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ON THE INEQUALITIES FOR THE VOLUME OF THE UNIT BALL Ω_n IN $\mathbb{R}^n$

Abstract. The inequalities about the volume of the unit ball Ω_n in $\mathbb{R}^n$ were studies by several authors, especially Horst Alzer has a great contribution to this topic. Thereafter many authors produced numerous papers on this topic. Motivated by the work of the several authors, we make a contribution to the topic by giving the new inequalities about the volume of the unit ball Ω_n. Our inequalities refine the recent results existing in the literature.

Key words: *inequalities, gamma function, psi function, volume of the unit ball*

2010 Mathematical Subject Classification: *26D07, 33B15*

1. Introduction. Let $\Omega_n = \pi^{n/2}/\Gamma(n/2+1)$ be the volume of the unit ball in $\mathbb{R}^n$, $n = 1, 2, \ldots$, where Γ is the Euler gamma function.

Since the last few decades, several authors got interested to study the volume of the unit ball Ω_n, and presented interesting monotonicity properties of it. The sequence Ω_n is not monotone itself, it attains its maximum at $n = 5$, see [1, p.264]. In [2], Anderson et al. proved that $\Omega_n^{1/n}$, $n = 1, 2, \ldots$ is strictly decreasing with $\lim\limits_{x\to\infty} \Omega_n^{1/n} = 0$. In [3], Anderson and Qiu showed that $\Omega_n^{1/(n\log n)}$, $n = 2, 3, \ldots$ is strictly decreasing with $\lim_{x\to\infty} \Omega_n^{1/(n\log n)} = e^{-1/2}$. Klain and Rota [4] proved that $n\Omega_n/\Omega_{n-1}$, $1, 2, \ldots$ is strictly increasing.

Below we recall some inequalities which are the consequences of some monotonicity theorems. In [2, 4], authors proved that

$$\Omega_{n+1}^{n/(n+1)} < \Omega_n, \quad n = 1, 2, \ldots, \tag{1}$$

and

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < 1 + \frac{1}{n}, \quad n = 1, 2, \ldots. \tag{2}$$

In [5], Alzer proved that for $n = 1, 2, \ldots$

$$\left(1+\frac{1}{n}\right)^{a_1} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1+\frac{1}{n}\right)^{b_1}, \tag{3}$$

$$a_2\Omega_{n+1}^{n/(n+1)} < \Omega_n < b_2\Omega_{n+1}^{n/(n+1)}, \tag{4}$$

with the best possible constants

$$a_1 = 2 - (\log\pi)/\log 2 = 0.3485\ldots, \quad b_1 = 1/2,$$

and

$$a_2 = 2/\sqrt{\pi} = 1.1283\ldots, \quad b_2 = \sqrt{e} = 1.6487\ldots.$$

Inequalities (3) and (4) were slightly improved by Mortici [6] for $n \geq 4$ as follows:

$$\left(1+\frac{1}{n}\right)^{1/2-1/(4n)} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1+\frac{1}{n}\right)^{1/2}, \tag{5}$$

$$\frac{k}{(2\pi)^{1/(2n)}} \leq \frac{\Omega_n}{\Omega_{n+1}^{n/(n+1)}} < \frac{\sqrt{e}}{(2\pi)^{1/(2n)}}, \tag{6}$$

where $k = 1.5714\ldots$, the first inequality in (6) turns to equality if and only if $n = 11$.

Recently, Yin [7] proved that for $n = 1, 2, \ldots$

$$\frac{(n+1)(n+1/6)}{(n+b)^2} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \frac{(n+1)(n+b/2)}{(n+1/3)^2}, \tag{7}$$

$$\frac{\sqrt{e}}{\sqrt[2n+2]{2\pi}}\frac{\left(\sqrt{n+4/3}\right)^{(2n+1)/(n+1)}}{\sqrt{(n+1)(n+1+b/2)}} < \frac{\Omega_n}{\Omega_{n+1}^{n/(n+1)}} < \tag{8}$$

$$< \frac{\sqrt{e}}{\sqrt[2n+2]{2\pi}}\frac{\left(\sqrt{n+1+b}\right)^{(2n+1)/(n+1)}}{\sqrt{(n+1)(n+7/6)}},$$

where $b = \sqrt[3]{391/30} - 2 = 0.3533\ldots$

For the detailed study of volume of the unit ball $\Omega_n \in \mathbb{R}^n$ and its related inequalities, we refer to [8–14].

