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ON INEQUALITIES RELATED TO SOME QUASI-CONVEX FUNCTIONS

Abstract. Estimations of errors in inequalities related to some quasi-convex functions in literature are simplified. Two new general inequalities for functions whose n-th derivatives for any positive integer n in absolute values are quasi-convex have been established. Some special cases are discussed with applications in numerical integration and special means.

Key words: *inequalities, quasi-convex function, Simpson type rule, numerical integration, special means*

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1. Introduction. It is well known that a function $f : [a, b] \to \mathbb{R}$ is called quasi-convex on $[a, b]$ if

$$f(\lambda x + (1-\lambda) y) \le \max\{f(x), f(y)\}$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$ (e.g., see [1] and [2]). Thus we see clearly that if $f : [a, b] \to \mathbb{R}$ is quasi-convex on $[a, b]$ then for any $t \in [a, b]$ we have

$$f(t) \le \max\{f(a), f(b)\}.$$

It should be noticed that any convex function is a quasi-convex function and there exist quasi-convex functions which are neither convex nor continuous (e.g., see [3] and [4]).

Along this paper, we consider a real interval $I \subset \mathbb{R}$, and denote that I° is the interior of I.

In [5–7], we see the following three inequalities for quasi-convex functions.

Theorem A. [7, Theorem 6] *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and $f' \in L^1[a,b]$. If $|f'|^q$ is quasi-convex on $[a,b]$ and $q \geq 1$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \leq \frac{5(b-a)}{36}\left(\max\{|f'(a)|^q, |f'(b)|^q\}\right)^{\frac{1}{q}}. \tag{1}$$

Theorem B. [5, Theorem 4] *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and $f'' \in L^1[a,b]$. If $|f''|^{\frac{p}{p-1}}$ is quasi-convex on $[a,b]$, for $p > 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \leq \frac{(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{\frac{1}{p}}\left(\max\{|f''(a)|^q, |f''(b)|^q\}\right)^{\frac{1}{q}}, \tag{2}$$

where $q = \dfrac{p}{p-1}$.

Theorem C. [6, Theorem 3] *Let $f'' : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function on I° such that $f''' \in L^1[a,b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'''|^q, q = \frac{p}{p-1}$ is quasi-convex on $[a,b]$, for some fixed $p > 1$, then the following inequality holds:*

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{6}\left[f(a) + 4\left(\frac{f(a)+f(b)}{2}\right) + f(b)\right]\right| \leq \leq \frac{2^{-\frac{1}{p}}(b-a)^4}{48}\left(B(p+1, 2p+1)\right)^{\frac{1}{p}}\left[\left(\max\{|f'''(a)|^q, \left|f'''\left(\frac{a+b}{2}\right)\right|^q\}\right)^{\frac{1}{q}} + \left(\max\{|f'''(b)|^q, \left|f'''\left(\frac{a+b}{2}\right)\right|^q\}\right)^{\frac{1}{q}}\right]. \tag{3}$$

It should be noticed that

$$\left(\max\{|f'(a)|^q, |f'(b)|^q\}\right)^{\frac{1}{q}} = \max\{|f'(a)|, |f'(b)|\},$$

which has been overlooked in the literature (see e.g., [3–13]). The inequalities (1), (2) and (3) have a uniform bound independent of q. Indeed, for any $q > 0$ and positive integer n, $|f^{(n)}|^q$ is quasi-convex on $[a,b]$ if and only if $|f^{(n)}|$ is quasi-convex on $[a,b]$. Thus, instead of Theorem A, Theorem B and Theorem C, we actually just have the following three theorems as:

Theorem 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and $f' \in L^1[a,b]$. If $|f'|$ is quasi-convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}&\left|\tfrac{1}{6}\left[f(a)+4f\left(\tfrac{a+b}{2}\right)+f(b)\right]-\tfrac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\\&\leq\tfrac{5(b-a)}{36}\max\{|f'(a)|,|f'(b)|\}.\end{aligned}\tag{4}$$

Theorem 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and $f'' \in L^1[a,b]$. If $|f''|$ is quasi-convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{(b-a)^2}{12}\max\{|f''(a)|,|f''(b)|\}.\tag{5}$$

Theorem 3. *Let $f'' : I \subset \mathbb{R} \to \mathbf{R}$ be an absolutely continuous function on I° such that $f''' \in L^1[a,b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'''|$ is quasi-convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}&\left|\int_a^b f(t)\,dt-\tfrac{b-a}{6}\left[f(a)+4\left(\tfrac{f(a)+f(b)}{2}\right)+f(b)\right]\right|\leq\\&\leq\tfrac{(b-a)^4}{1152}\left[\max\{|f'''(a)|,|f'''\left(\tfrac{a+b}{2}\right)|\}+\max\{|f'''\left(\tfrac{a+b}{2}\right)|,|f'''(b)|\}\right].\end{aligned}\tag{6}$$

In this work, we will derive two new general inequalities for functions whose nth derivatives for any positive integer n in absolute values are quasi-convex, which provide some generalizations of the above three inequalities and some other interesting inequalities as special cases. Some applications in numerical integration and to special means are also given.

