DOI: 10.15393/j3.art.2016.3071

UDC 517.51, 517.38

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ON INEQUALITIES OF HERMITE – HADAMARD TYPE INVOLVING AN s-CONVEX FUNCTION WITH APPLICATIONS

Abstract. Motivated by a recent paper, the author provides some new integral inequalities of Hermite – Hadamard type involving the product of an s-convex function and a symmetric function and applies these new established inequalities to construct inequalities for special means.

Key words: *Hermite – Hadamard's integral inequality, s-convex function, symmetric function, Hölder inequality, mean*

2010 Mathematical Subject Classification: *26D15*

1. Introduction. Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a convex function on an interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int\limits_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}$$

is well known in literature as the Hermite – Hadamard inequality for convex functions.

We recall that Hudzik and Maligranda in [1] defined a function $f : [0, \infty) \to \mathbf{R}$ to be called s-convex in the second sense if

$$f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds for all $x, y \in [0, \infty)$, $\lambda \in [0, 1]$ and for some fixed $s \in (0, 1]$. The class of s-convex functions in the second sense is usually denoted with K_s^2. It can be easily seen that for $s = 1$ s-convexity reduces to ordinary convexity of functions defined on $[0, \infty)$. It is proved in [1] that all functions from K_s^2, $s \in (0, 1)$ are nonnegative.

Example 1. [1] *Let* $s \in (0,1)$ *and* $a, b, c \in \mathbf{R}$. *Define the function* $f : [0, \infty) \to \mathbf{R}$ *as*

$$f(t) = \begin{cases} a, & t = 0, \\ bt^s + c, & t > 0. \end{cases}$$

It can be easily checked that

(i) *if* $b \geq 0$ *and* $0 \leq c \leq a$, *then* $f \in K_s^2$,

(ii) *if* $b > 0$ *and* $c < 0$, *then* $f \notin K_s^2$.

In the recent paper [2], Hua et al established the following integral inequalities of Hermite–Hadamard type involving the product of an s-convex function and a symmetric function.

Theorem 1.1-1.3. [2, Theorem 3.1, 3.2, 3.5.] *Let* $f : I \subseteq \mathbf{R} \to \mathbf{R}$ *be differentiable on* $\operatorname{int} I$ *and* $g : [a, b] \to [0, \infty)$ *be continuous and symmetric to* $\frac{a+b}{2}$ *for* $a, b \in I$ *with* $a < b$ *such that* $f' \in L^1[a, b]$. *If* $|f'|^q$ *is s-convex on* $[a, b]$ *for* $q \geq 1$ *and some fixed* $s \in (0, 1]$, *then*

$$\Big|\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx - \int_a^b f(x)g(x)\,dx\Big| \leq \gamma,$$

where γ *is the minimum of the following three values:*

$$\frac{(b-a)^2}{8}\Big[\frac{2^{1-s}}{(s+1)(s+2)}\Big]^{1/q}||g||_\infty\{[(1+s2^{s+1})|f'(a)|^q + |f'(b)|^q]^{1/q} +$$

$$+[|f'(a)|^q + (1+s2^{s+1})|f'(b)|^q]^{1/q}\},$$

$$\frac{(b-a)^2}{8}\Big[\frac{2}{(s+1)(s+2)}\Big]^{1/q}||g||_\infty\Big\{\Big[(s+1)|f'(a)|^q + \Big|f'\Big(\frac{a+b}{2}\Big)\Big|^q\Big]^{1/q} +$$

$$+\Big[\Big|f'\Big(\frac{a+b}{2}\Big)\Big|^q + (s+1)|f'(b)|^q]^{1/q}\Big\},$$

and

$$\frac{b-a}{2}\Big[\frac{|f'(a)|^q + |f'(b)|^q}{2}\Big]^{1/q}\int_0^1\Big[\int_{L(t)}^{U(t)} g(x)\,dx\Big]\,dt.$$

Theorem 1.4-1.6. [2, Theorem 3.3, 3.4, 3.6.] *Let* $f : I \subseteq \mathbf{R} \to \mathbf{R}$ *be differentiable on* $\operatorname{int} I$ *and* $g : [a, b] \to [0, \infty)$ *be continuous and symmetric to* $\frac{a+b}{2}$ *for* $a, b \in I$ *with* $a < b$. *If* $|f'|^q$ *for* $q > 1$ *is convex on* $[a, b]$, *then*

$$\Big|\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx - \int_a^b f(x)g(x)\,dx\Big| \leq \delta,$$

where δ is the minimum of the following three values:

$$\frac{(b-a)^2}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left[\frac{1}{2^s(s+1)}\right]^{1/q}||g||_\infty\{[(2^{s+1}-1)|f'(a)|^q+$$

$$+|f'(b)|^q]^{1/q}+[|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q]^{1/q}\},$$

$$\frac{(b-a)^2}{4(s+1)^{1/q}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}||g||_\infty\left\{\left[|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\right.$$

$$\left.+\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right]^{1/q}\right\},$$

and

$$\frac{b-a}{4}\left\{\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{1/q}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{1/q}\right\}\times$$

$$\times\left[\int_0^1\left[\int_{L(t)}^{U(t)}g(x)\,dx\right]^{q/(q-1)}dt\right]^{1-1/q}.$$

Within them,

$$L(t)=\frac{1+t}{2}a+\frac{1-t}{2}b=ta+(1-t)\frac{a+b}{2} \tag{1}$$

and

$$U(t)=\frac{1-t}{2}a+\frac{1+t}{2}b=tb+(1-t)\frac{a+b}{2}. \tag{2}$$

It should be noticed that we here have improved the expression of Theorem 3.6 in [2].