Our main results are as follows:

Theorem 1. *For $n = 1, 2, \ldots$ and $\beta = 1/1620$, we have*

$$Bl(n) < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < Bu(n),$$

where

$$Bl(n) = \frac{(1+n)\left(\cosh(1/n)\right)^{-n}}{n(1+32\beta/n^5)^4},$$

and

$$Bu(n) = \frac{(1+n)\left(\cosh(1/n)\right)^{-n}}{n^{11}(\beta+n^5)^{-2}}.$$

Theorem 2. *For $n = 1, 2, \ldots$ and $\beta = 1/1620$, we have*

$$B_l(n) < \frac{\Omega_n}{\Omega_{n+1}^{n/(n+1)}} < B_u(n),$$

where

$$B_l(n) = \frac{2^{(n+2n^2)/(4m)} e^{n/(2m)} m^{n/m} \pi^{-1/(2m)} n^{(18+35n)/(4m)}}{(\sinh(1/n))^{n/(4m)}((32\beta+n^5)(2\cosh(1/n))^{n(2n+1)/(4m)}}$$

and

$$B_u(n) = \frac{2^{(n+2n^2)/(4m)} e^{n/(2m)} m^{n/m} \pi^{-1/(2m)} n^{-(2+25n)/(4m)} (\beta+n^5)^{n/m}}{(\sinh(1/n))^{n/(4m)}(2\cosh(1/n))^{n(2n+1)/(4m)}},$$

with $m = 1 + n$.

2. Preliminaries and proofs. We recall the following Legendre duplication formula

$$\Gamma(2z) = \pi^{-1/2} 2^{2z-1} \Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \quad 2z \neq 0, -1, -2, \ldots, \tag{9}$$

see [15, 5.5.5].

Lemma 1. [16] *For all $x > 0$,*

$$C(\alpha, x) < \Gamma(1+x) < C(\beta, x),$$

where

$$C(a,x) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh\frac{1}{x}\right)^{x/2} \left(1 + \frac{a}{x^5}\right)$$

with the best possible constants $\alpha = 0$ and $\beta = 1/1620$.

Lemma 2. *For $n \geq 1$, the following identities hold true*

$$\frac{\Gamma\left((n+1)/2\right)\Gamma\left((n+3)/2\right)}{\Gamma\left((n+2)/2\right)^2} = \frac{\pi(n+1)(n!)^2}{2^{2n+1}\Gamma\left(\frac{n}{2}+1\right)^4}, \tag{10}$$

$$\frac{\Gamma((n+3)/2)^{n/(n+1)}}{\Gamma(n/2+1)} = \frac{2^{n+1}\Gamma((n+1)/2+1)^{n/(n+1)}}{\sqrt{\pi}(n+1)!}. \tag{11}$$

Proof. Letting $z = (n+1)/2$ and $z = (n+3)/2$ in (9) we get

$$\Gamma\left(\frac{n+1}{2}\right) = \frac{\sqrt{\pi}}{2^n}\frac{\Gamma(n+1)}{\Gamma((n+2)/2)} \tag{12}$$

and

$$\Gamma\left(\frac{n+3}{2}\right) = \frac{\sqrt{\pi}}{2^{2+n}}\frac{\Gamma(n+3)}{\Gamma((n+4)/2)}, \tag{13}$$

respectively. Now using (12) and (13) we get

$$\begin{aligned}\frac{\Gamma\left((n+1)/2\right)\Gamma\left((n+3)/2\right)}{\Gamma\left((n+2)/2\right)^2} &= \frac{\pi}{2^{2n+2}}\frac{\Gamma(n+1)\Gamma(n+3)}{\Gamma((n+2)/2)^3\Gamma((n+4)/2)} = \\ &= \frac{\pi}{2^{2n+2}}\frac{(n+2)!n!}{(n+2)\Gamma((n+4)/2)^4} = \\ &= \frac{\pi}{2^{2n+2}}\frac{(n+1)(n!)^2}{\Gamma((n+4)/2)^4}.\end{aligned}$$

