1}{4}(M-m)(N-n)$$

that holds for the Lebesgue integrable functions F and G that satisfy the conditions

$$m \le F(t) \le M \text{ and } n \le G(t) \le N$$

for almost every $t \in [a,b]$, we have

$$0 \le \frac{1}{b-a}\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right) u(t)\,dt =$$
$$= \frac{1}{b-a}\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right) u(t)\,dt -$$
$$- \frac{1}{b-a}\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right) dt \cdot \frac{1}{b-a}\int_a^b u(t)\,dt \le$$
$$\le \frac{1}{2}[u(b)-u(a)]$$

which implies that

$$\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)u(t)\,dt \le \frac{1}{2}(b-a)\left[u(b)-u(a)\right]. \tag{18}$$

A similar result holds for v.

Making use of the inequalities (14), (15), (16) and (18) we deduce the desired result (12). □

4. Applications for Čebyšev and (CBS)-Type Functionals. The following lemma is of interest in itself.

Lemma 1. *Let $F:[a,b]\times[a,b]\to\mathbb{C}$ be continuous on the rectangle $[a,b]\times[a,b]$ and let $h:[a,b]\to\mathbb{C}$ be an S-dominated function by the pair (u,v) which are monotonic nondecreasing on $[a,b]$. Then we have*

$$\left|\int_a^b\left(\int_a^b F(x,y)\,dh(y)\right)dh(x)\right|^2 \le \tag{19}$$

$$\le\left(\int_a^b\left(\int_a^b |F(x,y)|\,du(y)\right)du(x)\right)^{1/2}\times$$

$$\times\left(\int_a^b\left(\int_a^b |F(x,y)|\,dv(y)\right)dv(x)\right)^{1/2}\times$$

$$\times\left(\int_a^b\left(\int_a^b |F(x,y)|\,du(y)\right)dv(x)\right)^{1/2}\times$$

$$\times\left(\int_a^b\left(\int_a^b |F(x,y)|\,dv(y)\right)du(x)\right)^{1/2}.$$

Proof. Assume that x is fixed in $[a,b]$. If we apply Theorem 1 for the S-dominated function $h:[a,b]\to\mathbb{C}$ we have

$$\left|\int_a^b F(x,y)\,dh(y)\right|\le\left(\int_a^b|F(x,y)|\,du(y)\right)^{1/2}\left(\int_a^b|F(x,y)|\,dv(y)\right)^{1/2}. \tag{20}$$

Applying again Theorem 1 and utilizing (20) we have

$$\begin{aligned}
&\left|\int_a^b \left(\int_a^b F(x,y)\,dh(y)\right) dh(x)\right|^2 \leq \qquad (21)\\
&\leq \int_a^b \left|\int_a^b F(x,y)\,dh(y)\right| du(x) \int_a^b \left|\int_a^b F(x,y)\,dh(y)\right| dv(x) \leq \\
&\leq \int_a^b \left(\int_a^b |F(x,y)|\,du(y)\right)^{1/2} \left(\int_a^b |F(x,y)|\,dv(y)\right)^{1/2} du(x) \times \\
&\times \int_a^b \left(\int_a^b |F(x,y)|\,du(y)\right)^{1/2} \left(\int_a^b |F(x,y)|\,dv(y)\right)^{1/2} dv(x).
\end{aligned}$$

On making use of the Cauchy–Bunyakovsky–Schwarz inequality for the Riemann–Stieltjes integral of monotonic nondecreasing integrators we have for the integrator u

$$\begin{aligned}
&\int_a^b \left(\int_a^b |F(x,y)|\,du(y)\right)^{1/2} \left(\int_a^b |F(x,y)|\,dv(y)\right)^{1/2} du(x) \leq \qquad (22)\\
&\leq \left(\int_a^b \left(\int_a^b |F(x,y)|\,du(y)\right) du(x)\right)^{1/2} \times \\
&\times \left(\left(\int_a^b |F(x,y)|\,dv(y)\right) du(x)\right)^{1/2}
\end{aligned}$$

and for the integrator v

$$\begin{aligned}
&\int_a^b \left(\int_a^b |F(x,y)|\,du(y)\right)^{1/2} \left(\int_a^b |F(x,y)|\,dv(y)\right)^{1/2} dv(x) \leq \qquad (23)\\
&\leq \left(\int_a^b \left(\int_a^b |F(x,y)|\,du(y)\right) dv(x)\right)^{1/2} \times \\
&\times \left(\left(\int_a^b |F(x,y)|\,dv(y)\right) dv(x)\right)^{1/2}.
\end{aligned}$$