2. The Results.

Lemma. (see [14]) *Let $f : [a,b] \to \mathbb{R}$ be such that the $(n-1)$th derivative $f^{(n-1)}$ $(n \geq 1)$ is absolutely continuous on $[a,b]$ and $f^{(n)} \in L^1[a,b]$. Then* *we have the identity*

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\Big[\theta f(a) + 2(1-\theta) f\left(\tfrac{a+b}{2}\right) + \theta f(b)\Big] + \\ + \sum_{k=1}^{\frac{n-1}{2}} \frac{[1-(2k+1)\theta](b-a)^{2k+1}}{(2k+1)!2^{2k}} f^{(2k)}\left(\frac{a+b}{2}\right) + \\ + (-1)^n \int_a^b K_n(x,\theta) f^{(n)}(x)\,dx, \tag{7}$$

where $\theta \in [0,1]$ *and*

$$K_n(x,\theta) := \begin{cases} \dfrac{(x-a)^n}{n!} - \dfrac{\theta(b-a)(x-a)^{n-1}}{2(n-1)!}, & \text{if } x \in [a, \frac{a+b}{2}], \\ \dfrac{(x-b)^n}{n!} + \dfrac{\theta(b-a)(x-b)^{n-1}}{2(n-1)!}, & \text{if } x \in (\frac{a+b}{2}, b]. \end{cases} \tag{8}$$

Theorem 4. *Let* $f : [a,b] \to \mathbb{R}$ *be such that the* $(n-1)$*th derivative* $f^{(n-1)}$ $(n \geq 1)$ *is absolutely continuous on* $[a,b]$ *and* $f^{(n)} \in L^1[a,b]$. *If* $|f^{(n)}|$ *is quasi-convex on* $[a,b]$, *then we have*

$$\Big| \int_a^b f(x)\,dx - \frac{b-a}{2}[\theta f(a) + 2(1-\theta) f\left(\tfrac{a+b}{2}\right) + \theta f(b)] - \\ - \sum_{k=1}^{\frac{n-1}{2}} \frac{[1-(2k+1)\theta](b-a)^{2k+1}}{(2k+1)!\,2^{2k}} f^{(2k)}\left(\frac{a+b}{2}\right) \Big| \leq \\ \leq I(n,\theta)\max\{|f^{(n)}(a)|, |f^{(n)}(b)|\}, \tag{9}$$

where $\theta \in [0,1]$ *and*

$$I(n,\theta) = \begin{cases} \dfrac{[1-(n+1)\theta + 2n^n\theta^{n+1}](b-a)^{n+1}}{(n+1)!\,2^n}, & n < \frac{1}{\theta}, \\ \dfrac{[(n+1)\theta - 1](b-a)^{n+1}}{(n+1)!\,2^n}, & n \geq \frac{1}{\theta}. \end{cases} \tag{10}$$

Proof. From (7) of the Lemma, we have

$$\Big|\int_a^b f(x)\,dx - \tfrac{b-a}{2}\Big[\theta f(a) + 2(1-\theta) f\left(\tfrac{a+b}{2}\right) + \theta f(b)\Big] - \\ - \sum_{k=1}^{\frac{n-1}{2}} \frac{[1-(2k+1)\theta](b-a)^{2k+1}}{(2k+1)!\,2^{2k}} f^{(2k)}\left(\frac{a+b}{2}\right)\Big| = \\ = \Big|\int_a^b K_n(x,\theta) f^{(n)}(x)\,dx\Big| \le \int_a^b |K_n(x,\theta) f^{(n)}(x)|\,dx \le \\ \le \max\nolimits_{x\in[a,b]} |f^{(n)}(x)| \int_a^b |K_n(x,\theta)|\,dx \le \\ \le \max\{|f^{(n)}(a)|, |f^{(n)}(b)|\} \int_a^b |K_n(x,\theta)|\,dx. \tag{11}$$

By elementary calculus, it is not difficult to get the following results:

$$\int_a^b |K_n(x,\theta)|\,dx = \begin{cases} \dfrac{[1-(n+1)\theta + 2n^n\theta^{n+1}](b-a)^{n+1}}{(n+1)!\,2^n}, & n < \frac{1}{\theta}, \\ \dfrac{[(n+1)\theta - 1](b-a)^{n+1}}{(n+1)!\,2^n}, & n \ge \frac{1}{\theta}. \end{cases} \tag{12}$$

Consequently, the inequality (9) with (10) follows from (11) and (12). The proof is completed. □

Corollary 1. *Let $f:[a,b]\to\mathbb{R}$ be such that the $(n-1)$th derivative $f^{(n-1)}$ $(n \ge 1)$ is absolutely continuous on $[a,b]$ and $f^{(n)} \in L^1[a,b]$. If $|f^{(n)}|$ is quasi-convex on $[a,b]$, then we get a midpoint type inequality*

$$\Big|\int_a^b f(x)\,dx - \tfrac{b-a}{2} f\left(\tfrac{a+b}{2}\right) - \sum_{k=1}^{\frac{n-1}{2}} \frac{(b-a)^{2k+1}}{(2k+1)!\,2^{2k}} f^{(2k)}\left(\frac{a+b}{2}\right)\Big| \le \\ \le \frac{(b-a)^{n+1}}{(n+1)!\,2^n} \max\{|f^{(n)}(a)|, |f^{(n)}(b)|\},$$

a trapezoid type inequality

$$\Big|\int_a^b f(x)\,dx - \tfrac{b-a}{2}[f(a)+f(b)] + \sum_{k=1}^{\frac{n-1}{2}} \frac{k(b-a)^{2k+1}}{(2k+1)!\,2^{2k-1}} f^{(2k)}\left(\frac{a+b}{2}\right)\Big| \le \\ \le \frac{n(b-a)^{n+1}}{(n+1)!\,2^n} \max\{|f^{(n)}(a)|, |f^{(n)}(b)|\},$$

a Simpson type inequality

$$\left|\int_a^b f(x)\,dx-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]+\right.$$
$$\left.+\sum_{k=1}^{\frac{n-1}{2}}\frac{(k-1)(b-a)^{2k+1}}{3(2k+1)!\,2^{2k-1}}f^{(2k)}\left(\frac{a+b}{2}\right)\right|\leq$$

