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A NOTE ON THE EFFECT OF PROJECTIONS ON BOTH MEASURES AND THE GENERALIZATION OF q-DIMENSION CAPACITY

Abstract. In this paper, we are concerned both with the properties of the generalization of the L^q-spectrum relatively to two Borel probability measures and with the generalized q-dimension Riesz capacity. We are also interested in the study of their behaviors under orthogonal projections.

Key words: *orthogonal projection, Hausdorff measure and dimension, capacity, dimension spectra*

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1. Introduction. The notions of singularity exponents of spectra and generalized dimensions are the major components of the multifractal analysis. Recently, the projectional behavior of dimensions and multifractal spectra of measures have generated a large interest in the mathematical literature [1]–[9]. The first of these results was obtained by Marstrand in [7], where he proved that a Borel set E of the plane satisfies

$$\dim_H \pi_V(E) = \min\big(\dim_H E, 1\big), \qquad \text{for a.e. line } V,$$

where $\dim_H$ denotes the Hausdorff dimension (see [10]). This statement was later generalized by Kaufman [6]. Further more, in [8], Mattila proved that for a Borel measure μ of $\mathbb{R}^n$

$$\dim_H \pi_V(\mu) = \min\big(\dim_H \mu, m\big),$$

for a.e. vector subspace with dimension m.

Hunt and Kaloshin [11] introduced a new potential-theoretic definition of the dimension spectrum D_q of a probability measure for $q > 1$ and explained its relation with prior definitions. This definition was applied to prove that if $1 < q \leq 2$ and μ is a Borel probability measure with compact support in $\mathbb{R}^n$, then under almost every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, the q-dimension of the image of μ is $\min(m, D_q(\mu))$. In particular, the q-dimension of μ is preserved providing $m > D_q(\mu)$, for $1 < q \leq 2$. This results was later generalized by Bahroun and Bhouri in [12].

Readers familiar with potential theory will have encountered the definition of the s-capacity of a set E, i.e.

$$C_s(E) = \Big[\inf\big\{I_s(\mu): \quad \mu \in \mathcal{M}(E)\big\}\Big]^{-1}$$

where $\mathcal{M}(E)$ is the set of Radon measures μ with compact support on $E \subset \subset \mathbb{R}^n$ such that $0 < \mu(\mathbb{R}^n) < \infty$ and $I_s(\mu)$ is the s-energy of μ (see section 2). This makes us able to recall the defintion of capacitary dimension of a set E as follows

$$C(E) = \sup\big\{s: \ C_s(E) > 0\big\} = \inf\big\{s: \ C_s(E) = 0\big\}.$$

We point out, in contrast to the Hausdorff measures (denoted by $\mathcal{H}^s$), that any bounded set of $\mathbb{R}^n$ has finite s-capacity, for all $s > 0$. Especially, for $E \subset \mathbb{R}^n$ with $\mathcal{H}^s(E) < \infty$, we have $C_s(E) = 0$ and $C(E) \leq \dim_H(E)$. Matilla has compared, in [13, 14], the capacitary dimension of a Borel set E and its orthogonal projection.

In this paper, we study the behavior of the generalized L^q-spectrum relatively to two measures on $\mathbb{R}^n$ and compare it (the spectrum) to its correspondant under an orthogonal projection. Moreover, we focus on the generalized (s, q)-Riesz capacity of a subset of $\mathbb{R}^n$. We define the generalized q-dimension Riesz capacity and show that the q-dimension is preserved under almost every orthogonal projection.