Utilising (21)–(23) we deduce the desired result (19). □

When no confusion is possible, we write $\int_a^b f du$ instead of $\int_a^b f(x)\, du(x)$.

For the complex-valued functions p, f, g and h, ℓ defined on the interval $[a,b]$ we define the following *Čebyšev type functionals*

$$C(p,f,g;h) := \int_a^b p dh \int_a^b pfg dh - \int_a^b pf dh \int_a^b pg dh \tag{24}$$

and

$$\begin{aligned} C(p,f,g;h,\ell) = {} & \int_a^b p d\ell \int_a^b pfg dh + \int_a^b p dh \int_a^b pfg d\ell - \\ & - \int_a^b pf d\ell \int_a^b pg dh - \int_a^b pf dh \int_a^b pg d\ell \end{aligned} \tag{25}$$

provided that all the Riemann–Stieltjes integrals involved above exist.

We observe that

$$C(p,f,g;h,h) = 2C(p,f,g;h)$$

and

$$C(p,f,g;h,\ell) = C(p,f,g;\ell,h).$$

Theorem 4. *Let f, $g : [a,b] \to \mathbb{R}$ be continuous and synchronous on $[a,b]$, i. e.,*

$$(f(x) - f(y))(g(x) - g(y)) \geq 0$$

for any $x, y \in [a,b]$. If $h : [a,b] \to \mathbb{C}$ is an S-dominated function by the pair (u,v) which are monotonic nondecreasing on $[a,b]$, then for any continuous nonnegative function $p : [a,b] \to [0,\infty)$ we have

$$|C(p,f,g;h)|^2 \leq \frac{1}{2} C(p,f,g;u,v) [C(p,f,g;u)]^{1/2} [C(p,f,g;v)]^{1/2}. \tag{26}$$

Proof. Define the function $F : [a,b] \times [a,b] \to \mathbb{R}$ by

$$\begin{aligned} F(x,y) &:= p(x) p(y) (f(x) - f(y)) (g(x) - g(y)) = \\ &= p(y) p(x) f(x) g(x) + p(x) p(y) f(y) g(y) - \\ &- p(x) f(x) p(y) g(y) - p(y) f(y) p(x) g(x). \end{aligned} \tag{27}$$

We observe that, since p is nonnegative and f, g are synchronous, then $F(x,y) \geq 0$ for any $x, y \in [a,b]$. The function F is also continuous on the rectangle $[a,b] \times [a,b]$.

By simple calculation with the Riemann–Stieltjes integral we have

$$\int_a^b \left(\int_a^b F(x,y)\, dh(y) \right) dh(x) = 2C(p,f,g;h),$$

$$\int_a^b \left(\int_a^b |F(x,y)|\, du(y) \right) du(x) = 2C(p,f,g;u) \geq 0,$$

$$\int_a^b \left(\int_a^b |F(x,y)|\, dv(y) \right) dv(x) = 2C(p,f,g;v) \geq 0$$

and

$$\int_a^b \left(\int_a^b |F(x,y)|\, du(y) \right) dv(x) = \int_a^b \left(\int_a^b |F(x,y)|\, dv(y) \right) du(x) = \\ = C(p,f,g;u,v) \geq 0.$$

