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NECESSARY AND SUFFICIENT CONDITIONS OF (α, β)-ACCESSIBILITY OF DOMAIN IN NONSMOOTH CASE

Abstract. In [1], [2] (α, β)-accessible domains in $\mathbb{C}$ were defined and investigated, and the criterion of (α, β)-accessibility in a smooth case was obtained. (α, β)-accessible domains are starlike, γ-accessible ($\gamma = \min\{\alpha; \beta\}$), and satisfy the so-called "cone condition" (i. e. the domains are conically accessible from the interior), which is important for applications, such as the theory of integral representations of functions, imbedding theorems, the questions of the boundary behavior of functions, the solvability of Dirichlet problem. In this paper the author obtains the necessary and some sufficient conditions of (α, β)-accessibility of domain in nonsmooth case.

Key words: *α-accessible domain, (α, β)-accessible domain, cone condition*

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In [1, 2, 3] (α, β)-accessible domains were studied as a generalization of α-accessible domains (see [4]).

Definition 1. [1, 2] *Let $\alpha, \beta \in [0, 1)$, $D \subset \mathbb{C}$, $0 \in D$. A domain D is called (α, β)-accessible with respect to 0 if for each point $p \in \partial D$ there exists a number $r = r(p) > 0$ such that the cone*

$$K_+(p, \alpha, \beta, r) = \left\{ z \in \mathbb{C} : -\frac{\beta\pi}{2} \le \operatorname{Arg}(z - p) - \operatorname{Arg} p \le \frac{\alpha\pi}{2}, |z - p| \le r \right\}$$

is contained in $\mathbb{C} \backslash D$.

In [2], it was shown that the domain D is (α,β)-accessible with respect to 0 if and only if, for each point $p\in\partial D$, the unbounded cone

$$K_+(p,\alpha,\beta)=\left\{z\in\mathbb{C}:-\frac{\beta\pi}{2}\le \operatorname{Arg}(z-p)-\operatorname{Arg}p\le\frac{\alpha\pi}{2}\right\}$$

is contained in $\mathbb{C}\backslash D$.

If in the Definition 1 $\alpha=\beta$, then we obtain the definition of α-accessible domain with respect to 0 (see [4]).

An (α,β)-accessible domain is γ-accessible ($\gamma=\min\{\alpha,\beta\}$), and starlike with respect to 0, as a consequence (see [4]). In the case $\alpha=\beta=0$ the class of $(0,0)$–accessible domains coincides with the class of starlike with respect to 0 domains (see [5]).

Further, we assume that:

a) the function $F(x)$ is defined and continuous in $\mathbb{R}^2$;

b) the open set $D=\{x\in\mathbb{R}^2: F(x)<0\}$ contains 0;

c) there exist derivatives $\dfrac{\partial F}{\partial l}(p)$ at the points of the set level $S=\{p\in\in\mathbb{R}^2: F(p)=0\}$ in all directions $l\in(K_+(p,\alpha,\alpha)-p)\backslash\{0\}$.

In [5] the following necessary condition of α-accessibility was obtained:

Theorem 1. [5] *Let assumptions a), b), c) be satisfied. If D is an α-accessible domain for a certain $\alpha\in[0;1)$ then derivatives $\dfrac{\partial F}{\partial l}(p)$ are positive for any direction $l\in(K_+(p,\alpha,\alpha)-p)\backslash\{0\}$ and for any point $p\in S$.*

Regarding sufficiency the following theorems were received:

Theorem 2. [5] *Let assumptions a), b), c) be satisfied. If for a certain $\alpha\in[0;1)$ $\dfrac{\partial F}{\partial l}(p)>0$ for any direction $l\in(K_+(p,\alpha,\alpha)-p)\backslash\{0\}$ and for any point $p\in\partial D$ then D is an α-accessible domain.*

Theorem 3. [6] *Let assumptions a), b) be satisfied and D be bounded set. If for a certain $\alpha\in[0;1)$, an arbitrarily small $\delta>0$, and for a certain differentiable, strictly increasing numeric function Ψ in $\mathbb{R}^+=\{\xi\in\mathbb{R}:\xi>0\}$, such that $\lim\limits_{\xi\to+0}\Psi(\xi)=0\overset{def}{=}\Psi(0)$, we have*

$$\frac{\partial F}{\partial l}(p)>-\Psi'(\Psi^{-1}(-F(p)))\frac{2}{|p|}\Psi^{-1}(-F(p))\cos\frac{\alpha\pi}{2}$$

for any direction $l \in (K_+(p, \alpha, \alpha) - p) \setminus \{0\}$ and for any point $p \in D^\delta = \{x \in D : \rho(x, \partial D) < \delta\}$, then D is an α-accessible domain.

