Probl. Anal. Issues Anal. Vol. 4 (22), No. 2, 2015, pp. 73–92

DOI: 10.15393/j3.art.2015.2829

UDC 517.51

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INTEGRAL INEQUALITIES OF HERMITE – HADAMARD TYPE FOR $((\alpha, m), \log)$-CONVEX FUNCTIONS ON CO–ORDINATES

Abstract. The convexity of functions is a basic concept in mathematics and it has been generalized in various directions. Establishing integral inequalities of Hermite – Hadamard type for various convex functions is one of the main topics in the theory of convex functions and attracts a number of mathematicians for several centuries. Currently an amount of literature on integral inequalities of Hermite – Hadamard type for various convex functions has been accumulated. In the paper the authors introduce a new concept "$((\alpha, m), \log)$–convex functions on the co–ordinates on the rectangle of the plane" and establish new integral inequalities of the Hermite – Hadamard type for $((\alpha, m), \log)$-convex functions on the co–ordinates on the rectangle of the plane.

Key words: *convex function,* $((\alpha, m), \log)$*-convex function, co–ordinates, integral inequality of the Hermite – Hadamard type*

2010 Mathematical Subject Classification: *26A51, 26D15, 26D20, 26E60, 41A55*

1. Introduction. The convexity of functions is a very important and fundamental concept in mathematics and mathematical sciences. It has been being generalized to various forms and there is an amount of literature on integral inequalities of the Hermite – Hadamard type for various convex functions.

Let us recall some definitions and related conclusions.

Definition 1. *Let* $I \subseteq \mathbb{R} = (-\infty, \infty)$ *be an interval. A function* $f : I \to \mathbb{R}$ *is said to be convex if*

$$f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y)$$

holds for all $x, y \in I$ *and* $\lambda \in [0, 1]$.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}$$

is valid. This double inequality is well known in the literature as the Hermite–Hadamard integral inequality for convex functions.

Definition 2. *If a positive function $f : I \subseteq \mathbb{R} \to \mathbb{R}_+ = (0, \infty)$ satisfies*

$$f(\lambda x + (1-\lambda)y) \le f^{\lambda}(x) f^{1-\lambda}(y)$$

for all $\lambda \in [0, 1]$, then we call f a logarithmically convex function on I.

Remark 1. *It is well known that logarithmic convexity of a function f is equivalent to convexity of the function $\ln f$ or $\log_a f$ for $a > 1$.*

Definition 3. [1] *Let $f : [0, b] \to \mathbb{R}$ and $m \in (0, 1]$; if*

$$f(\lambda x + m(1-\lambda)y) \le \lambda f(x) + m(1-\lambda)f(y)$$

is valid for all $x, y \in [0, b]$ and $\lambda \in [0, 1]$, then we say that f is an m-convex function on $[0, b]$.

Remark 2. *The 1-convexity is equivalent to the ordinary convexity defined by Definition 1.*

Theorem 1. [2] *Let $f : \mathbb{R}_0 = [0, \infty) \to \mathbb{R}$ be m-convex and $m \in (0, 1]$. If $f \in L_1([a, b])$ for $0 \le a < b < \infty$, then*

$$\frac{1}{b-a} \int_a^b f(x)dx \le \min\left\{\frac{f(a) + mf(b/m)}{2}, \frac{mf(a/m) + f(b)}{2}\right\}.$$

Definition 4. [3] *Let $f : [0, b] \to \mathbb{R}$ and $(\alpha, m) \in (0, 1] \times (0, 1]$; if*

$$f(\lambda x + m(1-\lambda)y) \le \lambda^{\alpha} f(x) + m(1-\lambda^{\alpha})f(y)$$

is valid for all $x, y \in [0, b]$ and $\lambda \in [0, 1]$, then we say that f is an (α, m)-convex function on $[0, b]$.

Remark 3. *The $(1, m)$-convexity is equivalent to m-convexity. Any convex function for which $f(0) \le 0$ is m-convex for any $m \in (0, 1]$.*

Theorem 2. [4, Theorem 3.1] *Let $I \supseteq \mathbb{R}_0$ be an open real interval and let $f : I \to \mathbb{R}$ be a differentiable function on I such that $f' \in L([a,b])$ for $0 \le a < b < \infty$. If $(f')^q$ is (α,m)-convex on $[a,b]$ for some given numbers $\alpha, m \in (0,1]$ and $q \ge 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q} \min\Bigg\{\Bigg[v_1[f'(a)]^q + \\ + v_2 m\left[f'\left(\frac{b}{m}\right)\right]^q\Bigg]^{1/q}, \left[v_2 m\left[f'\left(\frac{a}{m}\right)\right]^q + v_1[f'(b)]^q\right]^{1/q}\Bigg\},$$

where $v_1 = \frac{1}{(\alpha+1)(\alpha+2)}\left(\alpha + \frac{1}{2^\alpha}\right)$ *and* $v_2 = \frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{\alpha^2+\alpha+2}{2} - \frac{1}{2^\alpha}\right)$.

Definition 5. [5, 6] *A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co–ordinates $\Delta = [a,b]\times[c,d] \subseteq \mathbb{R}^2$ with $a < b$ and $c < d$, if the partial mappings*

$$f_y : [a,b] \to \mathbb{R}, \quad f_y(u) = f(u,y) \quad \text{and} \quad f_x : [c,d] \to \mathbb{R}, \quad f_x(v) = f(x,v)$$

are convex for all fixed $x \in (a,b)$ and $y \in (c,d)$.

Remark 4. *A formal definition for convex functions on the co–ordinates may be restated [5, 6] as follows. A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co–ordinates $\Delta = [a,b]\times[c,d]$ with $a < b$ and $c < d$, if*

$$f(tx+(1-t)z, \lambda y + (1-\lambda)w) \le t\lambda f(x,y) + t(1-\lambda)f(x,w) + \\ + (1-t)\lambda f(z,y) + (1-t)(1-\lambda)f(z,w)$$

holds for all $t, \lambda \in [0,1]$ and $(x,y),(z,w) \in \Delta$.