In this work, corresponding to Theorems 1.1-1.6, we will further establish some integral inequalities of Hermite–Hadamard type involving the product of an s-convex function and a symmetric function in two different ways. Finally, applications to some special means of positive real numbers are considered.

2. Main Results.

Lemma 2.1. (see [3]) *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be differentiable on* $\operatorname{int} I$ *and $g : [a,b] \to [0,\infty)$ be continuous and symmetric to $\frac{a+b}{2}$ for $a,b \in I$ with*

$a < b$. If $f' \in L^1[a,b]$, then

$$\begin{aligned}
&\int_a^b f(x)g(x)\,dx - f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)\,dx = \\
&= \frac{b-a}{2}\int_0^1 \Big[\int_a^{L(t)} g(x)\,dx\Big][f'(U(t)) - f'(L(t))]\,dt,
\end{aligned} \tag{3}$$

where L and U are defined by (1) and (2). In particular, we then have

$$\begin{aligned}
&\Big|\int_a^b f(x)g(x)\,dx - f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)\,dx\Big| \le \\
&\le \frac{b-a}{2}\int_0^1 \Big[\int_a^{L(t)} g(x)\,dx\Big][|f'(L(t))| + |f'(U(t))|]\,dt
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
&\Big|\int_a^b f(x)g(x)\,dx - f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)\,dx\Big| \le \\
&\le \frac{(b-a)^2}{4}||g||_\infty \int_0^1 (1-t)[|f'(L(t)| + |f'(U(t)|]\,dt,
\end{aligned} \tag{5}$$

where $||g||_\infty = \sup\limits_{t\in[a,b]} g(t)$.

Proof. Since $g(x)$ is symmetric to $\frac{a+b}{2}$, then $\int_a^{L(t)} g(x)\,dx = \int_{U(t)}^b g(x)\,dx$ for all $t \in [0,1]$. So we have

$$\int_0^1 \Big[\int_a^{L(t)} g(x)\,dx\Big] f'(L(t))\,dt = \frac{2}{a-b}\int_0^1 \Big[\int_a^{L(t)} g(x)\,dx\Big]\,d[f(L(t))] =$$

$$= \frac{2}{a-b}\Big\{\Big[\int_a^{L(t)} g(x)\,dx\Big] f(L(t))\Big|_0^1 + \frac{b-a}{2}\int_0^1 f(L(t))g(L(t))\,dt\Big\} =$$

$$= \frac{2}{a-b}\Big\{ -f\Big(\frac{a+b}{2}\Big) \int_a^{\frac{a+b}{2}} g(x)\,dx + \int_a^{\frac{a+b}{2}} f(x)g(x)\,dx \Big\} \tag{6}$$

and

$$\int_0^1 \Big[\int_a^{L(t)} g(x)\,dx \Big] f'(U(t))\,dt = \int_0^1 \Big[\int_{U(t)}^b g(x)\,dx \Big] f'(U(t))\,dt =$$

$$= \frac{2}{b-a} \int_0^1 \Big[\int_{U(t)}^b g(x)\,dx \Big] d[f(U(t))] =$$

$$= \frac{2}{b-a}\Big\{ \Big[\int_{U(t)}^b g(x)\,dx \Big] f(U(t)) \Big|_0^1 + \frac{b-a}{2} \int_0^1 f(U(t))g(U(t))\,dt \Big\} =$$

$$= \frac{2}{b-a}\Big\{ -f\Big(\frac{a+b}{2}\Big) \int_{\frac{a+b}{2}}^b g(x)\,dx + \int_{\frac{a+b}{2}}^b f(x)g(x)\,dx \Big\}. \tag{7}$$

Consequently, inequality (3) follows from (6) and (7), and Lemma 2.1 is thus proved. □

Theorem 2.1. *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be differentiable on* $\operatorname{int} I$ *and $g : [a,b] \to [0,\infty)$ be continuous and symmetric to $\frac{a+b}{2}$ for $a, b \in I$ with $a < b$ such that $f' \in L^1[a,b]$. If $|f'|^q$ is s-convex on $[a,b]$ for $q \ge 1$ and some fixed $s \in (0,1]$, then*

$$\Big| \int_a^b f(x)g(x)\,dx - f\Big(\frac{a+b}{2}\Big) \int_a^b g(x)\,dx \Big| \le$$

$$\le \frac{(b-a)^2}{8} \Big(\frac{2^{1-s}}{s+2} \Big)^{1/q} ||g||_\infty \Big\{ \Big[\frac{2^{s+2}-s-3}{s+1} |f'(a)|^q + |f'(b)|^q \Big]^{1/q} +$$

$$+ \Big[|f'(a)|^q + \frac{2^{s+2}-s-3}{s+1} |f'(b)|^q \Big]^{1/q} \Big\}. \tag{8}$$