The proof of (11) follows similarly. □

Proof of Theorem 1. By definition and from (10), we get

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} = \frac{\Gamma\left((n+1)/2\right)\Gamma\left((n+3)/2\right)}{\Gamma\left((n+2)/2\right)^2} = \frac{\pi(n+1)(n!)^2}{2^{2n+1}\Gamma\left(\frac{n}{2}+1\right)^4}.$$

Now applying Lemma 1, we get

$$\frac{\pi(n+1)}{2^{2n+1}}\frac{C(\alpha,n)^2}{C(\beta,n/2)} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \frac{\pi(n+1)}{2^{2n+1}}\frac{C(\beta,n)^2}{C(\alpha,n/2)}.$$

After simple calculation and letting $\alpha = 0$ and $\beta = 1/1620$ we arrive at the proof. □

Proof of Theorem 2. It is easy to see that

$$G(n)=\frac{\Omega_n}{\Omega_{n+1}^{n/(n+1)}}=\left(\frac{n+1}{2}\right)^{\frac{n}{n+1}}\left(\frac{\sqrt{\pi}}{2^n}\right)^{\frac{n}{n+1}}\frac{\Gamma(n+1)^{n/(n+1)}}{\Gamma(n/2+1)^{(2n+1)/(n+1)}}.$$

Now by utilizing Lemma 1, we get

$$\frac{\xi_0\left(e^{-n}n^{n+1/2}\left(n\sinh\left(\frac{1}{n}\right)\right)^{n/2}\right)^{n/(n+1)}}{\left(2^{-3n/4}e^{-n/2}n^{(n-9)/2}\left(32b+n^5\right)\left(n\sinh\left(\frac{2}{n}\right)\right)^{n/4}\right)^{(2n+1)/(n+1)}}<G(n)<$$

$$<\frac{\xi_0\left(e^{-n}n^{n-9/2}\left(b+n^5\right)\left(n\sinh\left(\frac{1}{n}\right)\right)^{n/2}\right)^{n/(n+1)}}{\left(2^{-3n/4}e^{-n/2}n^{(n+1)/2}\left(n\sinh\left(\frac{2}{nx}\right)\right)^{n/4}\right)^{(2n+1)/(n+1)}},$$

where

$$\xi_0=\frac{2^{-(n(2n+1))/(2(n+1))}(n+1)^{n/(n+1)}}{\pi^{1/(2n+2)}},$$

and this completes the proof. □

In [17, Theorem 3.1], Alzer proved that for $n=1,2,\ldots$

$$c_1(n)=a\frac{(2\pi e)^{n/2}}{n^{(n-1)/2}}<(n+1)\Omega_n-n\Omega_{n+1}<b\frac{(2\pi e)^{n/2}}{n^{(n-1)/2}}=c_2(n), \quad (14)$$

with the best possible constant

$$a=\frac{1}{e}\left(4-\frac{9\pi}{8}\right)\left(\frac{2}{\pi e}\right)^{1/2}=0.0829\ldots \quad \text{and} \quad b=\frac{1}{\sqrt{\pi}}=0.5641\ldots$$

Our next theorem refines the inequalities given in (14), see Figure 1.