Definition 6. [7] *For some $(\alpha_1,m_1),(\alpha_2,m_2) \in (0,1]^2$, a function $f : [0,b]\times[0,d] \to \mathbb{R}$ is said to be (α_1,m_1)-(α_2,m_2)-convex on the co–ordinates $[0,b]\times[0,d]$, if*

$$f(ta+m_1(1-t)b, \lambda c + m_2(1-\lambda)d) \le t^{\alpha_1}\lambda^{\alpha_2} f(a,c) + \\ + m_2 t^{\alpha_1}(1-\lambda^{\alpha_2})f(a,d) + m_1(1-t^{\alpha_1})\lambda^{\alpha_2} f(b,c) + \\ + m_1 m_2 (1-t^{\alpha_1})(1-\lambda^{\alpha_2}) f(b,d)$$

holds for all $t, \lambda \in [0,1]$ and $(x,y),(z,w) \in [0,b]\times[0,d]$.

Definition 7. [8] *For* $m_1, m_2, \alpha \in (0,1]$ *and* $s \in [-1,1]$, *a function* $f : [0,b] \times [0,d] \to \mathbb{R}_0$ *is called* (α, m_1)-(s, m_2)*-convex on co–ordinates if*

$$\begin{aligned} f(tx + m_1(1-t)z, \lambda y + m_2(1-\lambda)w) &\le t^\alpha \lambda^s f(x,y) + \\ + m_1(1-t^\alpha)\lambda^s f(z,y) &+ m_2 t^\alpha (1-\lambda)^s f(x,w) + \\ &+ m_1 m_2 (1-t^\alpha)(1-\lambda)^s f(z,w) \end{aligned}$$

holds for all $(t,\lambda) \in [0,1] \times (0,1)$ *and* $(x,y),(z,w) \in [0,b] \times [0,d]$.

Theorem 3. [5, 6, Theorem 2.2] *Let* $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ *be convex on the co–ordinates* $\Delta = [a,b] \times [c,d]$ *with* $a < b$ *and* $c < d$. *Then*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \right. \\ &\quad \left. + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \le \\ &\le \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x,y)\,dy\,dx \le \\ &\le \frac{1}{4}\left[\frac{1}{b-a}\left(\int_a^b f(x,c)dx + \int_a^b f(x,d)dx\right) + \right. \\ &\quad \left. + \frac{1}{d-c}\left(\int_c^d f(a,y)dy + \int_c^d f(b,y)dy\right)\right] \le \\ &\le \frac{1}{4}\left[f(a,c) + f(b,c) + f(a,d) + f(b,d)\right]. \end{aligned}$$

For more information on integral inequalities of the Hermite – Hadamard type for convex functions on the co–ordinates, please refer to [9]–[12] and closely related references therein.

In this paper, we will introduce a new concept "$((\alpha, m), \log)$-convex functions on the co–ordinates on the rectangle of the plane" and establish some new integral inequalities of the Hermite – Hadamard type for $((\alpha, m), \log)$-convex functions on the co–ordinates on the rectangle of the plane.

2. A definition and a lemma.

Now we introduce the definition as follows.

Definition 8. *A mapping* $f : [0,b] \times [c,d] \to \mathbb{R}_+$ *is said to be* $((\alpha, m), \log)$-*convex on co–ordinates* $[0,b] \times [c,d]$ *with* $c, d \in \mathbb{R}$, $c < d$, *and* $b > 0$ *if*

$$f(tx+m(1-t)z,\lambda y+(1-\lambda)w)\le$$
$$\le t^\alpha[f(x,y)]^\lambda[f(x,w)]^{1-\lambda}+m(1-t^\alpha)[f(z,y)]^\lambda[f(z,w)]^{1-\lambda}$$

holds for all $t,\lambda\in[0,1]$ *and* $(x,y),(z,w)\in[0,b]\times[c,d]$ *and some* $m,\alpha\in$ $\in(0,1]$.

Remark 5. *The Definition 8 implies*

$$f(tx+m(1-t)z,\lambda y+(1-\lambda)w)\le t^\alpha[f(x,y)]^\lambda[f(x,w)]^{1-\lambda}+$$
$$+m(1-t^\alpha)[f(z,y)]^\lambda[f(z,w)]^{1-\lambda}\le$$
$$\le t^\alpha\lambda f(x,y)+t^\alpha(1-\lambda)f(x,w)+$$
$$+m(1-t^\alpha)\lambda f(z,y)+m(1-t^\alpha)(1-\lambda)f(z,w).$$

If the function f *is* $((\alpha,m),\log)$*-convex on co–ordinates* $[0,b]\times[c,d]$*, then it is* (α,m)*-*$(1,1)$*-convex on co–ordinates* $[0,b]\times[c,d]$ *(with* $(\alpha_1,m_1)=$ $=(\alpha,m)$ *and* $(\alpha_2,m_2)=(1,1)$ *in Definition 6 or with* $(\alpha,m_1)=(\alpha,m)$ *and* $(s,m_2)=(1,1)$ *in Definition 7).*

In order to prove our main results, we need the following lemma.

Lemma 1. *Let* $f:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}$ *have partial derivatives of the second order. If* $f''_{xy}\in L_1(\Delta)$*, then*

$$J(f,\Delta)\triangleq\frac{1}{(b-a)(d-c)}\Big\{4f(a,c)-2f(a,d)-2f(b,c)+f(b,d)-$$
$$-\frac{1}{b-a}\int_a^b[2f(x,c)-f(x,d)]dx-\frac{1}{d-c}\int_c^d[2f(a,y)-f(b,y)]dy+$$
$$+\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)dx\,dy\Big\}=$$
$$=\int_0^1\int_0^1(1+t)(1+\lambda)f''_{xy}(ta+(1-t)b,\lambda c+(1-\lambda)d)dt\,d\lambda.\quad(1)$$

Proof. Integration by parts gives

$$\int_0^1\int_0^1(1+t)(1+\lambda)f''_{xy}(ta+(1-t)b,\lambda c+(1-\lambda)d)dt\,d\lambda=$$
$$=\frac{1}{a-b}\int_0^1(1+\lambda)\Big[(1+t)f'_y(ta+(1-t)b,\lambda c+(1-\lambda)d)\Big|_{t=0}^{t=1}-$$

$$
\begin{aligned}
&- \int_0^1 f'_y f(ta+(1-t)b, \lambda c+(1-\lambda)d)dt\Bigg] d\lambda = \\
&= \frac{1}{(a-b)(c-d)}\Bigg[4f(a,c) - 2f(b,c) - 2f(a,d) + f(b,d) - \\
&- 2\int_0^1 f(a, \lambda c+(1-\lambda)d)d\lambda + \int_0^1 f(b, \lambda c+(1-\lambda)d)d\lambda - \\
&- 2\int_0^1 f(ta+(1-t)b, c)dt + \int_0^1 f(ta+(1-t)b, d)dt + \\
&+ \int_0^1\int_0^1 f(ta+(1-t)b, \lambda c+(1-\lambda)d)dt\,d\lambda\Bigg].
\end{aligned}
$$

Using substitutions $x = ta+(1-t)b$ and $y = \lambda c+(1-\lambda)d$ for $t, \lambda \in [0,1]$, we obtain (1). Lemma 1 is thus proved. □

3. Some integral inequalities of the Hermite – Hadamard type. Now we are in a position to establish some new integral inequalities of the Hermite – Hadamard type for differentiable $((\alpha, m), \log)$-convex functions on the co–ordinates on rectangle of the plane $\mathbb{R}_0 \times \mathbb{R}$.