$$\leq I\left(n,\frac{1}{3}\right)\max\{|f^{(n)}(a)|,|f^{(n)}(b)|\},$$

where

$$I\left(n,\frac{1}{3}\right)=\begin{cases}\frac{5}{36}, & n=1,\\ \frac{1}{81}, & n=2,\\ \frac{(n-2)(b-a)^{n+1}}{3(n+1)!2^n}, & n\geq 3\end{cases}$$

and an averaged midpoint-trapezoid type inequality

$$\left|\int_a^b f(x)\,dx-\frac{b-a}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]+\right.$$
$$\left.+\sum_{k=1}^{\frac{n-1}{2}}\frac{(2k-1)(b-a)^{2k+1}}{(2k+1)!\,2^{2k+1}}f^{(2k)}\left(\frac{a+b}{2}\right)\right|\leq$$

$$\leq I\left(n,\frac{1}{2}\right)\max\{|f^{(n)}(a)|,|f^{(n)}(b)|\},$$

where

$$I\left(n,\frac{1}{2}\right)=\begin{cases}\frac{1}{48}, & n=1,\\ \frac{(n-1)(b-a)^{n+1}}{(n+1)!2^{n+1}}, & n\geq 2.\end{cases}$$

Proof. Set $\theta=0,1,\frac{1}{3},\frac{1}{2}$ in (9) and (10). □

Remark 1. *For $n=1$, we have*

$$\left|\int_a^b f(x)\,dx-\tfrac{b-a}{2}\left[\theta f(a)+2(1-\theta)f\left(\tfrac{a+b}{2}\right)+\theta f(b)\right]\right|\leq$$
$$\leq\frac{1-2\theta+2\theta^2}{4}(b-a)^2\max\{|f'(a)|,|f'(b)|\}. \tag{13}$$

If we take $\theta = 0, 1, \frac{1}{3}, \frac{1}{2}$ *in (13), then we get a midpoint inequality*

$$\left| \int_a^b f(x)\, dx - (b-a) f\left(\frac{a+b}{2}\right) \right| \le \frac{(b-a)^2}{4} \max\{|f^{(n)}(a)|, |f^{(n)}(b)|\},$$

a trapezoid inequality

$$\left| \int_a^b f(x)\, dx - \frac{b-a}{2}[f(a) + f(b)] \right| \le \frac{(b-a)^2}{4} \max\{|f'(a)|, |f'(b)|\},$$

a Simpson inequality

$$\begin{aligned} &\left| \int_a^b f(x)\, dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \right| \le \\ &\le \frac{5(b-a)^2}{36} \max\{|f'(a)|, |f'(b)|\} \end{aligned} \tag{14}$$

which recapture the inequality (4), and an averaged midpoint-trapezoid inequality

$$\begin{aligned} &\left| \int_a^b f(x)\, dx - \frac{b-a}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right] \right| \le \\ &\le \frac{(b-a)^2}{8} \max\{|f'(a)|, |f'(b)|\}. \end{aligned}$$

Remark 2. *For* $n = 2$*, we have*

$$\begin{aligned} &\left| \int_a^b f(x)\, dx - \tfrac{b-a}{2}\left[\theta f(a) + 2(1-\theta) f\left(\tfrac{a+b}{2}\right) + \theta f(b)\right] \right| \le \\ &\le I(2,\theta) \max\{|f''(a)|, |f''(b)|\}, \end{aligned} \tag{15}$$

where

$$I(2,\theta) = \begin{cases} \frac{[1-3\theta+8\theta^3](b-a)^3}{24}, & n < \frac{1}{2}, \\ \frac{[3\theta-1](b-a)^3}{24}, & n \ge \frac{1}{2}. \end{cases} \tag{16}$$

If we take $\theta = 0, 1, \frac{1}{3}, \frac{1}{2}$ *in (15) and (16), then we get a midpoint inequality*

$$\left| \int_a^b f(x)\, dx - (b-a) f\left(\frac{a+b}{2}\right) \right| \le \frac{(b-a)^3}{24} \max\{|f''(a)|, |f''(b)|\},$$

a trapezoid inequality

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{2}[f(a)+f(b)] \right| \le \frac{(b-a)^3}{12} \max\{|f''(a)|,|f''(b)|\},$$

which recapture the inequality (5), a Simpson inequality

$$\begin{aligned} \left| \int_a^b f(x)\,dx - \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \right| &\le \\ \le \frac{(b-a)^3}{81} \max\{|f''(a)|,|f''(b)|\} \end{aligned} \tag{17}$$

and an averaged midpoint-trapezoid inequality

$$\begin{aligned} \left| \int_a^b f(x)\,dx - \frac{b-a}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right] \right| &\le \\ \le \frac{(b-a)^2}{48} \max\{|f''(a)|,|f''(b)|\}. \end{aligned}$$

Remark 3. *For $n=3$, we have*

$$\begin{aligned} \Big| \int_a^b f(x)\,dx - \tfrac{b-a}{2}\left[\theta f(a) + 2(1-\theta) f\left(\tfrac{a+b}{2}\right) + \theta f(b)\right] - \\ - \frac{(1-3\theta)(b-a)^3}{24} f''\left(\frac{a+b}{2}\right) \Big| \le \\ \le I(3,\theta) \max\{|f'''(a)|,|f'''(b)|\}, \end{aligned} \tag{18}$$

where

$$I(3,\theta) = \begin{cases} \frac{[1-4\theta+54\theta^4](b-a)^4}{192}, & n < \frac{1}{2}, \\ \frac{[4\theta-1](b-a)^4}{192}, & n \ge \frac{1}{2}. \end{cases} \tag{19}$$