2. Preliminaries. Let m be an integer with $0 < m < n$ and $G_{n,m}$ the Grassmannian manifold of all m-dimensional linear subspaces of $\mathbb{R}^n$. Denote by $\gamma_{n,m}$ the invariant Haar measure on $G_{n,m}$ such that $\gamma_{n,m}(G_{n,m}) = 1$. For $V \in G_{n,m}$, we define the projection map $\pi_V : \mathbb{R}^n \longrightarrow V$ as the usual orthogonal projection onto V. Then, the set $\{\pi_V, \ V \in G_{n,m}\}$ is compact in the space of all linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ and the identification of V with π_V induces a compact topology for $G_{n,m}$. Also, for a Borel probability measure μ with compact support $supp\mu \subset \mathbb{R}^n$ and for $V \in G_{n,m}$, we denote by μ_V, the projection of μ onto V, i.e.

$$\mu_V(A) = \mu(\pi_V^{-1}(A)) \quad \forall A \subseteq V.$$

Since μ is compactly supported and $supp\mu_V = \pi_V(supp\mu)$ for all $V \in$ $\in G_{n,m}$, then, for any continuous function $f\colon V \longrightarrow \mathbb{R}$, we have,

$$\int_V f d\mu_V = \int f(\pi_V(x))d\mu(x),$$

whenever these integrals exist.

From now on, we consider a compactly supported Borel probability measure μ on $\mathbb{R}^n$ with topological support S_μ and a Borel probability measure ν on S_μ. First, we give a generalization of the L^q-spectrum, for $q > 0$, relatively to two compactly supported Borel probability measures μ and ν, by

$$T_{\mu,\nu}(q) = \liminf_{r\to 0} \frac{1}{\log r} \log \int_{S_\mu} \mu(B(x,r))^q d\nu(x).$$

This quantity appears as a generalization of the q-spectral dimension defined, for $q > 0$, by

$$D_\mu(q) = \lim_{r\to 0} \frac{1}{q\log r} \log \int \mu(B(x,r))^q d\mu(x).$$

It is clear that, if $\mu = \nu$, then $T_{\mu,\mu}(q) = qD_\mu(q)$.

The q-spectral dimension $D_\mu(q)$ allows us to measure in certain cases the degree of singularity and in other ones the degree of regularity of measures (see, for example, [11], [15]–[21]).

The generalised L^q-dimension $T_{\mu,\nu}(q)$ is strictly related to the relative multifractal analysis, the multifractal variation measure, the relative Renyi dimention and multifractal variation for projections of measures developed by Olsen, Cole, Svetova and Selmi et al. [22]–[24], [2]. Other works were carried in this sense in probability and symbolic spaces [25]–[29]. We note that some researchers such as El Naschie [30]–[34], Ord et al. [35] have achieved many valuable results on the same subject and application.

For example if ν is a Gibbs measure for the measure μ, i.e. there exists a measure ν on S_μ, a constant $K > 1$ and $t_q \in \mathbb{R}$ such that for every $x \in S_\mu$ and every $0 < r < 0$

$$K^{-1}\mu(B(x,r))^q(2r)^{t_q} \le \nu(B(x,r)) \le K\mu(B(x,r))^q(2r)^{t_q}$$

$T_{\mu,\nu}(q)$ represents the C_μ function of Olsen's multifractal formalism [17]. In this case Bahroun and Bhouri compared the multifractal spectrum of a measure μ and its projections μ_V (see [12]).

In [12], Bahroun and Bhouri investigated the behaviour of the generalized L^q-spectrum relatively to μ and ν under orthogonal projection and proved that, for $q > 0$ and $\gamma_{n,m}$-almost every $V \in G_{n,m}$, we have the following

1) If $0 < q \leq 1$ and $T_{\mu,\nu}(q) \leq mq$, then $T_{\mu_V,\nu_V}(q) = T_{\mu,\nu}(q)$;
2) If $q > 1$ and $T_{\mu,\nu}(q) \leq m$, then $T_{\mu_V,\nu_V}(q) = T_{\mu,\nu}(q)$.