The first main result is Theorem 4.

Theorem 4. *Let $f : \mathbb{R}_0 \times \mathbb{R} \to \mathbb{R}$ be a partially differentiable function on $\mathbb{R}_0 \times \mathbb{R}$ and $f''_{xy} \in L_1([a, \frac{b}{m}] \times [c,d])$ with $0 \le a < b$, $c < d$, and $m \in (0,1]$. If $\left|f''_{xy}\right|^q$ is $((\alpha, m), \log)$-convex on co–ordinates $[0, \frac{b}{m}] \times [c,d]$ for $q \ge 1$ and some $\alpha \in (0,1]$, then*

$$
\begin{aligned}
|J(f,\Delta)| \le \left(\frac{3}{2}\right)^{2(1-1/q)} &\left[\frac{1}{2(\alpha+1)(\alpha+2)}\right]^{1/q} \times \\
&\times \Bigg\{2(2\alpha+3)F\left(\left|f''_{xy}(a,c)\right|^q, \left|f''_{xy}(a,d)\right|^q\right) + \\
&+ m\alpha(3\alpha+5)F\left(\left|f''_{xy}\left(\frac{b}{m}, c\right)\right|^q, \left|f''_{xy}\left(\frac{b}{m}, d\right)\right|^q\right)\Bigg\}^{1/q},
\end{aligned}
$$

where

$$
F(u,v) = \begin{cases} L(u,v) - \dfrac{L(u,v)-u}{\ln v - \ln u}, & u \ne v, \\ \dfrac{3}{2}u, & u = v. \end{cases}
$$

Proof. By Lemma 1 and Hölder's integral inequality, we have

$$|J(f,\Delta)| \le \int_0^1\int_0^1 (1+t)(1+\lambda)\left|f''_{xy}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|dt\,d\lambda \le$$
$$\le\left(\int_0^1\int_0^1 (1+t)(1+\lambda)dt\,d\lambda\right)^{1-1/q}\left[\int_0^1\int_0^1 (1+t)\times\right.$$
$$\left.\times(1+\lambda)\left|f''_{xy}f(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q dt\,d\lambda\right]^{1/q}. \qquad (2)$$

Using the $((\alpha,m),\log)$-convexity of $\left|f''_{xy}\right|^q$, we have

$$\int_0^1\int_0^1 (1+t)(1+\lambda)\left|f''_{xy}f(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q dt\,d\lambda \le$$
$$\le\int_0^1\int_0^1 (1+t)(1+\lambda)\left[t^\alpha\left|f''_{xy}(a,c)\right|^{q\lambda}\left|f''_{xy}(a,d)\right|^{q(1-\lambda)}+\right.$$
$$\left.+m(1-t^\alpha)\left|f''_{xy}\left(\frac{b}{m},c\right)\right|^{q\lambda}\left|f''_{xy}\left(\frac{b}{m},d\right)\right|^{q(1-\lambda)}\right]dt\,d\lambda=$$
$$=\left[\int_0^1(1+t)t^\alpha dt\right]\int_0^1(1+\lambda)\left|f''_{xy}(a,c)\right|^{q\lambda}\left|f''_{xy}(a,d)\right|^{q(1-\lambda)}d\lambda+m\left[\int_0^1(1+\right.$$
$$\left.+t)(1-t^\alpha)dt\right]\int_0^1(1+\lambda)\left|f''_{xy}\left(\frac{b}{m},c\right)\right|^{q\lambda}\left|f''_{xy}\left(\frac{b}{m},d\right)\right|^{q(1-\lambda)}d\lambda. \qquad (3)$$

Note that

$$\int_0^1\int_0^1(1+t)(1+\lambda)dt\,d\lambda=\left(\frac{3}{2}\right)^2,$$
$$\int_0^1(1+t)t^\alpha dt=\frac{2\alpha+3}{(\alpha+1)(\alpha+2)},$$
$$\int_0^1(1+t)(1-t^\alpha)dt=\frac{\alpha(3\alpha+5)}{2(\alpha+1)(\alpha+2)}.$$

1) If $\left|f''_{xy}(a,c)\right|^q = \left|f''_{xy}(a,d)\right|^q$ and $\left|f''_{xy}\left(\frac{b}{m},c\right)\right|^q = \left|f''_{xy}\left(\frac{b}{m},d\right)\right|^q$, we have

$$\int_0^1(1+\lambda)\left|f''_{xy}(a,c)\right|^{q\lambda}\left|f''_{xy}(a,d)\right|^{q(1-\lambda)}d\lambda=\frac{3}{2}\left|f''_{xy}(a,c)\right|^q,$$
$$\int_0^1(1+\lambda)\left|f''_{xy}\left(\frac{b}{m},c\right)\right|^{q\lambda}\left|f''_{xy}\left(\frac{b}{m},d\right)\right|^{q(1-\lambda)}d\lambda=\frac{3}{2}\left|f''_{xy}\left(\frac{b}{m},c\right)\right|^q.$$