Proof. Notice that $|f'|^q$ is s-convex on $[a,b]$, by (5) in Lemma 2.1 with the first equality in (1) and (2), and using the Hölder inequality, we have

$$\Big|\int_a^b f(x)g(x)\,dx - f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)\,dx\Big| \le$$

$$\le \frac{(b-a)^2}{4}||g||_\infty \int_0^1 (1-t)[|f'(L(t))| + |f'(U(t))|]\,dt \le$$

$$\le \frac{(b-a)^2}{4}||g||_\infty \Big[\int_0^1 (1-t)\,dt\Big]^{1-1/q} \times$$

$$\times\Big\{\Big[\int_0^1 (1-t)\Big(\Big(\frac{1+t}{2}\Big)^s |f'(a)|^q + \Big(\frac{1-t}{2}\Big)^s |f'(b)|^q\Big)\,dt\Big]^{1/q} +$$

$$+\Big[\int_0^1 (1-t)\Big(\Big(\frac{1-t}{2}\Big)^s |f'(a)|^q + \Big(\frac{1+t}{2}\Big)^s |f'(b)|^q\Big)\,dt\Big]^{1/q}\Big\} = \frac{(b-a)^2}{4}\times$$

$$\times||g||_\infty\Big(\frac{1}{2}\Big)^{1-1/q}\Big\{\Big[\frac{2^{s+2}-s-3}{2^s(s+1)(s+2)}|f'(a)|^q + \frac{1}{2^s(s+2)}|f'(b)|^q\Big]^{1/q} +$$

$$+\Big[\frac{1}{2^s(s+2)}|f'(a)|^q + \frac{2^{s+2}-s-3}{2^s(s+1)(s+2)}|f'(b)|^q\Big]^{1/q}\Big\} =$$

$$= \frac{(b-a)^2}{8}||g||_\infty\Big(\frac{2^{1-s}}{s+2}\Big)^{1/q}\Big\{\Big[\frac{2^{s+2}-s-3}{s+1}|f'(a)|^q +$$

$$+|f'(b)|^q\Big]^{1/q} + \Big[|f'(a)|^q + \frac{2^{s+2}-s-3}{s+1}|f'(b)|^q\Big]^{1/q}\Big\}.$$

Inequality (8) follows, and Theorem 2.1 is proved. □

Corollary 2.1.1. *Under conditions of Theorem 2.1,*
(i) if $q=1$, then

$$\Big|\int_a^b f(x)g(x)\,dx - f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)\,dx\Big| \le$$

$$\leq \frac{(b-a)^2}{2} ||g||_\infty \left[\frac{2^{s+1}-1}{2^s(s+1)(s+2)} \right] [|f'(a)| + |f'(b)|];$$

(ii) if $q = 1$ and $s = 1$, we have

$$\left| \int_a^b f(x)g(x)\,dx - f\Big(\frac{a+b}{2}\Big) \int_a^b g(x)\,dx \right| \leq \frac{(b-a)^2}{8} ||g||_\infty [|f'(a)| + |f'(b)|].$$

Corollary 2.1.2. *Under conditions of Theorem 2.1,*
(i) if $q = 1$ and $g(x) = 1$ for $x \in [a,b]$, then

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - f\Big(\frac{a+b}{2}\Big) \right| \leq \frac{(2^{s+1}-1)(b-a)}{2^{s+1}(s+1)(s+2)} [|f'(a)| + |f'(b)|];$$

(ii) if $q = 1$, $g(x) = 1$ for $x \in [a,b]$, and $s = 1$, we have

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - f\Big(\frac{a+b}{2}\Big) \right| \leq \frac{b-a}{8} [|f'(a)| + |f'(b)|], \tag{9}$$

and it should be noticed that inequality (9) first appeared in [4].