Theorem 3. *For $n=1,2,\ldots$ and $\beta=1/1620$, the following relations hold true*

$$b_1(n)<(n+1)\Omega_n-n\Omega_{n+1}<b_2(n),$$

where

$$b_1(n)=\frac{2^{3n/4}n^{-3n/4}(e\pi)^{n/2}(\sinh(2/n))^{-n/4}}{(n+1)\left(32\beta+n^5\right)}\xi,$$

$$\xi = \left(\frac{n^{9/2}(n+1)^2}{\sqrt{\pi}} - \frac{\sqrt{2}\left(32\beta + n^5\right)^2 (\cosh(1/n))^{n/2}}{n^4} \right),$$

and

$$b_2(n) = \frac{2^{3n/4} n^{-3n/4} (e\pi)^{n/2} (\sinh(2/n))^{-n/4}}{n+1} \times$$
$$\times \left(\frac{(n+1)^2}{\sqrt{n\pi}} - \frac{\sqrt{2} n^6 (\cosh(1/n))^{n/2}}{n^5 + \beta} \right).$$

Proof. Clearly,

$$(n+1)\Omega_n - n\Omega_{n+1} = \pi^{n/2} \left(\frac{2(n+1)}{n\Gamma(n/2)} - \frac{2^n n \Gamma(n/2)}{(n+1)\Gamma(n)} \right).$$

From Lemma 1, we get

$$\frac{2^{3n/4} e^{n/2} n^{(9-n)/2} (n+1) \left(n \sinh(2/n)\right)^{-n/4}}{\sqrt{\pi}\left(32\beta + n^5\right)} <$$
$$< \frac{n+1}{\Gamma(n/2+1)} <$$
$$< \frac{2^{3n/4} e^{n/2} n^{-(n+1)/2} (n+1) \left(n \sinh(2/n)\right)^{-n/4}}{\sqrt{\pi}},$$

and

$$\frac{2^{(n+2)/4} e^{n/2} n^{6-n/2} \left(n \sinh(1/n)\right)^{-n/2} \left(n \sinh(2/n)\right)^{n/4}}{(n+1)\left(\beta + n^5\right)} <$$
$$< \frac{2^{n+1} n \Gamma(n/2+1)}{(n+1)\Gamma(n+1)} <$$
$$< \frac{2^{(n+2)/4} e^{n/2} n^{-n/2-4} \left(32\beta + n^5\right) \left(n \sinh(1/n)\right)^{-n/2} \left(n \sinh(2/n)\right)^{n/4}}{n+1}.$$