If we take $\theta = 0, 1, \dfrac{1}{3}, \dfrac{1}{2}$ in (18) and (19), then we get a midpoint type inequality

$$\begin{aligned} \left| \int_a^b f(x)\,dx - (b-a) f\left(\tfrac{a+b}{2}\right) - \tfrac{(b-a)^3}{24} f''\left(\tfrac{a+b}{2}\right) \right| \le \\ \le \frac{(b-a)^4}{192} \max\{|f'''(a)|,|f'''(b)|\}, \end{aligned}$$

a trapezoid type inequality

$$\left| \int_a^b f(x)\, dx - \tfrac{b-a}{2}[f(a)+f(b)] + \tfrac{(b-a)^3}{12} f''\left(\tfrac{a+b}{2}\right) \right| \leq \frac{(b-a)^4}{64} \max\{|f'''(a)|, |f'''(b)|\},$$

a Simpson inequality

$$\left| \int_a^b f(x)\, dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \right| \leq \leq \frac{(b-a)^4}{576} \max\{|f'''(a)|, |f'''(b)|\} \tag{20}$$

and a midpoint-trapezoid type inequality

$$\left| \int_a^b f(x)\, dx - \frac{b-a}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right] + + \frac{(b-a)^3}{48} f''\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^4}{192} \max\{|f'''(a)|, |f'''(b)|\}.$$

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be such that the $(n-1)$th derivative $f^{(n-1)}$ $(n \geq 1)$ is absolutely continuous on $[a,b]$ and $f^{(n)} \in L^1[a,b]$. If $|f^{(n)}|$ is quasi-convex on $[a,b]$, then we have*

$$\begin{aligned} &\left| \int_a^b f(x)\, dx - \tfrac{b-a}{2}\left[\theta f(a) + 2(1-\theta) f\left(\tfrac{a+b}{2}\right) + \theta f(b)\right] - \right. \\ &\left. - \sum_{k=1}^{\frac{n-1}{2}} \frac{[1-(2k+1)\theta](b-a)^{2k+1}}{(2k+1)!\, 2^{2k}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| \leq \\ &\leq \frac{I(n,\theta)}{2}\left[\max\left\{\left|f^{(n)}(a)\right|, \left|f^{(n)}\left(\frac{a+b}{2}\right)\right|\right\} + \right. \\ &\left. + \max\{|f^{(n)}\left(\tfrac{a+b}{2}\right)|, |f^{(n)}(b)|\}\right], \end{aligned} \tag{21}$$

where $\theta \in [0,1]$ and $I(n,\theta)$ is as in (10).

Proof. From (7) of the Lemma, we have

$$
\begin{aligned}
&\Big| \int_a^b f(x)\,dx - \tfrac{b-a}{2}\Big[\theta f(a) + 2(1-\theta) f\left(\tfrac{a+b}{2}\right) + \theta f(b)\Big] - \\
&\quad - \sum_{k=1}^{\frac{n-1}{2}} \frac{[1-(2k+1)\theta](b-a)^{2k+1}}{(2k+1)!\,2^{2k}} f^{(2k)}\left(\frac{a+b}{2}\right)\Big| = \\
&= \Big| \int_a^b K_n(x,\theta) f^{(n)}(x)\,dx \Big| \le \int_a^b |K_n(x,\theta) f^{(n)}(x)|\,dx = \\
&= \int_a^{\frac{a+b}{2}} |K_n(x,\theta) f^{(n)}(x)|\,dx + \int_{\frac{a+b}{2}}^b |K_n(x,\theta) f^{(n)}(x)|\,dx \le \\
&\le \max_{x\in[a,\frac{a+b}{2}]} |f^{(n)}(x)| \int_a^{\frac{a+b}{2}} |K_n(x,\theta)|\,dx + \\
&+ \max_{x\in[\frac{a+b}{2},b]} |f^{(n)}(x)| \int_{\frac{a+b}{2}}^b |K_n(x,\theta)|\,dx \le \\
&\le \max\{|f^{(n)}(a)|, |f^{(n)}\left(\tfrac{a+b}{2}\right)|\} \int_a^{\frac{a+b}{2}} |K_n(x,\theta)|\,dx + \\
&+ \max\{|f^{(n)}\left(\tfrac{a+b}{2}\right)|, |f^{(n)}(b)|\} \int_{\frac{a+b}{2}}^b |K_n(x,\theta)|\,dx,
\end{aligned} \tag{22}
$$

Observe that

$$
\int_a^{\frac{a+b}{2}} |K_n(x,\theta)|\,dx = \int_{\frac{a+b}{2}}^b |K_n(x,\theta)|\,dx = \frac{1}{2}\int_a^b |K_n(x,\theta)|\,dx = \frac{I(n,\theta)}{2},
$$

the inequality (21) follows from (22). The proof is completed. □

Corollary 2. *Let $f:[a,b]\to\mathbb{R}$ be such that the $(n-1)$th derivative $f^{(n-1)}\,(n\ge 1)$ is absolutely continuous on $[a,b]$ and $f^{(n)}\in L^1[a,b]$. If $|f^{(n)}|$ is quasi-convex on $[a,b]$, then we get a midpoint type inequality*