In this article, analogs of theorems 1, 2 and 3 for (α, β)-accessible domain are obtained. In addition we have to replace the condition *c)* to the condition

d) there exist derivatives $\frac{\partial F}{\partial l}(p)$ at the points p of the set level S in all directions $l \in (K_+(p, \alpha, \beta) - p) \setminus \{0\}$.

The following theorem gives a necessary condition of (α, β)-accessibility.

Theorem 1'. *Let assumptions a), b) and d) be satisfied. If D is an (α, β)-accessible domain (with respect to 0) for certain $\alpha, \beta \in [0; 1)$ then*

$$\frac{\partial F}{\partial l}(p) \geq 0 \tag{1}$$

for any $p \in S$ and for any direction $l \in (K_+(p, \alpha, \beta) - p) \setminus \{0\}$.

Proof. Let $p \in S$. In the segment $[0; \mathrm{p}]$ there exists a point $p_0 \in \partial D$. Points $p + \rho l$, $\rho > 0$, $l \in (K_+(p, \alpha, \beta) - p)$, lie in the cone $K_+(p, \alpha, \beta) \subseteq K_+(p_0, \alpha, \beta)$. Since $K_+(p_0, \alpha, \beta) \cap D = \varnothing$, we have $K_+(p, \alpha, \beta) \cap D = \varnothing$. So $F(p + \rho l) \geq 0$. Therefore

$$\frac{\partial F}{\partial l}(p) = \lim_{\rho \to 0+} \frac{F(p + \rho l) - F(p)}{\rho} \geq 0.$$

□

Theorem 2'. *Let assumptions a), b) and d) be satisfied. If for certain $\alpha, \beta \in [0; 1)$*

$$\frac{\partial F}{\partial l}(p) > 0 \tag{2}$$

for any direction $l \in (K_+(p, \alpha, \beta) - p) \setminus \{0\}$ and for any point $p \in \partial D$ then D is an (α, β)-accessible domain.

Proof. Note that by Theorem 2 the set D under the given conditions is a γ-accessible domain, where $\gamma = \min\{\alpha, \beta\}$. It follows that D is starlike domain (see [4]).

Suppose that D is not an (α, β)-accessible domain. Then there exists $p \in \partial D$ such that $K_+(p, \alpha, \beta, \varepsilon) \cap D \neq \varnothing$ for any $\varepsilon > 0$. So there exists a sequence of distinct points $y_n \in D$ such that $y_n \in K_+(p, \alpha, \beta)$ and

$y_n \to p$ for $n \to \infty$. Notice, that these points y_n do not lie on the ray $\{pt : t \geq 1\}$. Otherwise, by the condition of starlikeness of domain D, the segment $[0, y_n] \subset D$, the point $p \in [0, y_n]$ and the ray $\{pt : t \geq 1\}$ lies in $\mathbb{R}^2 \backslash D$. This is a contradiction.

Draw a segment from each point y_n to point 0. The segment $[0, y_n]$ intersects one of the sides of the cone $K_+(p, \alpha, \beta)$ at the point z_n (but not the top of the cone). Choose the one of the two sides of the cone $K_+(p, \alpha, \beta)$, which contains an infinite number of points z_n. Since D is starlike ($y_n, 0 \in \in D$), it follows that $z_n \to p$ and $z_n \in D$. Let $l_0 = \dfrac{z_n - p}{|z_n - p|}$ be a unit vector. Note that l_0 is independent of n. Let $\rho_n = |z_n - p|$. Since $z_n \in D$, then $z_n = p + \rho_n l_0$ and $F(z_n) < 0$. Therefore,

$$\frac{F(z_n) - F(p)}{\rho_n} = \frac{F(p + \rho_n l_0) - F(p)}{\rho_n} < 0$$

$$\text{and} \quad \frac{\partial F}{\partial l_0}(p) = \lim_{\rho_n \to 0+} \frac{F(p + \rho_n l_0) - F(p)}{\rho_n} \leq 0.$$