Now by combining the above relations we get the proof. □

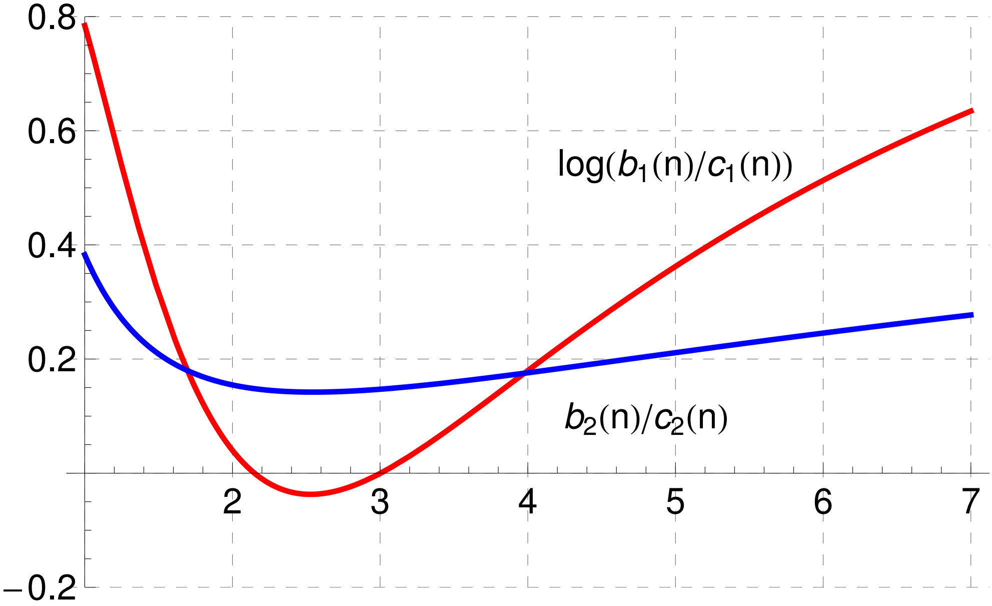

Figure 1: In this picture we plot $\log(b_1(n)/c_1(n))$ and b_2/c_2

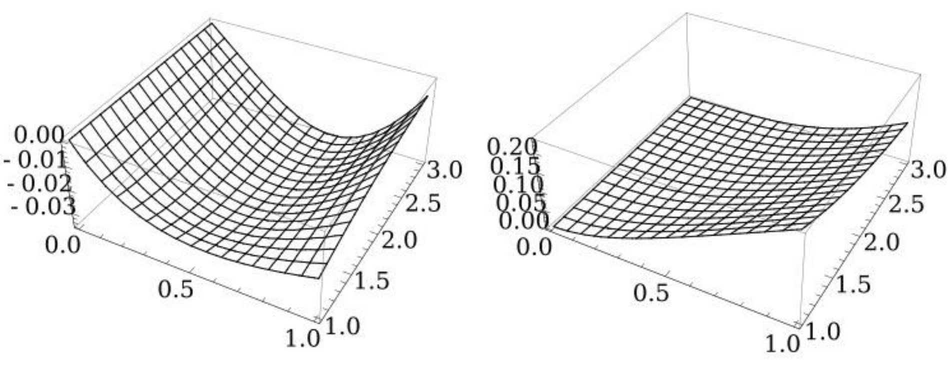

Figure 2: In this picture we plot $\log(b_1(n)/c_1(n))$ and b_2/c_2

3. Graphical and numerical comparison. We denote the lower and upper bounds of $F(n) = \Omega_n^2/(\Omega_{n-1}\Omega_{n+1})$ given in (3), (5), (7) by $Al(n)$, $Au(n)$, $Ml(n)$, $Mu(n)$, and $Yl(n)$, $Yu(n)$ respectively. In the following table we compare the above lower and upper bounds numerically. Recall that the inequalities in (5) are valid for $n \geq 4$.

n	$Ml(n)$	$Al(n)$	$Yl(n)$	$Bl(n)$	$F(n)$	$Bu(n)$	$Yu(n)$	$Au(n)$
1	–	1.2732	1.2740	1.1986	1.2732	1.2977	1.3238	1.4142
2	–	1.1518	1.1737	1.1768	1.1781	1.1797	1.1994	1.2247
3	–	1.1055	1.1264	1.1317	1.1318	1.1320	1.1436	1.1547
4	–	1.0809	1.0993	1.1044	1.1045	1.1045	1.1121	1.1180
5	1.0855	1.0656	1.0817	1.0865	1.0865	1.0865	1.0920	1.0954
6	1.0732	1.0552	1.0694	1.0738	1.0738	1.0738	1.0779	1.0801
7	1.0640	1.0476	1.0603	1.0643	1.0643	1.0643	1.0676	1.0690
8	1.0568	1.0419	1.0533	1.0570	1.0570	1.0570	1.0597	1.0607
9	1.0510	1.0374	1.0478	1.0512	1.0512	1.0512	1.0534	1.0541
10	1.0463	1.0338	1.0433	1.0464	1.0464	1.0464	1.0484	1.0488
11	1.0424	1.0308	1.0396	1.0425	1.0425	1.0425	1.0442	1.0445
12	1.0391	1.0283	1.0364	1.0392	1.0392	1.0392	1.0407	1.0408
13	1.0363	1.0262	1.0338	1.0363	1.0363	1.0363	1.0377	1.0377
14	1.0338	1.0243	1.0315	1.0339	1.0339	1.0339	1.0351	1.0351
15	1.0317	1.0227	1.0294	1.0317	1.0317	1.0317	1.0328	1.0328

It is clear from the above table that the bounds given in Theorem 1 refine the other bounds. We also see that the first inequality in (7) is not valid for $n = 1$, so we compare our bound Bl with Yl for $n = 2, 3, \ldots$, see Figure 2. Graphical and numerical comparison are made with the help of Mathematica® Software [18].

On the basis of the computer experiments we come up to the following conjecture, which refines the inequalities given in [19, 20].