The (s,q)-energy of μ relatively of ν, denoted by $I_{s,q}(\mu,\nu)$, is given by

$$I_{s,q}(\mu,\nu) = \int_{S_\mu} \Big(\int \frac{d\mu(y)}{\mid x-y \mid^{\frac{s}{q}}} \Big)^q d\nu(x).$$

This definition allows the application of some techniques developed by Bahroun and Bhouri in [12].

Remark. *It's clear that, if $q = 1$ and $\mu = \nu$, then the (s,q)-energy of μ relatively of ν reduces to the standard notion of the s-energy of μ, given by*

$$I_s(\mu) = \int \int \mid x-y \mid^{-s} d\mu(y) d\mu(x).$$

Frostman [36] showed that the Hausdorff dimension of a Borel subset E of $\mathbb{R}^n$ is the supremum of the positive reals s for which there exists a Borel probability measure μ charging E and for which the s-energy of μ is finite. This characterization is used by Kaufmann [6] and Mattila [8] to prove their results on the preservation of the Hausdorff dimension.

Proposition 1 generalizes this notion to the q-dimension spectrum $T_{\mu,\nu}(q)$, for $q > 0$, and thus allows the methods of potential theory to be applied to this part of the spectrum.

Proposition 1. [12] *For $q > 0$, we have*

1) $T_{\mu,\nu}(q) = \inf \Big\{ s \geq 0 \ : \ I_{s,q}(\mu,\nu) = \infty \Big\}$,
2) $T_{\mu,\nu}(q) = \sup \Big\{ s \geq 0 \ : \ I_{s,q}(\mu,\nu) < \infty \Big\}$.

Minkowski dimensions:

For a non-empty bounded subset E of $\mathbb{R}^n$ we define the upper Minkowski dimension as

$$\overline{\Delta}(E) \quad = \quad \inf \Big\{ s : \limsup_{r \to 0} N_r(E)\, r^s = 0 \Big\}$$

where $0 < r < \infty$ and $N_r(E)$ is the least number of balls with radius r needed to cover E. In a similar manner, we define the lower Minkowski dimension as

$$\underline{\Delta}(E) \quad = \quad \inf\Big\{s: \ \liminf_{r\to 0} N_r(E)\, r^s = 0\Big\}.$$

It is clear that $\dim_H(E) \le \underline{\Delta}(E) \le \overline{\Delta}(E)$. Whenever these two limits are equal, we call the common value the Minkowski dimension of E.

3. Projection results. In the following theorem, we investigate the relationship between the generalization of the L^q-spectrum relatively to two Borel probability measures μ and ν and study their behaviors under orthogonal projections.

Theorem 1. *For $q > 0$ and $\gamma_{n,m}$-a.e. $V \in G_{n,m}$, the following holds*

1) *If $0 < q \le 1$, then*

$$\min\Big(\big(1+\underline{\Delta}(S_\mu)\big)^{-1} T_{\mu,\nu}(q), mq\Big) \le T_{\mu_V,\nu_V}(q) \le T_{\mu,\nu}(q).$$

2) *If $q > 1$, then*

$$\min\Big(\big(1+\underline{\Delta}(S_\mu)\big)^{-1} T_{\mu,\nu}(q), m\Big) \le T_{\mu_V,\nu_V}(q) \le T_{\mu,\nu}(q).$$

Remark.

1) *The techniques used in the proof of the first assertion are similar to those of the proof of Theorem 3.1 in [11]. The proof of the second one is almost identical to that of assertion 2 of Theorem 2.1 in [12]. For more details, the reader can see the appendix.*
2) *In the case where $\underline{\Delta}(S_\mu) = 0$, we obtain the main theorem of Bahroun and Bhouri in [12].*

Generalization of the q-dimension capacity. Let μ be a locally finite Borel measure $\mathbb{R}^n$ and ν is a Borel probability measure on S_μ. For $E \subset \mathbb{R}^n$, we set

$$\mathcal{M}(E) = \Big\{\mu: \ supp\mu \subset E, \ supp\mu \text{ is compact}, \ \mu(E) = 1\Big\}.$$