$$
\begin{aligned}
&\Big| \int_a^b f(x)\,dx - \tfrac{b-a}{2} f\left(\tfrac{a+b}{2}\right) - \sum_{k=1}^{\frac{n-1}{2}} \frac{(b-a)^{2k+1}}{(2k+1)!\;2^{2k}} f^{(2k)}\left(\frac{a+b}{2}\right)\Big| \le \\
&\le \frac{(b-a)^{n+1}}{(n+1)!\;2^{n+1}}\Big[\max\Big\{\Big|f^{(n)}(a)\Big|, \Big|f^{(n)}\left(\frac{a+b}{2}\right)\Big|\Big\} + \\
&\quad + \max\{|f^{(n)}\left(\tfrac{a+b}{2}\right)|, |f^{(n)}(b)|\}\Big],
\end{aligned}
$$

a trapezoid type inequality

$$\left|\int\limits_a^b f(x)\,dx - \tfrac{b-a}{2}[f(a)+f(b)] + \sum_{k=1}^{\frac{n-1}{2}} \frac{k(b-a)^{2k+1}}{(2k+1)!\,2^{2k-1}} f^{(2k)}\left(\frac{a+b}{2}\right)\right| \leq$$
$$\leq \frac{n(b-a)^{n+1}}{(n+1)!\,2^{n+1}}\Big[\max\left\{\left|f^{(n)}(a)\right|,\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|\right\}+ + \max\{|f^{(n)}\left(\tfrac{a+b}{2}\right)|,|f^{(n)}(b)|\}\Big],$$

a Simpson type inequality

$$\left|\int\limits_a^b f(x)\,dx - \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]+ + \sum_{k=1}^{\frac{n-1}{2}} \frac{(k-1)(b-a)^{2k+1}}{3(2k+1)!\,2^{2k-1}} f^{(2k)}\left(\frac{a+b}{2}\right)\right| \leq$$
$$\leq \frac{I\left(n,\frac{1}{3}\right)}{2}\Big[\max\left\{\left|f^{(n)}(a)\right|,\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|\right\}+ + \max\{|f^{(n)}\left(\tfrac{a+b}{2}\right)|,|f^{(n)}(b)|\}\Big],$$

where

$$I\left(n,\frac{1}{3}\right)=\begin{cases}\frac{5}{36}, & n=1,\\ \frac{1}{81}, & n=2,\\ \frac{(n-2)(b-a)^{n+1}}{3(n+1)!2^{n}}, & n\geq 3,\end{cases}$$

and an averaged midpoint-trapezoid type inequality

$$\left|\int\limits_a^b f(x)\,dx - \frac{b-a}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]+ + \sum_{k=1}^{\frac{n-1}{2}} \frac{(2k-1)(b-a)^{2k+1}}{(2k+1)!\,2^{2k+1}} f^{(2k)}\left(\frac{a+b}{2}\right)\right| \leq$$
$$\leq \frac{I\left(n,\frac{1}{2}\right)}{2}\Big[\max\left\{\left|f^{(n)}(a)\right|,\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|\right\}+ + \max\{|f^{(n)}\left(\tfrac{a+b}{2}\right)|,|f^{(n)}(b)|\}\Big],$$

where

$$I\left(n,\frac{1}{2}\right)=\begin{cases}\frac{1}{48}, & n=1,\\ \frac{(n-1)(b-a)^{n+1}}{(n+1)!2^{n+1}}, & n\geq 2.\end{cases}$$

Proof. Set $\theta = 0, 1, \frac{1}{3}, \frac{1}{2}$ in (21) and (10). □

Remark 4. *For $n = 1$, we have*

$$\begin{aligned} &\left| \int_a^b f(x)\,dx - \frac{b-a}{2}\left[\theta f(a) + 2(1-\theta) f\left(\frac{a+b}{2}\right) + \theta f(b)\right]\right| \leq \\ &\leq \frac{1-2\theta+2\theta^2}{8}(b-a)^2\left[\max\left\{|f'(a)|, \left|f'\left(\frac{a+b}{2}\right)\right|\right\}+ \right. \\ &\left. + \max\left\{\left|f'\left(\tfrac{a+b}{2}\right)\right|, |f'(b)|\right\}\right]. \end{aligned} \tag{23}$$

If we take $\theta = 0, 1, \frac{1}{3}, \frac{1}{2}$ in (23), then we get a midpoint inequality

$$\begin{aligned} &\left| \int_a^b f(x)\,dx - (b-a) f\left(\tfrac{a+b}{2}\right)\right| \leq \\ &\leq \frac{(b-a)^2}{8}\left[\max\left\{|f'(a)|, \left|f'\left(\frac{a+b}{2}\right)\right|\right\}+ \right. \\ &\left. + \max\left\{\left|f'\left(\tfrac{a+b}{2}\right)\right|, |f'(b)|\right\}\right], \end{aligned}$$

a trapezoid inequality

$$\begin{aligned} &\left| \int_a^b f(x)\,dx - \tfrac{b-a}{2}[f(a) + f(b)]\right| \leq \\ &\leq \frac{(b-a)^2}{8}\left[\max\left\{|f'(a)|, \left|f'\left(\frac{a+b}{2}\right)\right|\right\}+ \right. \\ &\left. + \max\left\{\left|f'\left(\tfrac{a+b}{2}\right)\right|, |f'(b)|\right\}\right], \end{aligned}$$

a Simpson inequality

$$\begin{aligned} &\left| \int_a^b f(x)\,dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \leq \\ &\leq \frac{5(b-a)^2}{72}\left[\max\left\{|f'(a)|, \left|f'\left(\frac{a+b}{2}\right)\right|\right\}+ \right. \\ &\left. + \max\left\{\left|f'\left(\tfrac{a+b}{2}\right)\right|, |f'(b)|\right\}\right] \end{aligned}$$

and an averaged midpoint-trapezoid inequality

$$\begin{aligned}&\left|\int\limits_a^b f\left(x\right)\,dx-\frac{b-a}{4}\left[f\left(a\right)+2f\left(\frac{a+b}{2}\right)+f\left(b\right)\right]\right|\le\\&\le\frac{\left(b-a\right)^2}{16}\left[\max\left\{\left|f'\left(a\right)\right|,\left|f'\left(\frac{a+b}{2}\right)\right|\right\}+\right.\\&\left.+\max\left\{\left|f'\left(\tfrac{a+b}{2}\right)\right|,\left|f'\left(b\right)\right|\right\}\right].\end{aligned}$$