Conjecture. *The function*

$$f(x) = \frac{2}{x}\left(\left(\frac{\sinh(x)}{x}\right)^{1/x} - 1\right)$$

is increasing from $(0, 1)$ *onto* $(1/3, c)$. *In particular,*

$$\frac{x(6+x)^x}{6^x} \leq \sinh(x) \leq \frac{x(2+cx)^x}{2^x},$$

where $c = 2(\sinh(1) - 1) = 0.3504\ldots$

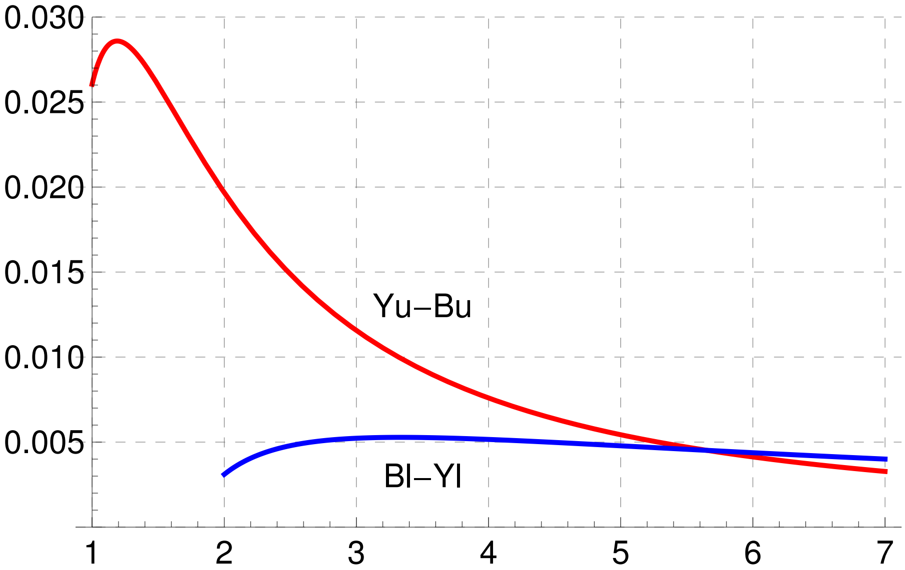

Figure 3: Here we plot $Yu(n) - Bu(n)$ and $Bl(n) - Yl(n)$, and see that the estimations given in Theorem 1 are better than the (7)

Again we denote the lower and upper bounds of $G(n) = \Omega_n/\Omega_{n+1}^{n/(n+1)}$ given in (4), (6), (8) by A_l, A_u, $M_l(n)$, $M_u(n)$ and $Y_l(n)$, $Y_u(n)$, respectively. The numerical comparison of these bounds is given in the following table. We do not include the comparison of $Y_n(n)$ in the following table, because it is an invalid bound.

n	A_l	$Y_l(n)$	$M_1(n)$	$B_l(n)$	$G(n)$	$B_u(n)$	$M_u(n)$	A_u
1	1.1284	0.9423	–	1.1032	1.1284	1.1364	–	1.6487
2	1.1284	1.0723	–	1.2084	1.2090	1.2097	–	1.6487
3	1.1284	1.1565	–	1.2651	1.2651	1.2653	–	1.6487
4	1.1284	1.2165	–	1.3069	1.3069	1.3069	–	1.6487
5	1.1284	1.2619	1.3076	1.3393	1.3393	1.3393	1.3719	1.6487
6	1.1284	1.2975	1.3482	1.3654	1.3654	1.3654	1.4146	1.6487
7	1.1284	1.3265	1.3781	1.3869	1.3869	1.3869	1.4459	1.64872
8	1.1284	1.3505	1.4009	1.4049	1.4049	1.4049	1.4698	1.6487
9	1.1284	1.3708	1.4189	1.4204	1.4204	1.4204	1.4887	1.6487
10	1.1284	1.3882	1.4334	1.4337	1.4337	1.4337	1.5040	1.6487
11	1.1284	1.4033	1.4455	1.4454	1.4454	1.4454	1.5166	1.6487
12	1.1284	1.4166	1.4556	1.4558	1.4558	1.4558	1.5272	1.6487
13	1.1284	1.4284	1.4642	1.4650	1.4650	1.4650	1.5362	1.6487
14	1.1284	1.4389	1.4716	1.4733	1.4733	1.4733	1.5440	1.6487
15	1.1284	1.4483	1.4780	1.4807	1.4807	1.4807	1.5507	1.6487

References

[1] Böhm J. and Hertel E. *Polyedergeometrie in n-dimensionalen Rümen konstanter Krümmung.* Birkhäuser, Basel, 1981.