2) If $\xi \triangleq \frac{\left|f''_{xy}(a,c)\right|^q}{\left|f''_{xy}(a,d)\right|^q} \neq 1$ and $\eta \triangleq \frac{\left|f''_{xy}(\frac{b}{m},c)\right|^q}{\left|f''_{xy}(\frac{b}{m},d)\right|^q} \neq 1$, integrating by parts, we can write

$$\int_0^1 (1+\lambda)\left|f''_{xy}(a,c)\right|^{q\lambda}\left|f''_{xy}(a,d)\right|^{q(1-\lambda)} d\lambda =$$
$$=\left|f''_{xy}(a,d)\right|^q \int_0^1 (1+\lambda)\xi^\lambda d\lambda = \frac{\left|f''_{xy}(a,d)\right|^q}{\ln\xi}\left[2\xi - 1 - \int_0^1 \xi^\lambda d\lambda\right] =$$
$$=\frac{\left|f''_{xy}(a,d)\right|^q}{\ln\xi}\left[2\xi - 1 - \frac{\xi-1}{\ln\xi}\right] = F\left(\left|f''_{xy}(a,c)\right|^q, \left|f''_{xy}(a,d)\right|^q\right)$$

and

$$\int_0^1 (1+\lambda)\left|f''_{xy}\left(\frac{b}{m},c\right)\right|^{q\lambda}\left|f''_{xy}\left(\frac{b}{m},d\right)\right|^{q(1-\lambda)} d\lambda =$$
$$=\left|f''_{xy}\left(\frac{b}{m},d\right)\right|^q \int_0^1 (1+\lambda)\eta^\lambda d\lambda = \frac{\left|f''_{xy}(\frac{b}{m},d)\right|^q}{\ln\eta}\left[2\xi - 1 - \int_0^1 \eta^\lambda d\lambda\right] =$$
$$=\frac{\left|f''_{xy}(\frac{b}{m},d)\right|^q}{\ln\xi}\left[2\eta - 1 - \frac{\xi-1}{\ln\eta}\right] = F\left(\left|f''_{xy}\left(\frac{b}{m},c\right)\right|^q, \left|f''_{xy}\left(\frac{b}{m},d\right)\right|^q\right). \tag{4}$$

By utilizing (2), (3), and (4), we obtain

$$|J(f,\Delta)| \le \int_0^1\int_0^1 (1+t)(1+\lambda)\left|f''_{xy}(ta+(1-t)b, \lambda c + (1-\lambda)d)\right| dt\, d\lambda +$$
$$+ (1-t)b, \lambda c + (1-\lambda)d)\Big|^q dt\, d\lambda\Big)^{1/q} \le$$
$$\le\left(\frac{3}{2}\right)^{2(1-1/q)}\left(\frac{1}{2(\alpha+1)(\alpha+2)}\right)^{1/q}\Bigg\{2(2\alpha+3)F\left(\left|f''_{xy}(a,c)\right|^q,\right.$$
$$\left.\left|f''_{xy}(a,d)\right|^q\right) + m\alpha(3\alpha+5)F\left(\left|f''_{xy}\left(\frac{b}{m},c\right)\right|^q, \left|f''_{xy}\left(\frac{b}{m},d\right)\right|^q\right)\Bigg\}^{1/q}.$$

This completes the proof of Theorem 4. □

If taking $q = 1$ in Theorem 4, we can derive the following corollary.

Corollary 1. *Under the conditions of Theorem 4, if* $q = m = \alpha = 1$, *then*

$$|J(f,\Delta)| \le \frac{1}{6}\Big\{5F\big(\big|f''_{xy}(a,c)\big|^q, \big|f''_{xy}(a,d)\big|^q\big)+ \\ + 4F\big(\big|f''_{xy}(b,c)\big|^q, \big|f''_{xy}(b,d)\big|^q\big)\Big\}.$$

Theorem 5. *Let* $f : \mathbb{R}_0 \times \mathbb{R} \to \mathbb{R}$ *be a partially differentiable function on* $\mathbb{R}_0 \times \mathbb{R}$ *and* $f''_{xy} \in L_1([a, \frac{b}{m}] \times [c,d])$ *with* $0 \le a < b$, $c < d$, *and* $m \in (0,1]$. *If* $\big|f''_{xy}\big|^q$ *is* $((\alpha,m),\log)$*-convex on co–ordinates* $[0, \frac{b}{m}] \times [c,d]$ *for* $q > 1$ *and* $\alpha \in (0,1]$, *then*

$$|J(f,\Delta)| \le \left(\frac{3(q-1)}{2(2q-1)}\big[2^{(2q-1)/(q-1)}-1\big]\right)^{1-1/q}\left[\frac{1}{2(\alpha+1)(\alpha+2)}\right]^{1/q} \times \\ \times \Big\{2(2\alpha+3)L\big(\big|f''_{xy}(a,c)\big|^q, \big|f''_{xy}(a,d)\big|^q\big)+ \\ + m\alpha(3\alpha+5)L\left(\left|f''_{xy}\left(\frac{b}{m},c\right)\right|^q, \left|f''_{xy}\left(\frac{b}{m},d\right)\right|^q\right)\Big\}^{1/q},$$

where $L(u,v)$ *is defined by*

$$L(u,v) = \int_0^1 u^t v^{1-t} dt = \begin{cases} \dfrac{v-u}{\ln v - \ln u}, & u \ne v, \\ u, & u = v. \end{cases} \tag{5}$$

Proof. By Lemma 1, Hölder's integral inequality, and the $((\alpha,m),\log)$-convexity of $\big|f''_{xy}\big|^q$, it follows that

$$|J(f,\Delta)| \le \left(\int_0^1\int_0^1 (1+t)(1+\lambda)^{q/(q-1)} dt\, d\lambda\right)^{1-1/q} \times \\ \times \left[\int_0^1\int_0^1 (1+t)\big|f''_{xy}f(ta+(1-t)b, \lambda c+(1-\lambda)d)\big|^q dt\, d\lambda\right]^{1/q} \le \\ \le \left(\int_0^1\int_0^1 (1+t)(1+\lambda)^{q/(q-1)} dt\, d\lambda\right)^{1-1/q} \times \\ \times\Big\{\int_0^1\int_0^1 (1+t)\Big[t^\alpha\big|f''_{xy}(a,c)\big|^{q\lambda}\big|f''_{xy}(a,d)\big|^{q(1-\lambda)}+$$

$$+m(1-t^{\alpha})\left|f''_{xy}\left(\frac{b}{m},c\right)\right|^{q\lambda}\left|f''_{xy}\left(\frac{b}{m},d\right)\right|^{q(1-\lambda)}\Bigg]dt\,d\lambda\Bigg\}^{1/q}=$$

$$=\left(\frac{3(q-1)}{2(2q-1)}\left[2^{(2q-1)/(q-1)}-1\right]\right)^{1-1/q}\times$$