Theorem 2.2. *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be differentiable on $\operatorname{int} I$ and $g : [a,b] \to [0,\infty)$ be continuous and symmetric to $\frac{a+b}{2}$ for $a, b \in I$ with $a < b$ such that $f' \in L^1[a,b]$. If $|f'|^q$ is s-convex on $[a,b]$ for $q \geq 1$ and some fixed $s \in (0,1]$, then*

$$\left| \int_a^b f(x)g(x)\,dx - f\Big(\frac{a+b}{2}\Big) \int_a^b g(x)\,dx \right| \leq \frac{(b-a)^2}{8} \left[\frac{2}{(s+1)(s+2)} \right]^{1/q} \times$$

$$\times ||g||_\infty \left\{ \left[|f'(a)|^q + (s+1) \left| f'\Big(\frac{a+b}{2}\Big) \right|^q \right]^{1/q} + \right.$$

$$\left. + \left[(s+1) \left| f'\Big(\frac{a+b}{2}\Big) \right|^q + |f'(b)|^q \right]^{1/q} \right\}. \tag{10}$$

Proof. Notice that $|f'|^q$ is s-convex on $[a,b]$, by (5) in Lemma 2.1 with the second equality in (1) and (2), and using the Hölder inequality in a different way, we have

$$\left| \int_a^b f(x)g(x)\,dx - f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)\,dx \right| \le$$

$$\le \frac{(b-a)^2}{4}||g||_\infty \int_0^1 (1-t)[|f'(L(t))| + |f'(U(t))|]\,dt =$$

$$= \frac{(b-a)^2}{4}||g||_\infty \int_0^1 (1-t)\left[\left|f'\left(ta + (1-t)\frac{a+b}{2}\right)\right| + \right.$$

$$\left. + \left|f'\left(tb + (1-t)\frac{a+b}{2}\right)\right|\right] dt \le \frac{(b-a)^2}{4}||g||_\infty \left[\int_0^1 (1-t)\,dt\right]^{1-1/q} \times$$

$$\times \left\{ \left[\int_0^1 (1-t)\left(t^s|f'(a)|^q + (1-t)^s\left|f'\Big(\frac{a+b}{2}\Big)\right|^q\right) dt\right]^{1/q} + \right.$$

$$\left. + \left[\int_0^1 (1-t)\left((1-t)^s\left|f'\Big(\frac{a+b}{2}\Big)\right|^q + t^s|f'(b)|^q\right) dt\right]^{1/q} \right\} = \frac{(b-a)^2}{4} \times$$

$$\times ||g||_\infty \Big(\frac{1}{2}\Big)^{1-1/q} \left\{ \left[\frac{1}{(s+1)(s+2)}|f'(a)|^q + \frac{1}{s+2}\left|f'\Big(\frac{a+b}{2}\Big)\right|^q\right]^{1/q} + \right.$$

$$\left. + \left[\frac{1}{s+2}\left|f'\Big(\frac{a+b}{2}\Big)\right|^q + \frac{1}{(s+1)(s+2)}|f'(b)|^q\right]^{1/q} \right\} = \frac{(b-a)^2}{8} \times$$

$$\times ||g||_\infty \left[\frac{2}{(s+1)(s+2)}\right]^{1/q} \left\{ \left[|f'(a)|^q + (s+1)\left|f'\Big(\frac{a+b}{2}\Big)\right|^q\right]^{1/q} + \right.$$

$$\left. + \left[(s+1)\left|f'\Big(\frac{a+b}{2}\Big)\right|^q + |f'(b)|^q\right]^{1/q} \right\}.$$

Inequality (10) follows, and Theorem 2.2 is proved. □

Corollary 2.2.1. *Under conditions of Theorem 2.2,*
(i) if $q = 1$*, then*

$$\left| \int_a^b f(x)g(x)\,dx - f\Big(\frac{a+b}{2}\Big) \int_a^b g(x)\,dx \right| \le$$

$$\le \frac{(b-a)^2}{4(s+1)(s+2)} ||g||_\infty \left[|f'(a)| + 2(s+1)\left| f'\Big(\frac{a+b}{2}\Big) \right| + |f'(b)| \right];$$

(ii) if $q = 1$ *and* $s = 1$*, we have*

$$\left| \int_a^b f(x)g(x)\,dx - f\Big(\frac{a+b}{2}\Big) \int_a^b g(x)\,dx \right| \le$$

$$\le \frac{(b-a)^2}{24} ||g||_\infty \left[|f'(a)| + 4\left| f'\Big(\frac{a+b}{2}\Big) \right| + |f'(b)| \right].$$

Corollary 2.2.2. *Under conditions of Theorem 2.2,*
(i) if $q = 1$ *and* $g(x) = 1$ *for* $x \in [a, b]$*, then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - f\Big(\frac{a+b}{2}\Big) \right| \le$$

$$\le \frac{b-a}{4(s+1)(s+2)} \left[|f'(a)| + 2(s+1)\left| f'\Big(\frac{a+b}{2}\Big) \right| + |f'(b)| \right];$$

(ii) if $q = 1$*,* $g(x) = 1$ *for* $x \in [a, b]$*, and* $s = 1$*, we have*

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - f\Big(\frac{a+b}{2}\Big) \right| \le$$

$$\le \frac{b-a}{24} \left[|f'(a)| + 4\left| f'\Big(\frac{a+b}{2}\Big) \right| + |f'(b)| \right], \tag{11}$$

and it should be noticed that inequality (11) can also be derived from (2.8) of [5].