Utilising inequality (19) we have

$$[2C(p,f,g;h)]^2 \leq [2C(p,f,g;u)]^{1/2} [2C(p,f,g;v)]^{1/2} \times \\ \times [C(p,f,g;u,v)]^{1/2} [C(p,f,g;u,v)]^{1/2},$$

which is clearly equivalent to (26). □

For the complex-valued functions p, f, g and h, ℓ defined on the interval $[a,b]$ we define the following *(CBS)-type functionals*

$$B(p,f,g;h,\ell) := \int_a^b p|f|^2 dh \int_a^b p|g|^2 d\ell + \int_a^b p|g|^2 dh \int_a^b p|f|^2 d\ell - \\ - \int_a^b pf\overline{g}dh \int_a^b pf\overline{g}d\ell - \int_a^b p\overline{f}gdh \int_a^b p\overline{f}gd\ell \tag{28}$$

and

$$B(p,f,g;h) := \frac{1}{2} B(p,f,g;h,h) = \tag{29}$$

$$= \int_a^b p\,|f|^2\,dh \int_a^b p\,|g|^2\,dh -$$
$$- \frac{1}{2}\left[\left(\int_a^b pf\overline{g}dh\right)^2 + \left(\int_a^b p\overline{f}gdh\right)^2\right].$$

If p is nonnegative and h is real-valued, then

$$\left(\int_a^b p\overline{f}gdh\right)^2 = \overline{\left(\int_a^b pf\overline{g}dh\right)^2},$$

which implies that

$$B\,(p,f,g;h) = \int_a^b p\,|f|^2\,dh \int_a^b p\,|g|^2\,dh - \operatorname{Re}\left(\int_a^b pf\overline{g}dh\right)^2.$$

Also, if p is nonnegative and f, g are real-valued, then

$$B\,(p,f,g;h) = \int_a^b pf^2dh \int_a^b pg^2dh - \left(\int_a^b pfgdh\right)^2.$$

The following result also holds.

Theorem 5. *Let $f,\ g : [a,b] \to \mathbb{C}$ be continuous on $[a,b]$. If $h : [a,b] \to \to \mathbb{C}$ is an S-dominated function by the pair (u,v), which are monotonic nondecreasing on $[a,b]$, then for any continuos nonnegative function $p : [a,b] \to [0,\infty)$ we have*

$$|B\,(p,f,g;h)|^2 \leq \frac{1}{2}B\,(p,f,g;u,v)\,[B\,(p,f,g;u)]^{1/2}\,[B\,(p,f,g;v)]^{1/2}. \tag{30}$$

Proof. Define the function $F : [a,b] \times [a,b] \to \mathbb{R}$ by

$$F\,(x,y) := p\,(x)\,p\,(y)\left|f\,(x)\,\overline{g\,(y)} - g\,(x)\,\overline{f\,(y)}\right|^2 = \tag{31}$$
$$= p\,(y)\,p\,(x)\left[|f\,(x)|^2\,|g\,(y)|^2 + |f\,(y)|^2\,|g\,(x)|^2 -\right.$$
$$\left. -f\,(x)\,\overline{g\,(x)}f\,(y)\,\overline{g\,(y)} - \overline{f\,(x)}g\,(x)\,\overline{f\,(y)}g\,(y)\right].$$

We observe that, since p is nonnegative, then $F(x,y) \geq 0$ for any $x, y \in$ $\in [a,b]$. The function F is also continuous on the rectangle $[a,b] \times [a,b]$.

By simple calculation with the Riemann–Stieltjes integral we have

$$\int_a^b \left(\int_a^b F(x,y)\, dh(y) \right) dh(x) = 2B(p,f,g;h),$$

$$\int_a^b \left(\int_a^b |F(x,y)|\, du(y) \right) du(x) = 2B(p,f,g;u) \geq 0,$$

$$\int_a^b \left(\int_a^b |F(x,y)|\, dv(y) \right) dv(x) = 2B(p,f,g;v) \geq 0$$

and

$$\int_a^b \left(\int_a^b |F(x,y)|\, du(y) \right) dv(x) = \int_a^b \left(\int_a^b |F(x,y)|\, dv(y) \right) du(x) = \\ = B(p,f,g;u,v) \geq 0.$$

