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ABOUT THE EQUALITY OF THE TRANSFORM OF LAPLACE TO THE TRANSFORM OF FOURIER

Abstract. We proved that the transform of Laplace does not have complex part on the complex axis for the wide class of functions in different situations. The main theorem is proved presenting a function as sum of two Laplace transforms. The transforms are defined in the left and right parts of the plain accordingly. Such presentation is proved to be unique. With help of the results we obtain equality of the transforms of Laplace and Fourier for some class of functions.

Key words: *Laplace transform, Fourier transforms, Dirichlet problem, new inverse of Laplace transform*

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1. Introduction. We consider the transform of Laplace in the form

$$LF_{-}u(x)(\cdot)(v),\ v \in [0, +\infty),$$

where, by definition,

$$LS(x)(\cdot)(v) = \int\limits_0^\infty e^{-vx} S(x)dx,\ v \in [0, \infty),$$

$$\int\limits_{-\infty}^{\infty} e^{\pm ixy}\, u(x)dx = F_{\pm}u(x)(\cdot)(y),\ y \in (-\infty, \infty).$$

The basic result of the article is formulated in Theorem 2:

$$(\pi/2)u(t) = LC^0 u(x)(\cdot)(t),\ t \in (0, \infty),$$

if $u(-x) = u(x),\ x \in (-\infty, \infty)$ (in the condition of Theorem 2), where

$$\int\limits_0^\infty \cos vx\, S(x)dx = C^0 S(x)(\cdot)(v),\ v \in (-\infty, \infty).$$

Similar results are considered in works [1] – [6] in connection with the problem of the inverse of transform of Laplace ([2], [3], [4]). Traditional methods, related to regular functions ([1], [3], [6] – [9]), are not applicable to functions $u(t) = u_2(x)$ without properties $u_2(-x) = u_2(x)$ or $u_2(-x) = -u_2(x)$. The direction is considered in the second part of proof of Theorem 1.

The main result (Theorem 1) follows from presenting function $u(ip)$ as a sum of Laplace transforms

$$\mathsf{L}(p) + (-\mathsf{L}_1(p)) = u(ip),$$

(the exact formulation is in Lemma 1), if the transforms are defined in the left and right parts of the complex plain accordingly, where, by definition,

$$\mathsf{L}_1(p) = (1/2\pi) \int\limits_{-\infty}^{\infty} u(x)/(p + ix)dx,\ \operatorname{Re} p \leq 0,$$

$$\mathsf{L}(p) = (1/2\pi) \int\limits_{-\infty}^{\infty} u(x)/(p + ix)dx,\ \operatorname{Re} p \geq 0.$$

With help of this presentation we obtain the equality of the transform of Laplace to the transform of Fourier (Theorem 2).

In general all results follow from the $\operatorname{Re} \mathsf{L}(is) = \mathsf{L}(is)$ equality of Theorem 1, if $s \in (-\infty, \infty)$.

The results of Theorem 1 are related to the problem of Dirichlet and the integrals of Poisson and Schwartz [7, p. 209]. This subject is studied in [1], [4]; we do not touch it here.

The proof follows from some main facts: in rather wide conditions we can use (for $s \in (-\infty, \infty)$)

$$2\pi \mathsf{L}(-is) = \int\limits_0^\infty e^{isx} dx \int\limits_{-\infty}^{\infty} e^{-itx} u(t)dt =$$

$$= \int\limits_0^\infty \cos sxdx \int\limits_0^\infty 2\cos txu(t)dt + i\int\limits_0^\infty \sin sxdx \int\limits_0^\infty 2\cos txu(t)dt =$$

$$= \pi u(s) + 2i\int\limits_0^\infty \sin sxdx \int\limits_0^\infty \cos txu(t)dt,\ u(-t) = u(t);$$

$$2\pi\mathsf{L}(-is) = \pi u(s) - 2i\int\limits_0^\infty \cos sxdx \int\limits_0^\infty \sin txu(t)dt,\ u(-t) = -u(t),$$

and we obtain the sum (Lemma 1)

$$\mathsf{L}(is) + (-\mathsf{L}_1(is)) = u(-s),\ s \in (-\infty,\infty),\ \mathsf{L}(p) + (-\mathsf{L}_1(p)) = u(ip),$$