$$\times\left\{\left(\int_0^1(1+t)t^{\alpha}dt\right)\int_0^1\left|f''_{xy}(a,c)\right|^{q\lambda}\left|f''_{xy}(a,d)\right|^{q(1-\lambda)}d\lambda+\right.$$

$$\left.+m\int_0^1(1+t)(1-t^{\alpha})dt\int_0^1\left|f''_{xy}\left(\frac{b}{m},c\right)\right|^{q\lambda}\left|f''_{xy}\left(\frac{b}{m},d\right)\right|^{q(1-\lambda)}d\lambda\right\}^{1/q}=$$

$$=\left(\frac{3(q-1)}{2(2q-1)}\left[2^{(2q-1)/(q-1)}-1\right]\right)^{1-1/q}\left(\frac{1}{2(\alpha+1)(\alpha+2)}\right)^{1/q}\times$$

$$\times\left\{2(2\alpha+3)L\left(\left|f''_{xy}(a,c)\right|^{q},\left|f''_{xy}(a,d)\right|^{q}\right)+\right.$$

$$\left.+m\alpha(3\alpha+5)L\left(\left|f''_{xy}\left(\frac{b}{m},c\right)\right|^{q},\left|f''_{xy}\left(\frac{b}{m},d\right)\right|^{q}\right)\right\}^{1/q}.$$

Theorem 5 is proved. □

Corollary 2. *Under the conditions of Theorem 5, if $m=\alpha=1$, then*

$$|J(f,\Delta)|\le\left(\frac{1}{3}\right)^{1/q}\left(\frac{3(q-1)}{2(2q-1)}\left[2^{(2q-1)/(q-1)}-1\right]\right)^{1-1/q}\times$$

$$\times\left\{5L\left(\left|f''_{xy}(a,c)\right|^{q},\left|f''_{xy}(a,d)\right|^{q}\right)+4L\left(\left|f''_{xy}(b,c)\right|^{q},\left|f''_{xy}(b,d)\right|^{q}\right)\right\}^{1/q}.$$

Theorem 6. *Let $f:\mathbb{R}_0\times\mathbb{R}\to\mathbb{R}$ be a partially differentiable function on $\mathbb{R}_0\times\mathbb{R}$ and $f''_{xy}\in L_1([a,\frac{b}{m}]\times[c,d])$ with $0\le a<b$, $c<d$, and $m\in(0,1]$. If $\left|f''_{xy}\right|^{q}$ is $((\alpha,m),\log)$-convex on co–ordinates $[0,\frac{b}{m}]\times[c,d]$ for $q>1$ and $\alpha\in(0,1]$, then*

$$|J(f,\Delta)|\le\left(\frac{3(q-1)}{2(2q-1)}\left[2^{(2q-1)/(q-1)}-1\right]\right)^{1-1/q}\left(\frac{1}{\alpha+1}\right)^{1/q}\times$$

$$\times\left\{F\left(\left|f''_{xy}(a,c)\right|^{q},\left|f''_{xy}(a,d)\right|^{q}\right)+\right.$$

$$\left.+\,m\alpha F\left(\left|f''_{xy}\left(\frac{b}{m},c\right)\right|^{q},\left|f''_{xy}\left(\frac{b}{m},d\right)\right|^{q}\right)\right\}^{1/q},$$

where $F(u,v)$ is given as in Theorem 4

Proof. By Lemma 1, Hölder's inequality, and the $((\alpha,m),\log)$-convexity of $\left|f''_{xy}\right|^q$, we get

$$|J(f,\Delta)| \le \int_0^1\int_0^1 (1+t)(1+\lambda)\left|f''_{xy}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right| dt\, d\lambda \le$$
$$\le \left(\int_0^1\int_0^1 (1+t)^{q/(q-1)}(1+\lambda)dtd\lambda\right)^{1-1/q}\times$$
$$\times\left[\int_0^1\int_0^1 (1+\lambda)\left|f''_{xy}f(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q dtd\lambda\right]^{1/q} \le$$
$$\le \left(\frac{3(q-1)}{2(2q-1)}\left[2^{(2q-1)/(q-1)}-1\right]\right)^{1-1/q}\times$$
$$\times\left\{\left(\int_0^1 t^\alpha dt\right)\int_0^1 (1+\lambda)\left|f''_{xy}(a,c)\right|^{q\lambda}\left|f''_{xy}(a,d)\right|^{q(1-\lambda)}d\lambda+\right.$$
$$\left.+m\int_0^1 (1-t^\alpha)dt\int_0^1(1+\lambda)\left|f''_{xy}\left(\frac{b}{m},c\right)\right|^{q\lambda}\left|f''_{xy}\left(\frac{b}{m},d\right)\right|^{q(1-\lambda)}d\lambda\right\}^{1/q} =$$
$$= \left(\frac{3(q-1)}{2(2q-1)}\left[2^{(2q-1)/(q-1)}-1\right]\right)^{1-1/q}\left(\frac{1}{\alpha+1}\right)^{1/q}\left[F\left(\left|f''_{xy}(a,c)\right|^q,\right.\right.$$
$$\left.\left.\left|f''_{xy}(a,d)\right|^q\right)+m\alpha F\left(\left|f''_{xy}\left(\frac{b}{m},c\right)\right|^q,\left|f''_{xy}\left(\frac{b}{m},d\right)\right|^q\right)\right]^{1/q}.$$

The proof of Theorem 6 is complete. □

Corollary 3. *Under the conditions of Theorem 6, if $m=\alpha=1$, then*

$$|J(f,\Delta)| \le \left(\frac{1}{2}\right)^{1/q}\left(\frac{3(q-1)}{2(2q-1)}\left[2^{(2q-1)/(q-1)}-1\right]\right)^{1-1/q}\times$$
$$\times\left\{F\left(\left|f''_{xy}(a,c)\right|^q,\left|f''_{xy}(a,d)\right|^q\right)+F\left(\left|f''_{xy}(b,c)\right|^q,\left|f''_{xy}(b,d)\right|^q\right)\right\}^{1/q}.$$