But this contradicts the condition $\dfrac{\partial F}{\partial l}(p) > 0$ for any direction $l \in \in (K_+(p, \alpha, \beta) - p)$ and for any point $p \in \partial D$. Thus, D is an (α, β)-accessible domain. $\square$

Note that the necessary (Theorem 1') and sufficient (Theorem 2') conditions of (α, β)-accessibility differ only by a sign of equality in inequalities (1) and (2). Below we represent another sufficient condition (an analog of Theorem 3), where in the right-hand side of an inequality there will be already negative expression, but it will be checked in the near-border zone D^δ, but not on the boundary of the domain D (compared to Theorem 2').

Theorem 3'. *Let assumptions a), b) be satisfied and D be a bounded set. If for certain $\alpha, \beta \in [0; 1)$, an arbitrarily small $\delta > 0$, and for a certain differentiable, strictly increasing numeric function Ψ in $\mathbb{R}^+$, such that $\lim\limits_{\xi \to +0} \Psi(\xi) = 0 \overset{def}{=} \Psi(0)$, derivatives*

$$\frac{\partial F}{\partial l}(p) > -\Psi'(\Psi^{-1}(-F(p))) \frac{2}{|p|^2} \Psi^{-1}(-F(p)) \left(\frac{l}{|l|}, p \right) \tag{3}$$

for any direction $l \in (K_+(p, \alpha, \beta) - p) \backslash \{0\}$ and for any point $p \in D^\delta$, then D is (α, β)-accessible domain.

Proof. For $t > 0$ denote $F_t(x) = F(x) + \Psi\left(t\,\frac{|x|^2}{2}\right)$ and $D_t = \{x \in \mathbb{R}^2 : F_t(x) < 0\}$. Suppose $x \in D_t$. It follows from the inequality $F(x) < -\Psi\left(t\,\frac{|x|^2}{2}\right) \leq 0$ that $x \in D$. Therefore $D_t \subset D$ for each $t > 0$. Note also that $0 \in D_t$, because $0 \in D$.

Let us show that there exists $T > 0$ such that for all $0 < t < T$ the level sets $S_t = \{x \in \mathbb{R}^n : F_t(x) = 0\}$ lie in D^δ.

At first, if $x \in S_t$, then $x \neq 0$ and $F(x) = -\Psi\left(t\,\frac{|x|^2}{2}\right) < 0$ for all $t > 0$. Hence, $S_t \subset D$ for all $t > 0$.

Secondly, for $0 < t_1 < t_2$ the inclusion $D_{t_2} \subset D_{t_1}$ is valid. Indeed, if $x \in D_{t_2}$, then $F(x) + \Psi\left(t_2\,\frac{|x|^2}{2}\right) < 0$. Therefore, from a strict increasing Ψ it follows that $F(x) + \Psi\left(t_1\,\frac{|x|^2}{2}\right) \leq F(x) + \Psi\left(t_2\,\frac{|x|^2}{2}\right) < 0$. Hence, $x \in D_{t_1}$.

Let us show that for a given $\delta > 0$ there exists a number $T > 0$ such that $S_t \subset D^\delta$ for each $t \in (0, T)$. Suppose that it is not true. Then there exists a sequence $y_n \in S_{t_n}$ such that the distance $\rho(y_n, S) > \delta$ as $t_n \to 0$.

Since D is bounded, then from the sequence $\{y_n\}$ it is possible to select a converging subsequence (denote it the same way), such that $y_n \to y_0 \in \mathbb{R}^2$.

Passing to the limit as $t_n \to 0$ in the equality $F(y_n) = -\Psi\left(t_n\,\frac{\|y_n\|^2}{2}\right)$, we see that $F(y_0) = -\Psi(0) = 0$. Hence, $y_0 \in S$. However, $\rho(y_n, S) > \delta$. Therefore, $\rho(y_0, S) \geq \delta$. This is a contradiction. Thus, there exists a number $T > 0$ such that $S_t \subset D^\delta$ for each $t \in (0, T)$.