We define the (s, q)-Riesz capacity of E for $s > 0$ and $q > 0$, by

$$C_{s,q}(E) = \Big[\inf_{\nu\in\mathcal{M}(E)} \Big\{\inf_{\mu\in\mathcal{M}(E)} \big\{I_{s,q}(\mu,\nu)\big\}\Big\}\Big]^{-1}.$$

Note that, in this definition, the potentials and the energies may be infinite: we adopt the convention $\frac{1}{\infty} = 0$. The generalized capacity $C_{s,q}$ is an outer measure on $\mathbb{R}^n$, this means that a Borel set E has a positive (s,q)-capacity if and only if there are two measures μ and ν in $\mathcal{M}(E)$ such that $I_{s,q}(\mu,\nu) < \infty$.

The generalised capacity $C_{s,q}(E)$ plays a role in the study of potential theory, for example one can compare the generalised capacity to the variational q-capacity, the relative q-capacity and the Riesz capacity in metric spaces (see [37]–[39]).

Now, we define the generalized q-dimension Riesz capacity by

$$C_q(E) = \sup\big\{s :\ C_{s,q}(E) > 0\big\} = \inf\big\{s :\ C_{s,q}(E) = 0\big\}. \tag{1}$$

In the following theorem, we show that $C_q(E)$ is preserved under almost every orthogonal projection.

Theorem 2. *Let $E \subset \mathbb{R}^n$. For $q > 0$ and $\gamma_{n,m}$-a.e. $V \in G_{n,m}$, one has the following:*

1) *If $0 < q \le 1$ and $C_q(E) \le mq$, then $C_q(\pi_V(E)) = C_q(E)$.*

2) *If $q > 1$ and $C_q(E) \le m$, then $C_q(\pi_V(E)) = C_q(E)$.*

Remark. *We can define an other type of (s,q)-Riesz capacity of E by setting, for $s > 0$ and $q > 0$,*

$$\widetilde{C}_{s,q}(E) = \Big[\inf_{\mu\in\mathcal{M}(E)} \Big\{I_{s,q}(\mu,\mu)\Big\}\Big]^{-1}.$$

This allows us to define the generalized q-dimension Riesz capacity by

$$\widetilde{C}_q(E) = \sup\big\{s :\ \widetilde{C}_{s,q}(E) > 0\big\} = \inf\big\{s :\ \widetilde{C}_{s,q}(E) = 0\big\}.$$

1) *Taking $q = 1$, $\widetilde{C}_{s,q}$ reduces to the standard notion of the s-Riesz capacity. Particularly, we obtain $\widetilde{C}_1(E) = C(E)$.*
2) *The generalized q-dimension Riesz capacity is preserved under almost every orthogonal projection. This means that $\widetilde{C}_q(E)$ satisfies the assertions of Theorem 2.*

To prove Theorem 2, we need some preliminary lemmas.

Lemma 1. *Let $0 < q \le 1$ and $0 < s < mq$.*

1) *There is a constant c, depending only on m, n and s, such that for $E \subset \mathbb{R}^n$,*

$$\int_V C_{s,q}(\pi_V(E))^{-1} d\gamma_{n,m}(V) \leq c\, C_{s,q}(E)^{-1}.$$

2) *If $C_{s,q}(E) > 0$, then $C_{s,q}(\pi_V(E)) > 0$, for $\gamma_{n,m}$-a.e. $V \in G_{n,m}$.*

Proof. We will prove Assertion 1). The second is its immediate consequence.

Let μ and ν be two compactly supported Radon measures on $\mathbb{R}^n$, such that

$$S_\mu, S_\nu \subset E \quad \text{and} \quad \mu(E) = \nu(E) = 1.$$

Then, μ_V and ν_V are two compactly supported Radon measures on $\mathbb{R}^m$, such that

$$S_{\mu_V}, S_{\nu_V} \subset \pi_V(E) \quad \text{and} \quad \mu_V(\pi_V(E)) = \nu_V(\pi_V(E)) = 1.$$

Consequently, $\quad C_{s,q}(\pi_V(E))^{-1} \leq I_{s,q}(\mu_V, \nu_V)$.