$ip \notin J$, where function $\mathsf{L}(p)$ is defined for $\mathrm{Re}\, p \geq 0$, and function $\mathsf{L}_1(p)$ with the same analytical expression is defined for $\mathrm{Re}\, p \leq 0$, J is a set of the special points of function $u(p)$; we use equalities

$$\mathsf{L}(-is) = \overline{\mathsf{L}(is)},\ \mathsf{L}(is) = -\overline{\mathsf{L}_1(is)}),\ u(-s) = u(s),\ s \in (-\infty,\infty).$$

In **the main second part of proof of Theorem 1** we consider function $u_2(p) = \mathsf{L}(p)$ in the form

$$l(is) + l(-is) = u_2(-s) = \mathsf{L}^*(is) + \mathsf{L}_1^*(is),\ s \in (-\infty,0),$$

where

$$\mathsf{L}^*(p) = (1/2\pi)\int\limits_{-\infty}^\infty u_2(x)/(p+ix)dx,\ \mathrm{Re}\, p \geq 0,$$

$\mathsf{L}(x) = u_2(x) = \mathrm{Re}\, u_2(x)$, $x \in (-\infty,+\infty)$, and function $\mathsf{L}_1^*(p)$ with the same analytical expression is defined for $\mathrm{Re}\, p \leq 0$, where, by definition,

$$l(p) = (1/2\pi)LC^0 2u_2(x)(\cdot)(p),\ \mathrm{Re}\, p \geq 0,$$

but $l(is) + l(-is) = l(-is) + l(-is)$, and we get

$$\mathsf{L}(is) = u_2(is) = u_2(-is) = \mathsf{L}(-is),\ s \in (-\infty,+\infty),$$

or (Theorem 1) $u(s)/2 = \mathrm{Re}\, u_2(is) = u_2(is) = \mathsf{L}(is)$, $s \in (-\infty,+\infty)$.

The main part of the article is the second part of proof of Theorem 1.

2. The transform of Laplace on the complex axis. By definition, the area of regularity of function $u(p)$ is G_u.

Theorem 1. *If $u(-p) = u(p)$, $p \in G_u$, $u(p)$ is regular in the C plane, except a finite number of points $J = \{z_1, \dots, z_k,\ \mathrm{Re}\ z_k \neq 0,\ \mathrm{Im}\ z_k \neq \neq 0,\ k \geq 2\}$, and*

$$u(0) = 0,\ |u(s)| \leq c/|s|^2,\ |s| \to \infty,\, s \in (-\infty, \infty),\, c < \infty,$$

where c is the constant, the equality

$$u(-y)/2 = \lim_{x \to 0+} (1/2\pi) \int_{-\infty}^{\infty} u(t)/((x+iy)+it)dt,\ y \in (-\infty, \infty),$$

takes place, and the equality

$$\pi u(it) = LF_{-}u(x)(\cdot)(t),\ t \in (-\infty, \infty),$$

takes place if

$$\lim_{x \to 0+} (1/2\pi) LF_{-}u(t)(\cdot)(x+iy) = (1/2\pi) LF_{-}u(t)(\cdot)(iy),\quad y \in (-\infty, \infty).$$

Proof. We use Lemma 1 (a part of Lemma 1 was proved in [2]–[5]. In the Appendix we give a new full proof).

Lemma 1. *1. Equality $\mathsf{L}(is) - \mathsf{L}_1(is) = u(-s)$, $s \in (-\infty, \infty)$, takes place if the function $u(s)$ is continuous for all $s \in (-\infty, \infty)$; equality*

$$\mathsf{L}(p) - \mathsf{L}_1(p) = u(ip),\ ip \in G_u,$$

takes place if $u(p)$ is regular in $\{p : |\mathrm{Im}\, p| < a\} \in G_u$, for a constant $a \in (0, +\infty)$.