Theorem 2.3. *Let* $f : I \subseteq \mathbf{R} \to \mathbf{R}$ *be differentiable on* $\operatorname{int} I$ *and* $g : [a, b] \to [0, \infty)$ *be continuous and symmetric to* $\frac{a+b}{2}$ *for* $a, b \in I$ *with*

$a < b$ such that $f' \in L^1[a,b]$. If $|f'|^q$ is s-convex on $[a,b]$ for $q > 1$ and some fixed $s \in (0,1]$, then

$$\Big|\int_a^b f(x)g(x)\,dx - f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)\,dx\Big| \le$$

$$\le \frac{(b-a)^2}{4}||g||_\infty\Big(\frac{q-1}{2q-1}\Big)^{1-1/q}\Big[\frac{1}{2^s(s+1)}\Big]^{1/q}\Big\{\Big[(2^{s+1}-1)|f'(a)|^q+$$
$$+|f'(b)|^q\Big]^{1/q}+\Big[|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q\Big]^{1/q}\Big\}. \qquad (12)$$

Proof. Notice that $|f'|^q$ is s-convex on $[a,b]$, by (5) in Lemma 2.1 with the first equality in (1) and (2), and using the Hölder inequality, we have

$$\Big|\int_a^b f(x)g(x)\,dx - f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)\,dx\Big| \le$$

$$\le \frac{(b-a)^2}{4}||g||_\infty\int_0^1(1-t)[|f'(L(t))|+|f'(U(t))|]\,dt \le$$

$$\le \frac{(b-a)^2}{4}||g||_\infty\Big[\int_0^1(1-t)^{q/q-1}\,dt\Big]^{1-1/q}\times$$

$$\times\Big\{\Big[\int_0^1\Big(\Big(\frac{1+t}{2}\Big)^s|f'(a)|^q+\Big(\frac{1-t}{2}\Big)^s|f'(b)|^q\Big)\,dt\Big]^{1/q}+$$

$$+\Big[\int_0^1\Big(\Big(\frac{1-t}{2}\Big)^s|f'(a)|^q+\Big(\frac{1+t}{2}\Big)^s|f'(b)|^q\Big)\,dt\Big]^{1/q}\Big\}=$$

$$=\frac{(b-a)^2}{4}||g||_\infty\Big(\frac{q-1}{2q-1}\Big)^{1-1/q}\Big[\frac{1}{2^s(s+1)}\Big]^{1/q}\{[(2^{s+1}-1)|f'(a)|^q+$$
$$+|f'(b)|^q]^{1/q}+[|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q]^{1/q}\}.$$

Inequality (12) follows, and Theorem 2.3 is proved. □

Corollary 2.3. *Under conditions of Theorem 2.3, if $s = 1$, then*

$$\Big| \int_a^b f(x)g(x)\,dx - f\Big(\frac{a+b}{2}\Big) \int_a^b g(x)\,dx \Big| \le$$

$$\le \frac{(b-a)^2}{4^{1+1/q}} ||g||_\infty \Big(\frac{q-1}{2q-1}\Big)^{1-1/q} \times$$

$$\times \{[3|f'(a)|^q + |f'(b)|^q]^{1/q} + [|f'(a)|^q + 3|f'(b)|^q]^{1/q}\}.$$

Theorem 2.4. *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be differentiable on $\operatorname{int} I$ and $g : [a,b] \to [0,\infty)$ be continuous and symmetric to $\frac{a+b}{2}$ for $a, b \in I$ with $a < b$ such that $f' \in L^1[a,b]$. If $|f'|^q$ is s-convex on $[a,b]$ for $q > 1$ and some fixed $s \in (0,1]$, then*

$$\Big| \int_a^b f(x)g(x)\,dx - f\Big(\frac{a+b}{2}\Big) \int_a^b g(x)\,dx \Big| \le \frac{(b-a)^2}{4(s+1)^{\frac{1}{q}}} \Big(\frac{q-1}{2q-1}\Big)^{1-\frac{1}{q}} ||g||_\infty \times$$

$$\times \left\{ \left[|f'(a)|^q + \left| f'\Big(\frac{a+b}{2}\Big) \right|^q \right]^{1/q} + \left[\left| f'\Big(\frac{a+b}{2}\Big) \right|^q + |f'(b)|^q \right]^{1/q} \right\}. \quad (13)$$

Proof. Notice that $|f'|^q$ is s-convex on $[a,b]$, by (4) in Lemma 2.1 with the second equality in (1) and (2), and using the Hölder inequality, we have

$$\Big| \int_a^b f(x)g(x)\,dx - f\Big(\frac{a+b}{2}\Big) \int_a^b g(x)\,dx \Big| \le$$

$$\le \frac{(b-a)^2}{4} ||g||_\infty \int_0^1 (1-t)[|f'(L(t))| + |f'(U(t))|]\,dt = \frac{(b-a)^2}{4} ||g||_\infty \times$$

$$\times \int_0^1 (1-t) \left[\left| f'\Big(ta + (1-t)\frac{a+b}{2}\Big) \right| + \left| f'\Big(tb + (1-t)\frac{a+b}{2}\Big) \right| \right] dt \le$$