By Fubini-Tonelli's theorem and the fact that $0 < q \leq 1$, we have

$$\begin{aligned}
\int_V C_{s,q}(\pi_V(E))^{-1} d\gamma_{n,m}(V) &\leq \int_V I_{s,q}(\mu_V, \nu_V) d\gamma_{n,m}(V) = \\
&= \int \int_V \left(\int \frac{d\mu(y)}{\mid \pi_V(x-y) \mid^{\frac{s}{q}}} \right)^q d\gamma_{n,m}(V) d\nu(x) \leq \\
&\leq \int \left(\int_V \int \frac{d\mu(y)}{\mid \pi_V(x-y) \mid^{\frac{s}{q}}} d\gamma_{n,m}(V) \right)^q d\nu(x) = \\
&= \int \left(\int \int_V \frac{d\gamma_{n,m}(V)}{\mid \pi_V(x-y) \mid^{\frac{s}{q}}} \mu(y) \right)^q d\nu(x).
\end{aligned}$$

Since $s < mq$,

$$\int_V \frac{d\gamma_{n,m}(V)}{\mid \pi_V(x-y) \mid^{\frac{s}{q}}} \leq \frac{c}{\mid x-y \mid^{\frac{s}{q}}} \tag{2}$$

where c is a constant depending only on m, n and s (see corollary 3.12 in [13]).

Hence

$$\int_V C_{s,q}(\pi_V(E))^{-1} d\gamma_{n,m}(V) \leq c\, I_{s,q}(\mu, \nu).$$

By taking the infimum over all such μ and ν, we are done. $\square$

Lemma 2. *Let $q>1$ and $0<s<m$.*

1) *There is a constant c depending only on m,n and s, such that for $E\subset\mathbb{R}^n$,*
$$\int_V C_{s,q}(\pi_V(E))^{-1}d\gamma_{n,m}(V)\le c\,C_{s,q}(E)^{-1}.$$
2) *If $C_{s,q}(E)>0$, then $C_{s,q}(\pi_V(E))>0$, for $\gamma_{n,m}$-a.e. $V\in G_{n,m}$.*

Proof. By Fubini-Tonelli's theorem, Minkowski's inequality, inequality (2) and the fact that $q>1$, we get
$$\int_V C_{s,q}(\pi_V(E))^{-1}d\gamma_{n,m}(V)\le\int_V I_{s,q}(\mu_V,\nu_V)d\gamma_{n,m}(V)=$$
$$=\int\int_V\left(\int\frac{d\mu(y)}{\mid\pi_V(x-y)\mid^{\frac{s}{q}}}\right)^q d\gamma_{n,m}(V)d\nu(x)\le$$
$$\le\int\left(\int\left[\int_V\frac{d\gamma_{n,m}(V)}{\mid\pi_V(x-y)\mid^s}\right]^{\frac{1}{q}}d\mu(y)\right)^q d\nu(x)\le c\,I_{s,q}(\mu,\nu).$$
$\square$

Lemma 3.

1) *Let $0<q\le 1$ and $0<s<mq$. There is a constant c depending only on m,n and s such that, for $E\subset\mathbb{R}^n$, we have*
$$c^{-1}\,C_{s,q}(E)\le\int_V C_{s,q}(\pi_V(E))d\gamma_{n,m}(V)\le C_{s,q}(E).$$
2) *Let $q>1$ and $0<s<m$. There is a constant c_1 depending only on m,n and s such that for $E\subset\mathbb{R}^n$, we have*
$$c_1^{-1}\,C_{s,q}(E)\le\int_V C_{s,q}(\pi_V(E))d\gamma_{n,m}(V)\le C_{s,q}(E).$$