Theorem 7. *Let $f:\mathbb{R}_0\times\mathbb{R}\to\mathbb{R}$ be a partially differentiable function on $\mathbb{R}_0\times\mathbb{R}$ and $f''_{xy}\in L_1([a,\frac{b}{m}]\times[c,d])$ with $0\le a<b$, $c<d$, and $m\in(0,1]$. If $\left|f''_{xy}\right|^q$ is $((\alpha,m),\log)$-convex on co–ordinates $[0,\frac{b}{m}]\times[c,d]$*

for $q>1$ *and* $\alpha\in(0,1]$*, then*

$$|J(f,\Delta)|\le\left(\frac{q-1}{2q-1}\left[2^{(2q-1)/(q-1)}-1\right]\right)^{2(1-1/q)}\left(\frac{1}{\alpha+1}\right)^{1/q}\times$$
$$\times\left\{L\left(\left|f''_{xy}(a,c)\right|^q,\left|f''_{xy}(a,d)\right|^q\right)+\right.$$
$$\left.+m\alpha L\left(\left|f''_{xy}\left(\frac{b}{m},c\right)\right|^q,\left|f''_{xy}\left(\frac{b}{m},d\right)\right|^q\right)\right\}^{1/q},$$

where $L(u,v)$ *is given by* (5)*.*

Proof. By Lemma 1, Hölder's inequality, and the $((\alpha,m),\log)$-convexity of $\left|f''_{xy}\right|^q$, we acquire

$$|J(f,\Delta)|\le\int_0^1\int_0^1(1+t)(1+\lambda)\left|f''_{xy}(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|dtd\lambda\le$$
$$\le\left(\int_0^1\int_0^1[(1+t)(1+\lambda)]^{q/(q-1)}dtd\lambda\right)^{1-1/q}\times$$
$$\times\left[\int_0^1\int_0^1\left|f''_{xy}f(ta+(1-t)b,\lambda c+(1-\lambda)d)\right|^q dtd\lambda\right]^{1/q}\le$$
$$\le\left(\frac{q-1}{2q-1}\left[2^{(2q-1)/(q-1)}-1\right]\right)^{2(1-1/q)}\times$$
$$\times\left\{\left(\int_0^1 t^\alpha dt\right)\int_0^1\left|f''_{xy}(a,c)\right|^{q\lambda}\left|f''_{xy}(a,d)\right|^{q(1-\lambda)}d\lambda+\right.$$
$$\left.+m\left(\int_0^1(1-t^\alpha)dt\right)\int_0^1\left|f''_{xy}\left(\frac{b}{m},c\right)\right|^{q\lambda}\left|f''_{xy}\left(\frac{b}{m},d\right)\right|^{q(1-\lambda)}d\lambda\right\}^{1/q}=$$

$$=\left(\frac{q-1}{2q-1}\left[2^{(2q-1)/(q-1)}-1\right]\right)^{2(1-1/q)}\left(\frac{1}{\alpha+1}\right)^{1/q}\left\{L\left(\left|f''_{xy}(a,c)\right|^q,\right.\right.$$
$$\left.\left.\left|f''_{xy}(a,d)\right|^q\right)+m\alpha L\left(\left|f''_{xy}\left(\frac{b}{m},c\right)\right|^q,\left|f''_{xy}\left(\frac{b}{m},d\right)\right|^q\right)\right\}^{1/q}.$$

The proof of Theorem 7 is complete. □

Corollary 4. *Under the conditions of Theorem 7, if $m=\alpha=1$, then*

$$|J(f,\Delta)| \le \left(\frac{1}{2}\right)^{1/q}\left(\frac{q-1}{2q-1}\left[2^{(2q-1)/(q-1)}-1\right]\right)^{2(1-1/q)}\times$$
$$\times\Big\{L\big(\left|f''_{xy}(a,c)\right|^q,\left|f''_{xy}(a,d)\right|^q\big)+L\big(\left|f''_{xy}(b,c)\right|^q,\left|f''_{xy}(b,d)\right|^q\big)\Big\}^{1/q}.$$

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Theorem 8. *Let $f:\mathbb{R}_0\times\mathbb{R}\to\mathbb{R}_+$ be integrable on $[a,\frac{b}{m^2}]\times[c,d]$ with $0\le a<b$, $c<d$, and $m\in(0,1]$. If f is $((\alpha,m),\log)$-convex on co–ordinates $[0,\frac{b}{m^2}]\times[c,d]$ for $\alpha\in(0,1]$, then*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\le$$
$$\le\frac{1}{2^{\alpha+1}}\left\{\frac{1}{b-a}\int_a^b\left[f\left(x,\frac{c+d}{2}\right)+m(2^\alpha-1)f\left(\frac{x}{m},\frac{c+d}{2}\right)\right]dx+\right.$$
$$+\frac{1}{d-c}\int_c^d\left(\left[f\left(\frac{a+b}{2},y\right)f\left(\frac{a+b}{2},c+d-y\right)\right]^{1/2}+\right.$$
$$\left.\left.+m(2^\alpha-1)\left[f\left(\frac{a+b}{2m},y\right)f\left(\frac{a+b}{2m},c+d-y\right)\right]^{1/2}\right)dy\right\}\le$$
$$\le\frac{1}{2^{\alpha+1}}\left\{\frac{1}{b-a}\int_a^b\left[f\left(x,\frac{c+d}{2}\right)+m(2^\alpha-1)f\left(\frac{x}{m},\frac{c+d}{2}\right)\right]dx+\right.$$
$$\left.+\frac{1}{d-c}\int_c^d\left[f\left(\frac{a+b}{2},y\right)+m(2^\alpha-1)f\left(\frac{a+b}{2m},y\right)\right]dy\right\}\le$$
$$\le\frac{1}{2^{2\alpha}(b-a)(d-c)}\int_c^d\int_a^b\left[f(x,y)+2m(2^\alpha-1)f\left(\frac{x}{m},y\right)+\right.$$
$$\left.+m^2(2^\alpha-1)^2f\left(\frac{x}{m^2},y\right)\right]dxdy,$$

where $L(u,v)$ is the logarithmic mean.