2.
$$\lim_{p \to iy, \mathrm{Re}\, p \geq 0} 2\pi \mathsf{L}(p) = 2\pi \mathsf{L}(iy) = \pi u(-y) - \int_{-\infty}^{\infty} (u(x-y)/ix)dx\,,$$

$$\lim_{p \to iy, \mathrm{Re}\, p \leq 0} 2\pi \mathsf{L}_1(p) = 2\pi \mathsf{L}_1(iy),\ L_1(iy) = -\overline{L(iy)},$$

$y \in (-\infty, \infty)$, in all points $y : du(s)/ds|_{s=y} < \infty$, if the function $u(s)$ is continuous for all $s \in (-\infty, \infty)$, $\mathrm{Re}\, u(s) = u(s)$, $s \in (-\infty, \infty)$, and

$$|u(s)| \leq c/|s|^{\delta},\ s \to \pm\infty,\ s \in (-\infty, \infty),\, \delta > 0,\, \delta = \mathrm{const}.$$

3. If $u(0) = 0$, and $du(s)/ds|_{s=0+} < \infty$, $du(s)/ds|_{s=0-} < \infty$ values are defined, but $du(s)/ds|_{s=0+} \neq du(s)/ds|_{s=0-}$, the second part of Lemma 1 takes place for $y = (-\infty, \infty)$ (and for $y = 0$).

From Lemma 1 in the conditions of Theorem 1 we obtain

$$u(ip) = \mathsf{L}(p) - \mathsf{L}_1(p), \;ip \notin J.$$

We will assume that

$$u(ip) = u_1(p) + u_2(p) = \mathsf{L}(p) - \mathsf{L}_1(p), \;ip \notin J,$$

where $u_1(p)$, $u_2(p)$ are new functions: $u_1(p)$ is regular in the left part of the complex plane (without points $\operatorname{Re} p = 0$), $u_1(p)$ is bounded in the left part of the complex plane (with all points $\operatorname{Re} p = 0$), and $u_1(p)$ is continuous in all points on the complex axis (from the left part of the plain); $u_2(p)$ is regular in the right part of the complex plane (without points $\operatorname{Re} p = 0$), $u_2(p)$ is bounded in the right part of the complex plane (with all points $\operatorname{Re} p = 0$), and $u_2(p)$ is continuous in all points on the complex axis (from the right part of the plain).

We get

$$u_2(p) = \mathsf{L}(p), \;u_1(p) = -\mathsf{L}_1(p), \;-\mathsf{L}_1(p) = \mathsf{L}(-p), \;p \notin J;$$

the fact we obtain from

$$u_2(p) - \mathsf{L}(p) = -u_1(p) + (-\mathsf{L}_1(p)) \equiv 0,$$

where the functionin the left-hand side has special points in the left part of the complex plane (see Lemma 1), and the function in the right-hand side has special points in the right part of the complex plane (see Lemma 1), and both functions are equal to 0 in ∞ (it is obvious from Lemma 1). By definition $u_2(p) - \mathsf{L}(p)$ is regular in the right part of the plain (with $\operatorname{Re} p = 0$), and $-u_1(p) + (-\mathsf{L}_1(p))$ is regular in the left part of the plain (with $\operatorname{Re} p = 0$). We obtain that both functions are regular in all points of the complex plain, and both functions are bounded by a constant in their domains (this is obvious from the definition), or both functions are zero ([7]).

As a result we have $\operatorname{Re} u_2(s) = u_2(s), \;s \in (-\infty, \infty)$, where

$$u_2(-s) = \mathsf{L}_1(-s) + u(-is) = -\mathsf{L}(s) + u(-is), \;u(is) = u(-is), \;s \in [0, \infty).$$

The second part of the proof of Theorem 1. We will prove that $\operatorname{Re} u_2(is) = u_2(is), s \in (-\infty, \infty)$, where $u_2(p) = \mathsf{L}(p)$.

With help of Lemma 1 we will repeat the previous reasoning for new pairs of functions (**function** $u_2(x)$ **can be neither odd nor even**)

$$2\pi \mathsf{L}^*(p) = \int_{-\infty}^{\infty} u_2(x)/(p+ix)dx,\ \operatorname{Re} p \geq 0,$$

$$2\pi \mathsf{L}_1^*(p) = \int_{-\infty}^{\infty} u_2(x)/(p+ix)dx,\ \operatorname{Re} p \leq 0.$$

We get

$$\mathsf{L}^*(is) + (-\mathsf{L}_1^*(is)) = u_2(-s),\ s \in (-\infty, \infty),\ \mathsf{L}^*(p) + (-\mathsf{L}_1^*(p)) = u_2(ip),$$

$ip \neq J$, (in the equality we use Lemma 1 with $u_2(x)$ instead of function $u(x)$, $\operatorname{Re} u_2(x) = u_2(x)$, $x \in (-\infty, \infty)$).