Remark 5. *For $n=2$, we have*

$$\begin{aligned}&\left|\int\limits_a^b f\left(x\right)\,dx-\tfrac{b-a}{2}\left[\theta f\left(a\right)+2\left(1-\theta\right)f\left(\tfrac{a+b}{2}\right)+\theta f\left(b\right)\right]\right|\le\\&\le\frac{I\left(2,\theta\right)}{2}\left[\max\left\{\left|f''\left(a\right)\right|,\left|f''\left(\frac{a+b}{2}\right)\right|\right\}+\right.\\&\left.+\max\left\{\left|f''\left(\tfrac{a+b}{2}\right)\right|,\left|f''\left(b\right)\right|\right\}\right],\end{aligned}\tag{24}$$

where $I\left(2,\theta\right)$ is as expressed in (16).

If we take $\theta=0,1,\frac{1}{3},\frac{1}{2}$ in (24) and (16), then we get a midpoint inequality

$$\begin{aligned}&\left|\int\limits_a^b f\left(x\right)\,dx-\left(b-a\right)f\left(\tfrac{a+b}{2}\right)\right|\le\\&\le\frac{\left(b-a\right)^3}{48}\left[\max\left\{\left|f''\left(a\right)\right|,\left|f''\left(\frac{a+b}{2}\right)\right|\right\}+\right.\\&\left.+\max\left\{\left|f''\left(\tfrac{a+b}{2}\right)\right|,\left|f''\left(b\right)\right|\right\}\right],\end{aligned}$$

a trapezoid inequality

$$\begin{aligned}&\left|\int\limits_a^b f\left(x\right)\,dx-\tfrac{b-a}{2}\left[f\left(a\right)+f\left(b\right)\right]\right|\le\\&\le\frac{\left(b-a\right)^3}{8}\left[\max\left\{\left|f''\left(a\right)\right|,\left|f''\left(\frac{a+b}{2}\right)\right|\right\}+\right.\\&\left.+\max\left\{\left|f''\left(\tfrac{a+b}{2}\right)\right|,\left|f''\left(b\right)\right|\right\}\right],\end{aligned}$$

a Simpson inequality

$$\left|\int_a^b f(x)\,dx-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\le$$
$$\le\frac{(b-a)^3}{162}\left[\max\left\{|f''(a)|,\left|f''\left(\frac{a+b}{2}\right)\right|\right\}+\right.$$
$$\left.+\max\left\{\left|f''\left(\tfrac{a+b}{2}\right)\right|,|f''(b)|\right\}\right]$$

and an averaged midpoint-trapezoid inequality

$$\left|\int_a^b f(x)\,dx-\frac{b-a}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\le$$
$$\le\frac{(b-a)^2}{96}\left[\max\left\{|f''(a)|,\left|f''\left(\frac{a+b}{2}\right)\right|\right\}+\right.$$
$$\left.+\max\left\{\left|f''\left(\tfrac{a+b}{2}\right)\right|,|f''(b)|\right\}\right].$$

Remark 6. *For $n=3$, we have*

$$\left|\int_a^b f(x)\,dx-\tfrac{b-a}{2}\left[\theta f(a)+2(1-\theta)f\left(\tfrac{a+b}{2}\right)+\theta f(b)\right]-\right.$$
$$\left.-\tfrac{(1-3\theta)(b-a)^3}{24}f''\left(\tfrac{a+b}{2}\right)\right|\le\tfrac{I(3,\theta)}{2}\left[\max\left\{|f'''(a)|,\left|f'''\left(\tfrac{a+b}{2}\right)\right|\right\}+\right. \tag{25}$$
$$\left.+\max\left\{\left|f'''\left(\tfrac{a+b}{2}\right)\right|,|f'''(b)|\right\}\right],$$

where $I(3,\theta)$ is as expressed in (19).

If we take $\theta=0,1,\dfrac{1}{3},\dfrac{1}{2}$ in (25) and (19), then we get a midpoint type inequality

$$\left|\int_a^b f(x)\,dx-(b-a)f\left(\frac{a+b}{2}\right)-\frac{(b-a)^3}{24}f''\left(\frac{a+b}{2}\right)\right|\le$$
$$\le\frac{(b-a)^4}{384}\left[\max\left\{|f'''(a)|,\left|f'''\left(\frac{a+b}{2}\right)\right|\right\}+\right.$$
$$\left.+\max\left\{\left|f'''\left(\tfrac{a+b}{2}\right)\right|,|f'''(b)|\right\}\right],$$

a trapezoid type inequality

$$\left|\int\limits_a^b f(x)\,dx - \tfrac{b-a}{2}[f(a)+f(b)] + \tfrac{(b-a)^3}{12} f''\left(\tfrac{a+b}{2}\right)\right| \le$$
$$\le \frac{(b-a)^4}{128}\left[\max\left\{|f'''(a)|,\left|f'''\left(\frac{a+b}{2}\right)\right|\right\}+\right.$$
$$\left.+\max\left\{\left|f'''\left(\tfrac{a+b}{2}\right)\right|,|f'''(b)|\right\}\right],$$