Proof. Using the $((\alpha,m),\log)$-convexity of f and by the GA inequality, we have

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)=\int_0^1 f\left(\frac{ta+(1-t)b+(1-t)a+tb}{2},\frac{c+d}{2}\right)dt\le$$

$$\le \int_0^1 \left[\frac{1}{2^\alpha} f\left(ta+(1-t)b, \frac{c+d}{2}\right) + \right.$$
$$\left. + m\left(1-\frac{1}{2^\alpha}\right) f\left(\frac{(1-t)a+tb}{m}, \frac{c+d}{2}\right)\right] dt =$$
$$= \frac{1}{2^\alpha(b-a)} \int_a^b \left[f\left(x, \frac{c+d}{2}\right) + m(2^\alpha-1) f\left(\frac{x}{m}, \frac{c+d}{2}\right)\right] dx =$$
$$= \frac{1}{2^\alpha(b-a)} \int_0^1 \int_a^b \left[f\left(x, \frac{c+d}{2}\right) + m(2^\alpha-1) f\left(\frac{x}{m}, \frac{c+d}{2}\right)\right] dx d\lambda \le$$
$$\le \frac{1}{2^{2\alpha}(b-a)} \int_0^1 \int_a^b \left\{\left[f(x, \lambda c+(1-\lambda)d) f(x, (1-\lambda)c+\lambda d)\right]^{1/2} + \right.$$
$$+ 2m(2^\alpha-1)\left[f\left(\frac{x}{m}, \lambda c+(1-\lambda)d\right) f\left(\frac{x}{m}, (1-\lambda)c+\lambda d\right)\right]^{1/2} +$$
$$\left. + m^2(2^\alpha-1)^2\left[f\left(\frac{x}{m^2}, \lambda c+(1-\lambda)d\right) f\left(\frac{x}{m^2}, (1-\lambda)c+\lambda d\right)\right]^{1/2}\right\} dx d\lambda =$$
$$= \frac{1}{2^{2\alpha}(b-a)(d-c)} \int_c^d \int_a^b \left\{\left[f(x,y) f(x, c+d-y)\right]^{1/2} + \right.$$
$$+ 2m(2^\alpha-1)\left[f\left(\frac{x}{m}, y\right) f\left(\frac{x}{m}, c+d-y\right)\right]^{1/2} +$$
$$\left. + m^2(2^\alpha-1)^2\left[f\left(\frac{x}{m^2}, y\right) f\left(\frac{x}{m^2}, c+d-y\right)\right]^{1/2}\right\} dx dy \le$$
$$\le \frac{1}{2^{2\alpha}(b-a)(d-c)} \int_c^d \int_a^b \left[f(x,y) + 2m(2^\alpha-1) f\left(\frac{x}{m}, y\right) + \right.$$
$$\left. + m^2(2^\alpha-1)^2 f\left(\frac{x}{m^2}, y\right)\right] dx dy.$$

Similarly, we acquire

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) = \int_0^1 f\left(\frac{a+b}{2}, \frac{\lambda c+(1-\lambda)d+(1-\lambda)c+\lambda d}{2}\right) d\lambda \le$$
$$\le \int_0^1 \left\{\frac{1}{2^\alpha}\left[f\left(\frac{a+b}{2}, \lambda c+(1-\lambda)d\right) f\left(\frac{a+b}{2}, (1-\lambda)c+\lambda d\right)\right]^{1/2} + m \times \right.$$
$$\left. \times \left(1-\frac{1}{2^\alpha}\right)\left[f\left(\frac{a+b}{2m}, \lambda c+(1-\lambda)d\right) f\left(\frac{a+b}{2m}, (1-\lambda)c+\lambda d\right)\right]^{1/2}\right\} d\lambda =$$

$$= \frac{1}{2^\alpha(d-c)}\int_c^d\left\{\left[f\left(\frac{a+b}{2},y\right)f\left(\frac{a+b}{2},c+d-y\right)\right]^{1/2}+\right.$$

$$\left.+m(2^\alpha-1)\left[f\left(\frac{a+b}{2m},y\right)f\left(\frac{a+b}{2m},c+d-y\right)\right]^{1/2}\right\}dy\le$$

$$\le\frac{1}{2^\alpha(d-c)}\int_c^d\left[f\left(\frac{a+b}{2},y\right)+m(2^\alpha-1)f\left(\frac{a+b}{2m},y\right)\right]dy\le$$

$$\le\frac{1}{2^{2\alpha}(d-c)}\int_c^d\int_0^1\left[f(ta+(1-t)b,y)+2m(2^\alpha-1)\times\right.$$

$$\left.\times f\left(\frac{(1-t)a+tb}{m},y\right)+m^2(2^\alpha-1)^2f\left(\frac{(1-t)a+tb}{m^2},y\right)\right]dtdy=$$

$$=\frac{1}{2^{2\alpha}(b-a)(d-c)}\int_c^d\int_a^b\left[f(x,y)+2m(2^\alpha-1)f\left(\frac{x}{m},y\right)+\right.$$

$$\left.+m^2(2^\alpha-1)^2f\left(\frac{x}{m^2},y\right)\right]dxdy.$$

Combining the above inequalities leads to Theorem 8. □

Theorem 9. *Let $f:\mathbb{R}_0\times\mathbb{R}\to\mathbb{R}_+$ be integrable on $[a,\frac{b}{m^2}]\times[c,d]$ with $0\le a<b$, $c<d$, and $m\in(0,1]$. If f is $((\alpha,m),\log)$-convex on co–ordinates $[0,\frac{b}{m^2}]\times[c,d]$ for $\alpha\in(0,1]$, then*

$$\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)dxdy\le\frac{1}{2^{\alpha+1}(b-a)}\times$$

$$\times\int_a^b\left\{L(f(x,c),f(x,d))+m(2^\alpha-1)L\left(f\left(\frac{x}{m},c\right),f\left(\frac{x}{m},d\right)\right)\right\}dx+$$

$$+\frac{1}{2(\alpha+1)(d-c)}\int_c^d\left[f(a,y)+m\alpha f\left(\frac{b}{m},y\right)\right]dy\le$$

$$\le\frac{1}{2^{\alpha+2}(b-a)}\int_a^b G(x,c,d)dx+$$

$$+\frac{1}{2(\alpha+1)(d-c)}\int_c^d\left[f(a,y)+m\alpha f\left(\frac{b}{m},y\right)\right]dy\le$$

$$\le\frac{1}{2^{\alpha+2}(\alpha+1)}\left\{G(a,c,d)+m\alpha G\left(\frac{b}{m},c,d\right)+2L(f(a,c),f(a,d))+\right.$$

$$+ 2m(2^\alpha - 1)L\left(f\left(\frac{a}{m}, c\right), f\left(\frac{a}{m}, d\right)\right) + 2m\alpha\left\{L\left(f\left(\frac{b}{m}, c\right),\right.\right.$$
$$\left.\left. f\left(\frac{b}{m}, d\right)\right) + 2m(2^\alpha - 1)L\left(f\left(\frac{b}{m^2}, c\right), f\left(\frac{b}{m^2}, d\right)\right)\right\}\right\} \le$$
$$\le \frac{1}{2^{\alpha+1}(\alpha+1)}\left[G(a, c, d) + m\alpha G\left(\frac{b}{m}, c, d\right)\right],$$

where $L(u, v)$ is the logarithmic mean and

$$G(x, c, d) = f(x, c) + f(x, d) + m(2^\alpha - 1)\left(f\left(\frac{x}{m}, c\right) + f\left(\frac{x}{m}, d\right)\right)$$

for $x \in \left[a, \frac{b}{m}\right]$.