Since $t\,\frac{|x|^2}{2} = \Psi^{-1}\left(-F(x)\right)$ for all $x \in S_t$, and $S_t \subset D^\delta$ for $0 < t < T$, it follows from the condition of Theorem 3' that

$$\begin{aligned} \frac{\partial F}{\partial l}(x) &> -\Psi'(\Psi^{-1}(-F(x)))\frac{2}{|x|^2}\Psi^{-1}(-F(x))\left(\frac{l}{|l|}, x\right) = \\ &= -\Psi'\left(t\,\frac{|x|^2}{2}\right) t \left(\frac{l}{|l|}, x\right) \end{aligned} \tag{4}$$

for all $x \in S_t$, $0 < t < T$ and $l \in (K_+(x, \alpha, \beta) - x) \setminus \{0\}$.

Note that

$$\frac{\partial\Psi}{\partial l}\left(t\,\frac{|x|^2}{2}\right)=\left(\operatorname{grad}\Psi\left(t\,\frac{|x|^2}{2}\right),\frac{l}{|l|}\right)=t\,\Psi'\left(t\,\frac{|x|^2}{2}\right)\left(x,\frac{l}{|l|}\right). \quad (5)$$

Relations (4) and (5) imply that for all $t\in(0,T)$ and $x\in S_t$ there exist derivatives in the directions $l\in(K_+(x,\alpha,\beta)-x)\setminus\{0\}$ such that

$$\frac{\partial F_t}{\partial l}(x)=\frac{\partial F}{\partial l}(x)+\frac{\partial\Psi}{\partial l}\left(t\,\frac{|x|^2}{2}\right)>0.$$

Then by Theorem 2', the domain D_t is (α,β)-accessible for each $t\in(0,T)$.

Let us show that for each $x_0\in D$ there exists such domain D_t, $t\in$ $\in(0,T)$, that $x_0\in D_t$.

Suppose $x_0\in D$, $x_0\neq 0$ and $F(x_0)=-C<0$. Then, for all t such that $0<t<\dfrac{2\Psi^{-1}(C)}{\|x_0\|^2}=t_0$, we have that $x_0\in D_t$. This is because

$$F_t(x_0)=F(x_0)+\Psi\left(t\,\frac{|x_0|^2}{2}\right)<-C+\Psi\left(t_0\,\frac{|x_0|^2}{2}\right)=-C+C=0.$$

Hence, $x_0\in D_t$ for all $t<t_0$. Therefore, $D=\bigcup\limits_{0<t<T}D_t$, where each domain D_t is (α,β)-accessible. In [2], it was shown that the union of (α,β)-accessible domains is an (α,β)-accessible domain. That proves the theorem. □

If in the conditions of the Theorem 3' we take $\Psi(\xi)=\xi^n$, $n\in\mathbb{N}$, then the right-hand side of (3) has the form

$$-\Psi'(\Psi^{-1}(-F(p)))\frac{2}{|p|^2}\Psi^{-1}(-F(p))\left(\frac{l}{|l|},p\right)=$$
$$=-n\left(\Psi^{-1}(-F(p))\right)^{n-1}\frac{2}{|p|^2}\left(-F(p)\right)^{\frac{1}{n}}\left(\frac{l}{|l|},p\right)=$$
$$=-n\left(-F(p)\right)^{\frac{n-1}{n}}\frac{2}{|p|^2}\left(-F(p)\right)^{\frac{1}{n}}\left(\frac{l}{|l|},p\right)=\frac{2nF(p)}{|p|^2}\left(\frac{l}{|l|},p\right).$$

Then, in particular, we obtain the following sufficient condition of (α,β)-accessibility of a domain

Corollary 1. *Let assumptions a), b) be satisfied and D be a bounded set. If for certain $\alpha,\beta\in[0;1)$, $n\in\mathbb{N}$, an arbitrarily small $\delta>0$, and any point*

$p \in D^\delta$ there exist derivatives in the directions $l \in (K_+(p,\alpha,\beta)-p)\setminus\{0\}$ such that

$$\frac{\partial F}{\partial l}(p) > \frac{2nF(p)}{|p|^2}\left(\frac{l}{|l|},p\right) \tag{6}$$

for any point $p \in D^\delta$, then D is an (α,β)-accessible domain.