a Simpson inequality

$$\left|\int\limits_a^b f(x)\,dx - \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \le$$
$$\le \frac{(b-a)^4}{1152}\left[\max\left\{|f'''(a)|,\left|f'''\left(\frac{a+b}{2}\right)\right|\right\}+\right.$$
$$\left.+\max\left\{\left|f'''\left(\tfrac{a+b}{2}\right)\right|,|f'''(b)|\right\}\right]$$

which recapture the inequality (6) and an averaged midpoint-trapezoid type inequality

$$\left|\int\limits_a^b f(x)\,dx - \frac{b-a}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]+\right.$$
$$\left.+\frac{(b-a)^3}{48} f''\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^4}{384}\left[\max\left\{|f'''(a)|,\left|f'''\left(\frac{a+b}{2}\right)\right|\right\}+\right.$$
$$\left.+\max\left\{\left|f'''\left(\tfrac{a+b}{2}\right)\right|,|f'''(b)|\right\}\right].$$

3. Applications in numerical integration. We restrict further considerations to the Simpson quadrature rule.

Theorem 6. *Let* $\pi = \{x_0 = a < x_1 < \cdots < x_n = b\}$ *be a given subdivision of the interval* $[a,b]$ *such that* $h_i = x_{i+1} - x_i = h = \frac{b-a}{n}$ *and let the assumptions of Theorem 1 hold. Then we have*

$$\left|\int\limits_a^b f(t)\,dt - \tfrac{h}{6}\sum_{i=0}^{n-1}\left[f(x_i)+4f\left(\tfrac{x_i+x_{i+1}}{2}\right)+f(x_{i+1})\right]\right| \le$$
$$\le \frac{5(b-a)^2}{36n}\max\{|f'(a)|,|f'(b)|\}. \tag{26}$$

Proof. From the inequality (14) in Remark 1 we obtain

$$\begin{aligned}\Big|\int\limits_{x_i}^{x_{i+1}} f(t)\,dt-\tfrac{h}{6}\Big[f(x_i)+4f\Big(\tfrac{x_i+x_{i+1}}{2}\Big)+f(x_{i+1})\Big]\Big|\le\\ \le\frac{5h^2}{36}\max\{|f'(x_i)|,|f'(x_{i+1})|\}\le\\ \le\frac{5(b-a)^2}{36n^2}\max\{|f'(a)|,|f'(b)|\}.\end{aligned}\tag{27}$$

By summing (27) over i from 0 to $n-1$, we get

$$\begin{aligned}\sum_{i=0}^{n-1}\Big|\int\limits_{x_i}^{x_{i+1}} f(t)\,dt-\tfrac{h}{6}\Big[f(x_i)+4f\Big(\tfrac{x_i+x_{i+1}}{2}\Big)+f(x_{i+1})\Big]\Big|\le\\ \le\frac{5(b-a)^2}{36n}\max\{|f'(a)|,|f'(b)|\}.\end{aligned}\tag{28}$$

Consequently, the inequality (26) follows from (28). □

Theorem 7. *Let $\pi=\{x_0=a<x_1<\cdots<x_n=b\}$ be a given subdivision of the interval $[a,b]$ such that $h_i=x_{i+1}-x_i=h=\frac{b-a}{n}$ and let the assumptions of Theorem 2 hold. Then we have*

$$\begin{aligned}\Big|\int\limits_a^b f(t)\,dt-\tfrac{h}{6}\sum_{i=0}^{n-1}\Big[f(x_i)+4f\Big(\tfrac{x_i+x_{i+1}}{2}\Big)+f(x_{i+1})\Big]\Big|\le\\ \le\frac{(b-a)^3}{81n^2}\max\{|f''(a)|,|f''(b)|\}.\end{aligned}\tag{29}$$

Proof. From the inequality (17) in Remark 2 we obtain

$$\begin{aligned}\Big|\int\limits_{x_i}^{x_{i+1}} f(t)\,dt-\tfrac{h}{6}\Big[f(x_i)+4f\Big(\tfrac{x_i+x_{i+1}}{2}\Big)+f(x_{i+1})\Big]\Big|\le\\ \le\frac{h^3}{81}\max\{|f''(x_i)|,|f''(x_{i+1})|\}\le\\ \le\frac{(b-a)^3}{81n^3}\max\{|f''(a)|,|f''(b)|\}.\end{aligned}\tag{30}$$

By summing (30) over i from 0 to $n-1$, we get

$$\begin{aligned}\sum_{i=0}^{n-1}\Big|\int\limits_{x_i}^{x_{i+1}} f(t)\,dt-\tfrac{h}{6}\Big[f(x_i)+4f\Big(\tfrac{x_i+x_{i+1}}{2}\Big)+f(x_{i+1})\Big]\Big|\le\\ \le\frac{(b-a)^3}{81n^2}\max\{|f''(a)|,|f''(b)|\}.\end{aligned}\tag{31}$$

Consequently, the inequality (29) follows from (31). □

Theorem 8. *Let $\pi = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be a given subdivision of the interval $[a,b]$ such that $h_i = x_{i+1} - x_i = h = \frac{b-a}{n}$ and let the assumptions of Theorem 3 hold. Then we have*

$$\begin{aligned}\left|\int_a^b f(t)\,dt - \frac{h}{6}\sum_{i=0}^{n-1}\left[f(x_i) + 4f\left(\frac{x_i+x_{i+1}}{2}\right) + f(x_{i+1})\right]\right| \leq \\ \leq \frac{(b-a)^4}{576n^3}\max\{|f'''(a)|, |f'''(b)|\}.\end{aligned} \tag{32}$$