Proof. Since $L(x, y) \le \frac{x+y}{2}$ for $x, y > 0$, from the $((\alpha, m), \log)$-convexity of f and by the GA inequality, we obtain

$$\frac{1}{(b-a)(d-c)}\int_c^d \int_a^b f(x, y)dxdy =$$
$$= \frac{1}{b-a}\int_0^1 \int_a^b f(x, \lambda c + (1-\lambda)d)dxd\lambda \le$$
$$\le \frac{1}{2^\alpha(b-a)}\int_0^1 \int_a^b \left\{ [f(x, c)f(x, d)]^\lambda + \right.$$
$$\left. + m(2^\alpha - 1)\left[f\left(\frac{x}{m}, c\right)f\left(\frac{x}{m}, d\right)\right]^{1-\lambda}\right\}dxd\lambda =$$
$$= \frac{1}{2^\alpha(b-a)}\int_a^b \left\{L(f(x, c), f(x, d)) + \right.$$
$$\left. + m(2^\alpha - 1)L\left(f\left(\frac{x}{m}, c\right), f\left(\frac{x}{m}, d\right)\right)\right\}dx \le$$
$$\le \frac{1}{2^{\alpha+1}(b-a)}\int_a^b G(x, c, d)dx \le$$
$$\le \frac{1}{2^{\alpha+1}}\int_0^1 \left[t^\alpha G(a, c, d) + m(1-t^\alpha)G\left(\frac{b}{m}, c, d\right)\right]dt =$$
$$= \frac{1}{2^{\alpha+1}(\alpha+1)}\left[G(a, c, d) + m\alpha G\left(\frac{b}{m}, c, d\right)\right]$$

and

$$\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)dxdy =$$
$$= \frac{1}{d-c}\int_c^d\int_0^1 f(ta+(1-t)b,y)dtdy \le$$
$$\le \frac{1}{d-c}\int_c^d\int_0^1 \left[t^\alpha f(a,y)+m(1-t^\alpha)f\left(\frac{b}{m},y\right)\right]dtdy =$$
$$= \frac{1}{(\alpha+1)(d-c)}\int_c^d \left[f(a,y)+m\alpha f\left(\frac{b}{m},y\right)\right]dy \le$$
$$\le \frac{1}{2^\alpha(\alpha+1)}\int_0^1 \Bigg\{[f(a,c)]^\lambda [f(a,d)]^{1-\lambda} +$$
$$+ m(2^\alpha-1)\left[f\left(\frac{a}{m},c\right)\right]^\lambda \left[f\left(\frac{a}{m},d\right)\right]^{1-\lambda} +$$
$$+ m\alpha\Bigg\{\left[f\left(\frac{b}{m},c\right)\right]^\lambda \left[f\left(\frac{b}{m},d\right)\right]^{1-\lambda} +$$
$$+ m(2^\alpha-1)\left[f\left(\frac{b}{m^2},c\right)\right]^\lambda \left[f\left(\frac{b}{m^2},d\right)\right]^{1-\lambda}\Bigg\}\Bigg\}d\lambda =$$
$$= \frac{1}{2^\alpha(\alpha+1)}\Bigg\{L(f(a,c),f(a,d)) + m(2^\alpha-1)L\left(f\left(\frac{a}{m},c\right),f\left(\frac{a}{m},d\right)\right)+$$
$$+ m\alpha\Bigg\{L\left(f\left(\frac{b}{m},c\right),f\left(\frac{b}{m},d\right)\right)+$$
$$+ m(2^\alpha-1)L\left(f\left(\frac{b}{m^2},c\right),f\left(\frac{b}{m^2},d\right)\right)\Bigg\}\Bigg\} \le$$
$$\le \frac{1}{2^{\alpha+1}(\alpha+1)}\left[G(a,c,d)+m\alpha G\left(\frac{b}{m},c,d\right)\right].$$

Combining the above inequalities results in Theorem 9. □

Corollary 5. *Under the conditions of Theorems 8 and 9, if $\alpha = m = 1$, we have*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \le \frac{1}{2}\Bigg\{\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+$$

$$+\frac{1}{d-c}\int_c^d\left[f\left(\frac{a+b}{2},y\right)f\left(\frac{a+b}{2},c+d-y\right)\right]^{1/2}dy\Bigg\}\le$$

$$\le\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\le$$

$$\le\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)dxdy\le$$

$$\le\frac{1}{4}\left[\frac{1}{b-a}\int_a^b 2L(f(x,c),f(x,d))dx+\frac{1}{d-c}\int_c^d\big[f(a,y)+f(b,y)\big]dy\right]\le$$

$$\le\frac{1}{4}\left[\frac{1}{b-a}\int_a^b\big[f(x,c)+f(x,d)\big]dx+\frac{1}{d-c}\int_c^d\big[f(a,y)+f(b,y)\big]dy\right]\le$$

$$\le\frac{1}{8}\big[f(a,c)+f(a,d)+f(b,c)+f(b,d)+$$

$$+2L(f(a,c),f(a,d))+2L(f(b,c),f(b,d))\big]\le$$

$$\le\frac{1}{4}\big[f(a,c)+f(a,d)+f(b,c)+f(b,d)\big].$$

Acknowledgment. This work was partially supported by the National Natural Science Foundation of China under Grant no. 11361038, by the Foundation of the Research Program of Science and Technology at Universities of Inner Mongolia Autonomous Region under Grant no. NJZY14192, and by the Inner Mongolia Autonomous Region Natural Science Foundation Project under Grant no. 2015MS0123, China. The authors appreciate anonymous referees for their careful corrections to and valuable comments on the original version of this paper.

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Received June 23, 2015.
In revised form, October 26, 2015.

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