Utilising the inequality (19) we have

$$[2B(p,f,g;h)]^2 \leq [2B(p,f,g;u)]^{1/2} [2B(p,f,g;v)]^{1/2} \times \\ \times [B(p,f,g;u,v)]^{1/2} [B(p,f,g;u,v)]^{1/2},$$

which is clearly equivalent to (30). $\square$

5. Applications for Selfadjoint Operators. We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$. Let $A \in \mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda \in \mathbb{R}$ as follows

$$\varphi_\lambda(s) := \begin{cases} 1, \text{ for } -\infty < s \leq \lambda, \\ \\ 0, \text{ for } \lambda < s < +\infty. \end{cases}$$

Then for every $\lambda \in \mathbb{R}$ the operator

$$E_\lambda := \varphi_\lambda(A)$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [58, p. 256]:

Let A be a bounded selfadjoint operator on the Hilbert space H.

$$\text{Denote m} = \min\{\lambda \,|\lambda \in \mathrm{Sp}\,(\mathrm{A})\} =: \min \mathrm{Sp}\,(\mathrm{A}) \text{ and}$$

$$M = \max\{\lambda \,|\lambda \in \mathrm{Sp}\,(A)\} =: \max \mathrm{Sp}\,(A)\,.$$

Then there exists a family of projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, called the spectral family of A, with the following properties:

a) $E_\lambda \le E_{\lambda'}$ for $\lambda \le \lambda'$;

b) $E_{m-0} = 0, E_M = I$ and $E_{\lambda+0} = E_\lambda$ for all $\lambda \in \mathbb{R}$.

We have the representation

$$A = \int_{m-0}^{M} \lambda dE_\lambda.$$

More generally, for any continuous complex-valued function φ defined on $\mathbb{R}$ and for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left\| \varphi\,(A) - \sum_{k=1}^{n} \varphi\,(\lambda_k')\left[E_{\lambda_k} - E_{\lambda_{k-1}}\right] \right\| \le \varepsilon$$

whenever

$$\begin{cases} \lambda_0 < m = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = M, \\ \lambda_k - \lambda_{k-1} \le \delta \text{ for } 1 \le k \le n, \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \le k \le n \end{cases}$$

this means that

$$\varphi\,(A) = \int_{m-0}^{M} \varphi\,(\lambda)\, dE_\lambda,$$

where the integral is of the Riemann–Stieltjes type.

With the above assumptions for A, E_λ and φ we have the representations

$$\varphi\,(A)\,x = \int_{m-0}^{M} \varphi\,(\lambda)\, dE_\lambda x \text{ for all } x \in H$$

and

$$\langle \varphi(A)x,y\rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x, y\rangle \quad \text{for all } x,y \in H.$$

In particular,

$$\langle \varphi(A)x,x\rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x, x\rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(A)x\|^2 = \int_{m-0}^{M} |\varphi(\lambda)|^2\, d\|E_\lambda x\|^2 \quad \text{for all } x \in H.$$

Utilising Theorem 1 we can prove easily the following Schwarz type inequality:

Proposition 2. *Let A be a bounded selfadjoint operator on the Hilbert space H. Denote $m := \min\{\lambda \,|\lambda \in \mathrm{Sp}(A)\} = \min \mathrm{Sp}(A)$ and $M :=$ $:= \max\{\lambda \,|\lambda \in \mathrm{Sp}(A)\} = \max \mathrm{Sp}(A)$. If $f : \mathbb{R} \to \mathbb{C}$ is a continuous function on $[m,M]$, then we have the inequality*

$$|\langle f(A)x,y\rangle|^2 \le \langle |f(A)|\,x,x\rangle \langle |f(A)|\,y,y\rangle \tag{32}$$

for any $x,y \in H$.

Proof. Assume $\varepsilon > 0$ and for fixed $x,y \in H$ define the functions $h,u,v :$ $[m-\varepsilon, M] \to \mathbb{C}$ given by

$$h(t) := \langle E_t x, y\rangle,\ u(t) := \langle E_t x, x\rangle \text{ and } v(t) := \langle E_t y, y\rangle$$

where $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of the bounded selfadjoint operator A.