$$\leq \frac{(b-a)^2}{4}||g||_\infty \left[\int_0^1 (1-t)^{q/q-1}\, dt \right]^{1-1/q} \times$$

$$\times \left\{ \left[\int_0^1 \left(t^s |f'(a)|^q + (1-t)^s \left| f'\left(\frac{a+b}{2}\right) \right|^q \right) dt \right]^{1/q} + \right.$$

$$\left. + \left[\int_0^1 \left((1-t)^s \left| f'\left(\frac{a+b}{2}\right) \right|^q + t^s |f'(b)|^q \right) dt \right]^{1/q} \right\} =$$

$$= \frac{(b-a)^2}{4}||g||_\infty \left(\frac{q-1}{2q-1} \right)^{1-1/q} \left(\frac{1}{s+1} \right)^{1/q} \left\{ \left[|f'(a)|^q + \right. \right.$$

$$\left. \left. + \left| f'\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} + \left[\left| f'\left(\frac{a+b}{2}\right) \right|^q + |f'(b)|^q \right]^{1/q} \right\}.$$

Inequality (13) follows, and Theorem 2.4 is proved. □

Corollary 2.4. *Under conditions of Theorem 2.4, if $s = 1$, then*

$$\left| \int_a^b f(x)g(x)\, dx - f\left(\frac{a+b}{2}\right) \int_a^b g(x)\, dx \right| \leq$$

$$\leq \frac{(b-a)^2}{2^{2+1/q}}||g||_\infty \left(\frac{q-1}{2q-1} \right)^{1-1/q} \times$$

$$\times \left\{ \left[|f'(a)|^q + \left| f'\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} + \left[\left| f'\left(\frac{a+b}{2}\right) \right|^q + |f'(b)|^q \right]^{1/q} \right\}.$$

Theorem 2.5. *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be differentiable on $\operatorname{int} I$ and $g : [a,b] \to [0,\infty)$ be continuous and symmetric to $\frac{a+b}{2}$ for $a, b \in I$ with $a < b$. If $|f'|^q$ for $q \geq 1$ is convex on $[a,b]$, then*

$$\left| \int_a^b f(x)g(x)\, dx - f\left(\frac{a+b}{2}\right) \int_a^b g(x)\, dx \right| \leq$$

$$\leq (b-a)\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{1/q}\int_0^1\left[\int_a^{L(t)} g(x)\,dx\right]dt. \tag{14}$$

Proof. Notice that $|f'|^q$ is convex on $[a,b]$, by (4) in Lemma 2.1 and using the Hölder inequality, we have

$$\left|\int_a^b f(x)g(x)\,dx - f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)\,dx\right| \leq \frac{b-a}{2}\times$$

$$\times\left\{\int_0^1\left[\int_a^{L(t)} g(x)\,dx\right]|f'(L(t))|\,dt + \int_0^1\left[\int_a^{L(t)} g(x)\,dx\right]|f'(U(t))|\,dt\right\}\leq$$

$$\leq \frac{b-a}{2}\left\{\int_0^1\left[\int_a^{L(t)} g(x)\,dx\right]dt\right\}^{1-\frac{1}{q}}\left\{\left(\int_0^1\left[\int_a^{L(t)} g(x)\,dx\right]|f'(L(t))|^q\,dt\right)^{\frac{1}{q}}+\right.$$

$$\left.+\left(\int_0^1\left[\int_a^{L(t)} g(x)\,dx\right]|f'(U(t))|^q\,dt\right)^{\frac{1}{q}}\right\}. \tag{15}$$

From the power-mean inequality $(a^r+b^r)\leq 2^{1-r}(a+b)^r$ for $a,b>0$ and $r\leq 1$ and convexity of $|f'|^q$ on $[a,b]$, with the second equality in (1) and (2), we obtain

$$\left(\int_0^1\left[\int_a^{L(t)} g(x)\,dx\right]|f'(L(t))|^q\,dt\right)^{\frac{1}{q}}+\left(\int_0^1\left[\int_a^{L(t)} g(x)\,dx\right]|f'(U(t))|^q\,dt\right)^{\frac{1}{q}}\leq$$

$$\leq 2^{1-1/q}\left\{\int_0^1\left[\int_a^{L(t)} g(x)\,dx\right]\left[|f'(L(t))|^q+|f'(U(t))|^q\right]dt\right\}^{1/q}\leq 2^{1-1/q}\times$$

$$\times\left\{\int_0^1\left[\int_a^{L(t)} g(x)\,dx\right]\left[t|f'(a)|^q+2(1-t)\left|f'\Big(\frac{a+b}{2}\Big)\right|^q+t|f'(b)|^q\right]dt\right\}^{1/q}\leq$$

$$\leq 2^{1-1/q}\Big\{\int_0^1\Big[\int_a^{L(t)} g(x)\,dx\Big]\Big[t|f'(a)|^q+(1-t)(|f'(a)|^q+|f'(b)|^q)+t\times$$

$$\times|f'(b)|^q\Big]\,dt\Big\}^{\frac{1}{q}}=2^{1-\frac{1}{q}}[|f'(a)|^q+|f'(b)|^q]^{\frac{1}{q}}\Big\{\int_0^1\int_a^{L(t)} g(x)\,dx\,dt\Big\}^{\frac{1}{q}}. \quad (16)$$