[2] Anderson G. D., Vamanamurthy M. K. and Vuorinen M. *Special functions of quasiconformal theory.* Exposition. Math., 1989, vol. 7, pp. 97–136.

[3] Anderson G. D. and Qiu S.-L. *A monotoneity property of the gamma function.* Proc. Amer. Soc., 1997, vol. 125, pp. 3355–3362.

[4] Klain D. A. and Rota G.-C. *A continuous analogue of Sperner's theorem.* Comm. Pure Appl. Math., 1997, vol. 50, pp. 205–223.

[5] Alzer H. *Inequalities for the volume of the unit ball in* $\mathbb{R}^n$. J. Math. Anal. Appl., 2000, vol. 252, pp. 353–363.

[6] Mortici C. *Monotonicity properties of the volume of the unit ball in* $\mathbb{R}^n$. Optim. Lett., 2010, vol. 4, pp. 457–464.

[7] Yin L. *Several inequalities for the volume of the unit ball in* $\mathbb{R}^n$. Bull. Malays. Math. Sci. Soc., 2014, vol. 37, no. 4, pp. 1177–1183.

[8] Abramowitz M. and Stegun I., eds. *Handbook of mathematical functions with formulas, graphs and mathematical tables.* National Bureau of Standards, Dover, New York, 1965.

[9] Anderson G. D., Barnard R. W., Richards K. C., Vamanamurthy M. K. and Vuorinen M. *Inequalities for zero–balanced hypergeometric functions.* Trans. Amer. Math. Soc., 1995, vol. 347, pp. 1713–1723.

[10] Anderson G. D., Qiu S.-L., Vamanamurthy M. K. and Vuorinen M. *Generalized elliptic integrals and modular equation.* Pacific J. Math., 2000, vol. 192, no. 1, pp. 1–37.

[11] Anderson G. D., Vamanamurthy M. K. and Vuorinen M. *Conformal invariants, inequalities and quasiconformal maps.* J. Wiley, 1997, 505 pp.

[12] Alzer H. *On some inequalities for the gamma and psi functions.* Math. Comput., 1997, vol. 66, pp. 373–389.

[13] Alzer H. *Sharp inequalities for the digamma and polygamma functions.* Forum Math., 2004, vol. 16, pp. 181–221.

[14] Bhayo B. A., Sándor J. *Inequalities connecting generalized trigonometric functions with their inverses.* Probl. Anal. Issues Anal., 2013, vol. 2 (20), no. 2, pp. 82–90. DOI: 10.15393/j3.art.2013.2385.

[15] Olver F. W. J., Lozier D. W., Boisvert R. F., and Clark C. W., eds. *NIST handbook of mathematical functions.* Cambridge University Press, Cambridge, 2010.

[16] Alzer H. *Sharp upper and lower bounds for the gamma function.* Proc. Royal Soc. Edinburg, 2009, vol. 139, iss. 04, pp. 709–718.

[17] Alzer H. *Inequalities for the volume of the unit ball in* $\mathbb{R}^n$ *II.* Mediterr. J. Math., 2008, vol. 5, pp. 395–413.

[18] Ruskeepää H. *Mathematica® Navigator.* 3rd ed. Academic Press, 2009.

[19] Klén R., Visuri M., and Vuorinen M. *On Jordan type inequalities for hyperbolic functions.* J. Ineq. Appl., vol. 2010, 14 pp. DOI: 10.1155/2010/362548.

[20] Zhu L. *New inequalities of Shafer-Fink Type for arc hyperbolic sine.* J. Inequal. Appl., 2008, Art. ID 368275, 5 pp. DOI: 10.1155/2008/368275.

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