For $t,s \in [m-\varepsilon, M]$ with $t > s$ by utilizing the Schwarz inequality for nonnegative operators P

$$|\langle Px,y\rangle|^2 \le \langle Px,x\rangle \langle Py,y\rangle,$$

we have

$$|h(t)-h(s)|^2 = |\langle (E_t - E_s)x,y\rangle|^2 \le$$
$$\le \langle (E_t - E_s)x,x\rangle \langle (E_t - E_s)y,y\rangle = (u(t)-u(s))(v(t)-v(s)),$$

which shows that h is S-dominated by the monotonic nondecreasing functions (u,v) on $[m-\varepsilon,M]$.

Applying Theorem 1 to f, h, u and v on $[m-\varepsilon,M]$ we have

$$\left|\int_{m-\varepsilon}^{M} f(t)\,d(\langle E_t x,y\rangle)\right|^2 \le \int_{m-\varepsilon}^{M} |f(t)|\,d(\langle E_t x,x\rangle)\int_{m-\varepsilon}^{M} |f(t)|\,d(\langle E_t y,y\rangle) \tag{33}$$

for any $x,y\in H$.

Letting $\varepsilon\to 0+$ in (33) and utilizing the representation of continuous functions of selfadjoint operators, we deduce the desired result (32). □

For continuous functions p, f, g, the selfadjoint operator A and $x,y\in$ $\in H$ we define the functionals

$$\begin{aligned} C(p,f,g;A,x,y) := {} & \langle p(A)x,y\rangle\langle p(A)f(A)g(A)x,y\rangle - \\ & -\langle p(A)f(A)x,y\rangle\langle p(A)g(A)x,y\rangle, \end{aligned}$$

$$\begin{aligned} C(p,f,g;A,x) &:= C(p,f,g;A,x,x) = \\ &= \langle p(A)x,x\rangle\langle p(A)f(A)g(A)x,x\rangle - \\ &-\langle p(A)f(A)x,x\rangle\langle p(A)g(A)x,x\rangle, \end{aligned}$$

and

$$D(p,f,g;A,x,y) :=$$

$$:= \langle p(A)x,x\rangle\langle p(A)f(A)g(A)y,y\rangle + \langle p(A)y,y\rangle\langle p(A)f(A)g(A)x,x\rangle -$$

$$-\langle p(A)g(A)x,x\rangle\langle p(A)f(A)y,y\rangle - \langle p(A)g(A)y,y\rangle\langle p(A)f(A)x,x\rangle.$$

The following result holds:

Proposition 3. *Let A be a bounded selfadjoint operator on the Hilbert space H. Denote $m := \min\{\lambda\,|\,\lambda\in\mathrm{Sp}(A)\} = \min\mathrm{Sp}(A)$ and $M :=$ $:= \max\{\lambda\,|\,\lambda\in\mathrm{Sp}(A)\} = \max\mathrm{Sp}(A)$. Assume that $f,g:\mathbb{R}\to\mathbb{R}$ are continuous and synchronous functions on $[m,M]$ and $p:\mathbb{R}\to\mathbb{R}$ is a nonnegative continuous function on $[m,M]$. Then for any $x,y\in H$ we have*

$$\begin{aligned} &|C(p,f,g;A,x,y)|^2 \le \\ &\le \frac{1}{2}D(p,f,g;A,x,y)\,[C(p,f,g;A,x)]^{1/2}\,[C(p,f,g;A,y)]^{1/2}. \end{aligned} \tag{34}$$

The proof is similar to the one from Proposition 2 may be obtained by utilizing the integral inequality from Theorem 4. The details are omitted.

A simpler version of the above inequality (34) is as follows:

Corollary 2. *Let the assumptions of Proposition 3 for A, f and g be valid. Then for any $x, y \in H$ with $\|x\| = \|y\| = 1$ we have*

$$\begin{aligned} &|\langle x, y\rangle \langle f(A) g(A) x, y\rangle - \langle f(A) x, y\rangle \langle g(A) x, y\rangle|^2 \leq \\ &\leq \frac{1}{2}\left[\langle f(A) g(A) y, y\rangle + \langle f(A) g(A) x, x\rangle - \right. \\ &\left. - \langle g(A) x, x\rangle \langle f(A) y, y\rangle - \langle g(A) y, y\rangle \langle f(A) x, x\rangle\right] \times \\ &\times \left[\langle f(A) g(A) x, x\rangle - \langle f(A) x, x\rangle \langle g(A) x, x\rangle\right]^{1/2} \times \\ &\times \left[\langle f(A) g(A) y, y\rangle - \langle f(A) y, y\rangle \langle g(A) y, y\rangle\right]^{1/2}. \end{aligned} \tag{35}$$