Note that inequality (6) is equivalent to

$$\frac{1}{-F(p)}\frac{\partial(-F(p))}{\partial l} < \frac{2n}{|p|^2}\left(\frac{l}{|l|},p\right),$$

and so (6) can be rewritten as

$$\frac{\partial \ln(-F(p))}{\partial l} < \frac{2n}{|p|^2}\left(\frac{l}{|l|},p\right).$$

In the conditions of Theorem 3' let us consider the function $\eta = \Psi(\xi) = e^{-1/\xi}$. The function $\xi = \Psi^{-1}(\eta) = -\dfrac{1}{\ln\eta}$ is an inverse function to η. In this case the right part of (3) has the form

$$-\Psi'(\Psi^{-1}(-F(p)))\frac{2}{|p|^2}\Psi^{-1}(-F(p))\left(\frac{l}{|l|},p\right) =$$

$$= -\frac{1}{\left(\Psi^{-1}(-F(p))\right)^2}\, e^{-1/(\Psi^{-1}(-F(p)))}\frac{2}{|p|^2}\Psi^{-1}(-F(p))\left(\frac{l}{|l|},p\right) =$$

$$= -\left(\ln(-F(p))\right)^2 e^{\ln(-F(p))}\frac{2}{|p|^2}\left(-\frac{1}{\ln(-F(p))}\right)\left(\frac{l}{|l|},p\right) =$$

$$= \frac{-2F(p)\ln(-F(p))}{|p|^2}\left(\frac{l}{|l|},p\right).$$

Thus, we obtain

Corollary 2. *Let assumptions a), b) be satisfied and D be a bounded set. If for certain $\alpha,\beta \in [0;1)$ and an arbitrarily small $\delta > 0$, and for any point $p \in D^\delta$ there exist derivatives in the directions $l \in (K_+(p,\alpha,\beta)-p)\setminus\{0\}$ such that*

$$\frac{\partial F}{\partial l}(p) > \frac{-2F(p)\ln(-F(p))}{|p|^2}\left(\frac{l}{|l|},p\right) \tag{7}$$

for any point $p \in D^\delta$, then D is an (α,β)-accessible domain.

Note that inequality (7) is equivalent to

$$\frac{\partial \ln(-\ln(-F(p)))}{\partial l} > \frac{-2}{|p|^2}\left(\frac{l}{|l|}, p\right).$$

Indeed, dividing inequality (7) by $F(p)$, we get

$$\frac{1}{-F(p)}\frac{\partial(-F(p))}{\partial l} < \frac{-2\ln(-F(p))}{|p|^2}\left(\frac{l}{|l|}, p\right),$$

and so (7) can be rewritten as

$$\frac{\partial \ln(-F(p))}{\partial l} < \frac{-2\ln(-F(p))}{|p|^2}\left(\frac{l}{|l|}, p\right).$$

Dividing the last inequality by $\ln(-\mathrm{F}(\mathrm{p}))$ and taking into account that $\ln(-\mathrm{F}(\mathrm{p})) < 0$ in D^δ for sufficiently small δ we obtain

$$\frac{1}{-\ln(-F(p))}\frac{\partial(-\ln(-F(p)))}{\partial l} > \frac{-2}{|p|^2}\left(\frac{l}{|l|}, p\right).$$

It follows that inequality (7) is equivalent to

$$\frac{\partial \ln(-\ln(-F(p)))}{\partial l} > \frac{-2}{|p|^2}\left(\frac{l}{|l|}, p\right).$$

Note also that if we introduce on the plane a complex structure everywhere in the text of the article, then the inner product $\left(\frac{l}{|l|}, p\right)$ can be written as $\frac{\mathrm{Re}\left\{\bar{l}\cdot p\right\}}{|l|}$, because

$$\left(\frac{l}{|l|}, p\right) = |p|\cos(\arg p - \arg l) =$$

$$|p|\cos(\arg p)\cos(\arg l) + |p|\sin(\arg p)\sin(\arg l) =$$

$$= \mathrm{Re}\, p\ \mathrm{Re}\,\frac{l}{|l|} + \mathrm{Im}\, p\ \mathrm{Im}\,\frac{l}{|l|} = \frac{\mathrm{Re}\left\{\bar{l}\cdot p\right\}}{|l|}.$$

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