Inequality (14) follows from (15) and (16), and Theorem 2.5 is proved. □

Corollary 2.5.1. *Under conditions of Theorem 2.5, if* $q=1$, *then*

$$\Big|\int_a^b f(x)g(x)\,dx-f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)\,dx\Big|\leq$$

$$\leq\frac{b-a}{2}[|f'(a)|+|f'(b)|]\int_0^1\Big[\int_a^{L(t)} g(x)\,dx\Big]\,dt.$$

Theorem 2.6. *Let* $f: I\subseteq\mathbf{R}\to\mathbf{R}$ *be differentiable on* $\operatorname{int} I$ *and* $g:[a,b]\to[0,\infty)$ *be continuous and symmetric to* $\frac{a+b}{2}$ *for* $a,b\in I$ *with* $a<b$. *If* $|f'|^q$ *for* $q>1$ *is convex on* $[a,b]$, *then*

$$\Big|\int_a^b f(x)g(x)\,dx-f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)\,dx\Big|\leq$$

$$\leq\frac{b-a}{2}\Big\{\Big[\frac{3|f'(a)|^q+|f'(b)|^q}{4}\Big]^{1/q}+\Big[\frac{|f'(a)|^q+3|f'(b)|^q}{4}\Big]^{1/q}\Big\}\times$$

$$\times\Big\{\int_0^1\Big[\int_a^{L(t)} g(x)\,dx\Big]^{q/q-1}dt\Big\}^{1-1/q}. \quad (17)$$

Proof. Notice that $|f'|^q$ is convex on $[a,b]$, by (4) in Lemma 2.1 with the second equality in (1) and (2), and using the Hölder inequality, we have

$$\Big|\int_a^b f(x)g(x)\,dx-f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)\,dx\Big|\leq\frac{b-a}{2}\Big\{\int_0^1\Big[\int_a^{L(t)} g(x)\,dx\Big]\times$$

$$\times[|f'(L(t))| + |f'(U(t))|]\,dt\Bigg\} \le \frac{b-a}{2}\left\{\int_0^1\left[\int_a^{L(t)} g(x)\,dx\right]^{\frac{q}{q-1}} dt\right\}^{1-\frac{1}{q}} \times$$

$$\times\left\{\left[\int_0^1 |f'(L(t))|^q\,dt\right]^{\frac{1}{q}} + \left[\int_0^1 |f'(U(t))|^q\,dt\right]^{\frac{1}{q}}\right\} \le \frac{b-a}{2}\times$$

$$\times\left\{\int_0^1\left[\int_a^{L(t)} g(x)\,dx\right]^{\frac{q}{q-1}} dt\right\}^{1-\frac{1}{q}}\left\{\left(\int_0^1\left[t|f'(a)|^q + (1-t)\times\right.\right.\right.$$

$$\left.\left.\times\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]dt\right)^{\frac{1}{q}} + \left(\int_0^1\left[(1-t)\left|f'\left(\frac{a+b}{2}\right)\right|^q + t|f'(a)|^q\right]dt\right)^{\frac{1}{q}}\Bigg\} \le$$

$$\le \frac{b-a}{2}\left\{\int_0^1\left[\int_a^{L(t)} g(x)\,dx\right]^{\frac{q}{q-1}} dt\right\}^{1-\frac{1}{q}}\left\{\left(\int_0^1\left[\frac{1+t}{2}|f'(a)|^q +\right.\right.\right.$$

$$\left.\left.+\frac{1-t}{2}|f'(b)|^q\right]dt\right)^{\frac{1}{q}} + \left(\int_0^1\left[\frac{1-t}{2}|f'(a)|^q + \frac{1+t}{2}|f'(b)|^q\right]dt\right)^{\frac{1}{q}}\Bigg\} =$$

$$= \frac{b-a}{2}\left\{\left[\frac{3|f'(a)|^q + |f'(b)|^q}{4}\right]^{\frac{1}{q}} + \left[\frac{|f'(a)|^q + 3|f'(b)|^q}{4}\right]^{\frac{1}{q}}\right\}\times$$

$$\times\left\{\int_0^1\left[\int_a^{L(t)} g(x)\,dx\right]^{\frac{q}{q-1}} dt\right\}^{1-\frac{1}{q}}.$$

Inequality (17) follows, and Theorem 2.6 is proved. □

3. Applications to special means. Now we apply some of the above inequalities of Hermite–Hadamard type involving the product of an s-convex function and a symmetric function to construct inequalities for special means.