Remark 1. *If we take, as an example, $f(t) = t^p$ and $g(t) = t^q$ for $p, q > 0$ then for any positive operator A we have from (35) the inequality*

$$\begin{aligned} &\left|\langle x, y\rangle \left\langle A^{p+q} x, y\right\rangle - \langle A^p x, y\rangle \langle A^q x, y\rangle\right|^2 \leq \\ &\leq \frac{1}{2}\left[\left\langle A^{p+q} y, y\right\rangle + \left\langle A^{p+q} x, x\right\rangle - \langle A^q x, x\rangle \langle A^p y, y\rangle - \langle A^q y, y\rangle \langle A^p x, x\rangle\right] \times \\ &\times \left[\left\langle A^{p+q} x, x\right\rangle - \langle A^p x, x\rangle \langle A^q x, x\rangle\right]^{1/2} \left[\left\langle A^{p+q} y, y\right\rangle - \langle A^p y, y\rangle \langle A^q y, y\rangle\right]^{1/2}, \end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

6. Applications for Unitary Operators. Let $(H, \langle\cdot, \cdot\rangle)$ be a complex Hilbert space. We recall that the bounded linear operator $U : H \to H$ on H is *unitary* iff $U^* = U^{-1}$.

It is well known that (see for instance [58, pp. 275–276]), if U is a unitary operator, then there exists a family of *projections* $\{E_\lambda\}_{\lambda \in [0, 2\pi]}$, called the *spectral family* of U with the following properties:

a) $E_\lambda \leq E_\mu$ for $0 \leq \lambda \leq \mu \leq 2\pi$;
b) $E_0 = 0$ and $E_{2\pi} = 1_H$ (the *identity operator* on H);
c) $E_{\lambda+0} = E_\lambda$ for $0 \leq \lambda < 2\pi$;
d) $U = \int_0^{2\pi} e^{i\lambda} dE_\lambda$ where the integral is of the *Riemann–Stieltjes* type.

Moreover, if $\{F_\lambda\}_{\lambda \in [0, 2\pi]}$ is a family of projections satisfying the requirements a)-d) above for the operator U, then $F_\lambda = E_\lambda$ for all $\lambda \in \in [0, 2\pi]$.

Also, for each continuous complex-valued function $f:\mathcal{C}(0,1)\to\mathbb{C}$ on the complex unit circle $\mathcal{C}(0,1)$, we have

$$f(U)=\int_0^{2\pi} f\left(e^{i\lambda}\right)dE_\lambda,$$

where the integral is taken in the Riemann–Stieltjes sense.

In particular, we have the equalities

$$f(U)x=\int_0^{2\pi} f\left(e^{i\lambda}\right)dE_\lambda x,$$

$$\langle f(U)x,y\rangle=\int_0^{2\pi} f\left(e^{i\lambda}\right)d\langle E_\lambda x,y\rangle$$

and

$$\|f(U)x\|^2=\int_0^{2\pi}\left|f\left(e^{i\lambda}\right)\right|^2 d\|E_\lambda x\|^2,$$

for any $x,y\in H$.

Proposition 4. *Let U be a unitary operator on a Hilbert space H. Then for each continuous complex-valued function $f:\mathcal{C}(0,1)\to\mathbb{C}$ on the complex unit circle $\mathcal{C}(0,1)$, we have*

$$|\langle f(U)x,y\rangle|^2\le\langle|f(U)|x,x\rangle\langle|f(U)|y,y\rangle \tag{36}$$

for any $x,y\in H$.