Proof. From the inequality (20) in Remark 3 we obtain

$$\begin{aligned}\left|\int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{h}{6}\left[f(x_i) + 4f\left(\frac{x_i+x_{i+1}}{2}\right) + f(x_{i+1})\right]\right| \leq \\ \leq \frac{h^4}{576}\max\{|f'''(x_i)|, |f'''(x_{i+1})|\} \leq \\ \leq \frac{(b-a)^4}{576n^4}\max\{|f'''(a)|, |f'''(b)|\}.\end{aligned} \tag{33}$$

By summing (33) over i from 0 to $n-1$, we get

$$\begin{aligned}\sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{h}{6}\left[f(x_i) + 4f\left(\frac{x_i+x_{i+1}}{2}\right) + f(x_{i+1})\right]\right| \leq \\ \leq \frac{(b-a)^4}{576n^3}\max\{|f'''(a)|, |f'''(b)|\}.\end{aligned} \tag{34}$$

Consequently, the inequality (32) follows from (34). □

4. Applications to special means. We now consider the applications of the Simpson inequalities (14), (17) and (20) to the following special means:

(1) The arithmetic mean: $A(a,b) := \dfrac{a+b}{2},\ a,b \geq 0.$

(2) The Geometric mean: $G(a,b) := \sqrt{ab},\ a,b \geq 0.$

(3) The harmonic mean: $H(a,b) := \dfrac{2ab}{a+b},\ a,b > 0.$

(4) The logarithmic mean: $L(a,b) := \dfrac{b-a}{\ln b - \ln a},\ a \neq b,\ a,b > 0.$

(5) The identric mean: $I(a,b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}$, $a \neq b$, $a,b > 0$.

(6) The p-logarithmic mean: $L_p(a,b) = \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}$, $a \neq b$, $a,b > > 0$, $p \neq -1, 0$.

Using the Simpson inequalities (14), (17) and (20), some new inequalities are derived for the above means.

Proposition 1. *Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbf{N}$, $n \geq 3$. Then we have*

$$\left|\frac{1}{3}A(a^n,b^n) + \frac{2}{3}A^n(a,b) - L_n^n(a,b)\right| \leq \frac{5n(b-a)b^{n-1}}{36},$$

$$\left|\frac{1}{3}A(a^n,b^n) + \frac{2}{3}A^n(a,b) - L_n^n(a,b)\right| \leq \frac{n(n-1)(b-a)^2 b^{n-2}}{81}$$

and

$$\left|\frac{1}{3}A(a^n,b^n) + \frac{2}{3}A^n(a,b) - L_n^n(a,b)\right| \leq \frac{n(n-1)(n-2)(b-a)^3 b^{n-3}}{576}.$$

Proof. The assertion follows from applying the inequalities (14), (17) and (20) to the mapping $f(x) = x^n, x \in [a,b]$ and $n \in \mathbf{N}$ which implies that $|f'(x)| = nx^{n-1}$, $|f''(x)| = n(n-1)x^{n-2}$ and $|f'''(x)| = = n(n-1)(n-2)x^{n-3}$ are quasi-convex on $[a,b]$. □

Proposition 2. *Let $a, b \in \mathbb{R}$, $0 < a < b$. Then we have*

$$\left|\frac{1}{3}H^{-1}(a,b) + \frac{2}{3}A^{-1}(a,b) - L^{-1}(a,b)\right| \leq \frac{5(b-a)}{36a^2},$$

$$\left|\frac{1}{3}H^{-1}(a,b) + \frac{2}{3}A^{-1}(a,b) - L^{-1}(a,b)\right| \leq \frac{(b-a)^2}{81a^3}$$

and

$$\left|\frac{1}{3}H^{-1}(a,b) + \frac{2}{3}A^{-1}(a,b) - L^{-1}(a,b)\right| \leq \frac{(b-a)^3}{288a^4}.$$

Proof. The assertion follows from applying the inequalities (14), (17) and (20) to the mapping $f(x) = \frac{1}{x}, x \in [a,b]$ which implies that $|f'(x)| = \frac{1}{x^2}$, $|f''(x)| = \frac{2}{x^3}$ and $|f'''(x)| = \frac{6}{x^4}$ are quasi-convex on $[a,b]$. □

Proposition 3. *Let $a, b \in \mathbb{R}$, $0 < a < b$. Then we have*

$$\left|\frac{1}{3}\Big[\ln G\left(a,b\right)+2\ln A\left(a,b\right)\Big]-\ln I\left(a,b\right)\right| \leq \frac{5\left(b-a\right)}{36a},$$

$$\left|\frac{1}{3}\Big[\ln G\left(a,b\right)+2\ln A\left(a,b\right)\Big]-\ln I\left(a,b\right)\right| \leq \frac{\left(b-a\right)^2}{81a^2}$$

and

$$\left|\frac{1}{3}\Big[\ln G\left(a,b\right)+2\ln A\left(a,b\right)\Big]-\ln I\left(a,b\right)\right| \leq \frac{\left(b-a\right)^3}{288a^3}.$$

Proof. The assertion follows from applying the inequalities (14), (17) and (10) to the mapping $f\left(x\right)=\ln x, x\in\left[a,b\right]$ which implies that $\left|f'\left(x\right)\right|=\frac{1}{x}$, $\left|f''\left(x\right)\right|=\frac{1}{x^2}$ and $\left|f'''\left(x\right)\right|=\frac{2}{x^3}$ are quasi-convex on $[a,b]$. □

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