We use the new function

$$u_0(x) = u_2(|x|),\ x \in (-\infty, \infty).$$

With help of Lemma 1 we get

$$l(is) + l_1(is) = u_2(|s|) = u_2(-s),\ s \in (-\infty, 0),$$

(**not for** $s \in (-\infty, \infty)$), where

$$2\pi l(p) = \int_{-\infty}^{\infty} u_2(|x|)/(p+ix)dx,\ \operatorname{Re} p \geq 0,$$

$$2\pi l_1(p) = -\int_{-\infty}^{\infty} u_2(|x|)/(p+ix)dx,\ \operatorname{Re} p \leq 0.$$

For all positive $-s$ we obtain :

$$u_2(-s) = \mathsf{L}^*(is) + (-\mathsf{L}_1^*(is)) = l(is) + l_1(is) = u_2(|s|),\ -s \in (0, +\infty),$$

and (as in Lemma 1, but only for positive $-s \in (0, +\infty)$) we get ([7])

$$u_2(ip) = \mathsf{L}^*(p) + (-\mathsf{L}_1^*(p)) = l(p) + l_1(p),$$

in the area of joint regularity of the sum $u_2(ip)$, $p \notin J$. (We use the theorem about analytical continuation across a line ([7]), where functions $\mathsf{L}^*(p), \mathsf{L}_1^*(p), l(p), l_1(p)$ are continuous on the complex axis in their domains).

But $l_1(p) = l(-p), \operatorname{Re} p < 0$ (it is obvious from the values on $(-\infty, 0)$), and we get

$$l(p) + l_1(p) = l(p) + l(-p) = u_2(ip) = \mathsf{L}^*(p) + (-\mathsf{L}_1^*(p)),\ ip \neq J.$$

From the equality

$$l(p) - \mathsf{L}^*(p) = -l(-p) + (-\mathsf{L}_1^*(p)),\ ip \neq J,$$

with help of continuity of functions $l(p), l_1(p) = l(-p)$ on the complex axis and regularity of function $-\mathsf{L}^*(p), \mathsf{L}_1^*(p)$ for all $ip \neq J$, $p \in (-i\infty, i\infty)$ we obtain that

$$l(p) = \mathsf{L}^*(p) - l(-p) + (-\mathsf{L}_1^*(p)),\ \operatorname{Re} p \leq 0$$

is the analytical continuation of function $l(p)$ across all points of the complex axis from the right part of the plain to the left part of the plain (theorem about analytical continuation across the line [7]).

We have proved $u_2(-pi) = u_2(pi) = l(p) + l(-p)$, $ip \notin J$, and

$$\operatorname{Re} u_2(is) = u_2(is) = \mathsf{L}(is) = \operatorname{Re} \mathsf{L}(is), s \in (-\infty, \infty)$$

(for the function $u(-p) = u(p)$ ([7])).

We can see

$$u(-y)/2 = \operatorname{Re} \mathsf{L}(iy) = \mathsf{L}(iy) = \lim_{x \to 0+} \mathsf{L}(x + iy),\ y \in (-\infty, \infty),$$

(from Lemma 1), and (changing limits of integration ([10]))

$$u(-y)/2 = (1/2\pi) LF\_u(t)(\cdot)(iy) = \mathsf{L}(iy),\ u(-y) = u(y),\ y \in (-\infty, \infty),$$

if $\lim_{x \to 0+} (1/2\pi) LF\_u(t)(\cdot)(x + iy) = (1/2\pi) LF\_u(t)(\cdot)(iy),\ y \in (-\infty, \infty)$.
Theorem 1 is proved. $\square$

From Theorem 1 and formula

$$LF_{-}u(x)(\cdot)(it) = 2LC^0u(x)(\cdot)(t),\ t \in (-\infty, \infty),$$

we obtain Theorem 2.