For positive numbers $a > 0$ and $b > 0$, define

(i) the arithmetic mean:

$$A(a,b) = \frac{a+b}{2}$$

and

(ii) the generalized logarithmic mean:

$$L_r(a,b) = \left[\frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)}\right]^{1/r}, \quad r \neq -1, 0.$$

Let

$$f(x) = \frac{x^{s+1}}{s+1} \tag{18}$$

for $x > 0$, $s > 0$, and $q \geq 1$. If $0 < sq \leq 1$ and $0 < s \leq 1$, we have

$$|f'(tx + (1-t)y)|^q \leq t^{sq}x^{sq} + (1-t)^{sq}y^{sq} \leq t^s|f'(x)|^q + (1-t)^s|f'(y)|^q$$

for $x, y > 0$ and $t \in [0,1]$. At this time, it is easy to verify that the function $|f'(x)|^q = x^{sq} \in K_s^2$ for $x \in [a,b]$.

For $b > a > 0$, define

$$g(x) = \left(x - \frac{a+b}{2}\right)^2 \tag{19}$$

for $x \in [a,b]$. Applying Theorems 2.1–2.4 to the concrete functions (18) and (19) straightforwardly yields the following inequalities involving special means A and L_r.

Theorem 3.1. *For $b > a > 0$, $q \geq 1$, and $0 < s \leq 1$ such that $0 < sq \leq 1$, we have*

$$\Big|(b-a)^2A^{s+1}(a,b) - 12\Big[L_{s+3}^{s+3}(a,b) - 2A(a,b)L_{s+2}^{s+2}(a,b)+$$

$$+A^2(a,b)L_{s+1}^{s+1}(a,b)\Big]\Big| \leq \frac{3 \times 2^{(1-s)/q}(b-a)^3(s+1)^{1-1/q}}{8(s+2)^{1/q}}\times$$

$$\times\{[(2^{s+2} - s - 3)a^{sq} + (s+1)b^{sq}]^{1/q} + [(s+1)a^{sq} + (2^{s+2} - s - 3)b^{sq}]^{1/q}\}$$

and

$$\Big|(b-a)^2A^{s+1}(a,b) - 12\Big[L_{s+3}^{s+3}(a,b) - 2A(a,b)L_{s+2}^{s+2}(a,b)+$$

$$+A^2(a,b)L_{s+1}^{s+1}(a,b)\Big]\Big| \leq \frac{3 \times 2^{1/q}(b-a)^3(s+1)^{1-1/q}}{8(s+2)^{1/q}}\times$$

$$\times\{[a^{sq}+(s+1)A^{sq}(a,b)]^{1/q}+[(s+1)A^{sq}(a,b)+b^{sq}]^{1/q}\}.$$

Corollary 3.1.1. *For $b>a>0$ and $0<s\le 1$, we have*

$$\bigg|(b-a)^2A^{s+1}(a,b)-12\bigg[L_{s+3}^{s+3}(a,b)-2A(a,b)L_{s+2}^{s+2}(a,b)+ +A^2(a,b)L_{s+1}^{s+1}(a,b)\bigg]\bigg|\le\frac{3(b-a)^3}{2^{s+1}(s+2)}(2^{s+1}-1)(a^s+b^s)$$

and

$$\bigg|(b-a)^2A^{s+1}(a,b)-12\bigg[L_{s+3}^{s+3}(a,b)-2A(a,b)L_{s+2}^{s+2}(a,b)+ +A^2(a,b)L_{s+1}^{s+1}(a,b)\bigg]\bigg|\le\frac{3(b-a)^3}{2(s+2)}[A(a^s,b^s)+(s+1)A^s(a,b)].$$

Theorem 3.2. *For $b>a>0$, $q>1$, and $0<s<1$ such that $0<sq\le 1$, we have*

$$\bigg|(b-a)^2A^{s+1}(a,b)-12\bigg[L_{s+3}^{s+3}(a,b)-2A(a,b)L_{s+2}^{s+2}(a,b)+ +A^2(a,b)L_{s+1}^{s+1}(a,b)\bigg]\bigg|\le\frac{3(b-a)^3(s+1)^{1-1/q}}{2^{2+s/q}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\times$$
$$\times\{[(2^{s+1}-1)a^{sq}+b^{sq}]^{1/q}+[a^{sq}+(2^{s+1}-1)b^{sq}]^{1/q}\}$$

and

$$\bigg|(b-a)^2A^{s+1}(a,b)-12\bigg[L_{s+3}^{s+3}(a,b)-2A(a,b)L_{s+2}^{s+2}(a,b)+ +A^2(a,b)L_{s+1}^{s+1}(a,b)\bigg]\bigg|\le\frac{3(b-a)^3(s+1)^{1-1/q}}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\times$$
$$\times\{[a^{sq}+A^{sq}(a,b)]^{1/q}+[A^{sq}(a,b)+b^{sq}]^{1/q}\}.$$

Acknowledgment. The author would like to thank the referees for their helpful comments and suggestions.

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Received March 27, 2016.
In revised form, June 22, 2016.
Accepted June 15, 2016.

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