Proof. Let $\{E_\lambda\}_{\lambda\in[0,2\pi]}$ be the spectral family of the unitary operator U. For fixed $x,y\in H$ define the functions $h,u,v:[0,2\pi]\to\mathbb{C}$ given by

$$h(t):=\langle E_t x,y\rangle,\ u(t):=\langle E_t x,x\rangle\ \text{ and }\ v(t):=\langle E_t y,y\rangle.$$

For $t,s\in[0,2\pi]$, with $t>s$, by utilizing the Schwarz inequality for nonnegative operators P

$$|\langle Px,y\rangle|^2\le\langle Px,x\rangle\langle Py,y\rangle,$$

we have

$$|h(t)-h(s)|^2=|\langle(E_t-E_s)x,y\rangle|^2\le$$
$$\le\langle(E_t-E_s)x,x\rangle\langle(E_t-E_s)y,y\rangle=(u(t)-u(s))(v(t)-v(s)),$$

which shows that h is S-dominated by the monotonic nondecreasing functions (u, v) on $[0, 2\pi]$.

Applying Theorem 1 to $f(e^{it})$, h, u and v on $[0, 2\pi]$ we have

$$\left| \int_0^{2\pi} f\left(e^{it}\right) d\left(\langle E_t x, y\rangle\right) \right|^2 \leq$$

$$\leq \int_0^{2\pi} \left| f\left(e^{it}\right) \right| d\left(\langle E_t x, x\rangle\right) \int_0^{2\pi} \left| f\left(e^{it}\right) \right| d\left(\langle E_t y, y\rangle\right)$$

for any $x, y \in H$.

Utilising the representation of continuous functions of unitary operators, we deduce the desired result (36). □

For the complex-valued functions f, g defined on the complex unit circle $\mathcal{C}(0, 1)$ and the unitary operator U on the Hilbert space H we define the following functionals:

$$\begin{aligned} &D(f, g; U, x, y) := \\ &:= \left\langle |f(U)|^2 x, x \right\rangle \left\langle |g(U)|^2 y, y \right\rangle + \left\langle |g(U)|^2 x, x \right\rangle \left\langle |f(U)|^2 y, y \right\rangle - \\ &- \langle f(U)\, \overline{g}(U)\, x, x\rangle \langle f(U)\, \overline{g}(U)\, y, y\rangle - \left\langle \overline{f}(U)\, g(U)\, x, x\right\rangle \left\langle \overline{f}(U)\, g(U)\, y, y\right\rangle, \end{aligned}$$

$$\begin{aligned} B(f, g; U, x, y) &:= \left\langle |f(U)|^2 x, y \right\rangle \left\langle |g(U)|^2 x, y \right\rangle - \\ &- \frac{1}{2}\left[\langle f(U)\, \overline{g}(U)\, x, y\rangle^2 + \left\langle \overline{f}(U)\, g(U)\, x, y\right\rangle^2 \right] \end{aligned}$$

and

$$\begin{aligned} &B(f, g; U, x) := B(f, g; U, x, x) = \\ &= \left\langle |f(U)|^2 x, x \right\rangle \left\langle |g(U)|^2 x, x \right\rangle - \operatorname{Re} \langle f(U)\, \overline{g}(U)\, x, x\rangle^2, \end{aligned}$$

where $x, y \in H$.

Proposition 5. *Let U be a unitary operator on a Hilbert space H. Then for every continuous complex-valued functions $f, g : \mathcal{C}(0, 1) \to \mathbb{C}$ on the complex unit circle $\mathcal{C}(0, 1)$, we have*

$$|B(f, g; U, x, y)|^2 \leq \frac{1}{2} D(f, g; U, x, y) \left[B(f, g; U, x)\right]^{1/2} \left[B(f, g; U, y)\right]^{1/2} \tag{37}$$

for any $x, y \in H$.

The proof follows from Theorem 5 applied to the functions $f\left(e^{it}\right)$, $g\left(e^{it}\right)$, $p(t) = 1$, $h(t) := \langle E_t x, y\rangle$, $u(t) := \langle E_t x, x\rangle$ and $v(t) := \langle E_t y, y\rangle$ where $\{E_\lambda\}_{\lambda\in[0,2\pi]}$ is the spectral family of the unitary operator U and $t \in [0, 2\pi]$. The details are omitted.

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