Theorem 2. *In the conditions of Theorem 1*

$$\pi u(t) = 2LC^0u(x)(\cdot)(t),\ t \in (0, \infty),$$

if $u(p) = f(p^4)$, $p \notin J$, where function $f(z)$ is regular for all $z = p^4$, $p \notin J$.

3. Appendix.

Proof of Lemma 1. From [10]

$$2\pi(\mathsf{L}(x+iy) - \mathsf{L}_1(-x+iy)) =$$

$$= \int_{-\infty}^{\infty} [u(t)/(x+iy+it)]dt - \int_{-\infty}^{\infty} [u(t)/(-x+iy+it)]dt =$$

$$= \int_{-\infty}^{\infty} [u(t_1-y)2x/(x^2+t_1^2)]dt_1 = 2\int_{-\infty}^{\infty} [u(t_2x-y)/(1+t_2^2)]dt_2 \to 2\pi u(-y),$$

$x \to 0$, $y \in (-\infty, \infty)$.

The first part of Lemma 1 is proved.

From [10]

$$2\pi\mathsf{L}(x+iy) = \int_{-\infty}^{\infty} [u(t)/(x+iy+it)]dt = \int_{-\infty}^{\infty} [u(t_1-y)/(x+it_1)]dt_1 =$$

$$= \int_{-\infty}^{\infty} [u(t_1-y)x/(x^2+t_1^2)]dt_1 - i\int_{-\infty}^{\infty} [u(t_1-y)t_1/(x^2+t_1^2)]dt_1 =$$

$$= \int_{-\infty}^{\infty} [u(xt_2-y)/(1+t_2^2)]dt_2 -$$

$$-i\int_{-\infty}^{\infty} (u(t_1-y)/t_1)dt_1 + i\int_{-\infty}^{\infty} [(u(t_1-y)/t_1)x^2/(x^2+t_1^2)]dt_1 \to$$

$$\to \pi u(-y) - i \int_{-\infty}^{\infty} (u(t_1 - y)/t_1) dt_1, \ y \in (-\infty, \infty);$$

we use

$$2x \int_{-\infty}^{\infty} \frac{(u(t_1 - y)/t_1)x}{(x^2 + t_1^2)} dt_1 = x \int_{-\infty}^{\infty} \frac{((u(t_1 - y) - u(-t_1 - y))/t_1)x}{(x^2 + t_1^2)} dt_1 =$$

$$= x \int_{-\infty}^{\infty} [2U(t_1 - y - 2\theta t_1)x/(x^2 + t_1^2)] dt_1 =$$

$$= x \int_{-\infty}^{\infty} [2U(xt_2 - y - 2\theta_1 xt_2)/(1 + t_2^2)] dt_2 = x\pi(U(-y) + o(1)) \to 0, \ x \to 0,$$

$$2U(-y) < \infty, \ y \in (-\infty, \infty), \ 0 \le \theta \le 1, \ 0 \le \theta_1 \le 1,$$

where we use notation $du(t)/dt = U(t)$, (if $du(t)/dt = U(t)$ is defined for all $t \in (-\infty, \infty)$), and the second part of Lemma 1 is proved.

If $u(0) = 0$, and $du(s)/ds|_{s=0+}, du(s)/ds|_{s=0-}$, $s \in (-\infty, \infty)$, values are defined, we use for $y = 0$

$$\int_{-\infty}^{\infty} [(u(t_1)/t_1)x^2/(x^2 + t_1^2)] dt_1 = x \int_{-\infty}^{0} [([u(t_1) - u(0)]/t_1)x/(x^2 + t_1^2)] dt_1 +$$

$$+ x \int_{0}^{\infty} [([u(t_1) - u(0)]/t_1)x/(x^2 + t_1^2)] dt_1 = \Delta(x) \to 0, \ x \to 0;$$

$$\operatorname{Im} 2\pi \mathsf{L}(x + i0) = - \int_{-\infty}^{\infty} (u(t_1 - 0)/t_1) dt_1 + \Delta(x) \to$$

$$\to - \int_{-\infty}^{\infty} ([u(t_1) - u(0)]/t_1) dt_1, \ x \to 0.$$

For $\mathsf{L}_1(-x + iy) \to \mathsf{L}_1(iy)$, $-x = \operatorname{Re} p \le 0$, the equalities are proved in the similar way. $\square$

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