Proof.
1) Since $\gamma_{n,m}$ is an invariant Radon probability measure on $G_{n,m}$, using Hölder's inequality and Lemma 1, we obtain
$$1=\left(\int_V d\gamma_{n,m}(V)\right)^2=$$

$$= \left(\int_V \left(C_{s,q}(\pi_V(E))\right)^{\frac{1}{2}} \left(C_{s,q}(\pi_V(E))\right)^{-\frac{1}{2}} d\gamma_{n,m}(V) \right)^2 \leq$$

$$\leq \int_V C_{s,q}(\pi_V(E)) d\gamma_{n,m}(V) \int_V \left(C_{s,q}(\pi_V(E))\right)^{-1} d\gamma_{n,m}(V) \leq$$

$$\leq c \left(C_{s,q}(E)\right)^{-1} \int_V C_{s,q}(\pi_V(E)) d\gamma_{n,m}(V).$$

Hence

$$c^{-1}\, C_{s,q}(E) \leq \int_V C_{s,q}(\pi_V(E)) d\gamma_{n,m}(V).$$

Now, Fix $V \in G_{n,m}$. For all $s > 0$ and $q > 0$, we have

$$I_{s,q}(\mu_V, \nu_V) = \int \left(\int \frac{d\mu(u)}{\mid u - v \mid^{\frac{s}{q}}} \right)^q d\nu(v) =$$

$$= \int \left(\int \frac{d\mu(y)}{\mid \pi_V(x-y) \mid^{\frac{s}{q}}} \right)^q d\nu(x) \geq \int \left(\int \frac{d\mu(y)}{\mid x - y \mid^{\frac{s}{q}}} \right)^q d\nu(x) =$$

$$= I_{s,q}(\mu, \nu) \geq C_{s,q}(E)^{-1}.$$

By taking the infimum over all the measures μ_V and ν_V, we get

$$C_{s,q}(\pi_V(E)) \leq C_{s,q}(E).$$

2) The proof is similar to that of assertion 1). □

Proof of Theorem 2.
1) Take $0 < q \leq 1$ and $s < C_q(E) \leq mq$. From (1), we have $C_{s,q}(E) > 0$. Lemma 1 yields $C_{s,q}(\pi_V(E)) > 0$ for $\gamma_{n,m}$-a.e. $V \in G_{n,m}$. The definition of C_q implies that $s \leq C_q(\pi_V(E))$ for $\gamma_{n,m}$-a.e. V.
The second inequality is a consequence of assertion 1) of Lemma 3.
Assertion 2) is a consequence of Lemma 2 and Assertion 2) of Lemma 3.□

4. Appendix.

Proof of Theorem 1. We first need the following lemma.

Lemma 4. *Suppose that $s < m$ and there exists a constant C, depending only on n, m and s, such that for all $x, y \in \mathbb{R}^n \setminus \{0\}$ and $\rho > 0$,*

$$\int_V \frac{d\gamma_{n,m}(V)}{\mid \pi_V(x-y) \mid^s} \leq \frac{C}{\min\left\{ \mid x - y \mid, 1 \right\}^{s(1+\rho)}}.$$

Proof of lemma 4. It is a consequence of corollary 3.12 in [13]. □

1) Fix $0 < q \leq 1$ and choose $\rho > \underline{\Delta}(S_\mu)$.

We will prove that for $0 \leq s < \big(1+\rho\big)^{-1}\underline{T}_{\mu,\nu}(q) \leq mq$,

$$I_{s(1+\rho),q}(\mu,\nu) < \infty \quad \Rightarrow \quad I_{s,q}(\mu_V,\nu_V) < \infty,$$

for $\gamma_{n,m}$-a.e. $V \in G_{n,m}$. The result follows from the fact that ρ can be arbitrarily chosen close to $\underline{\Delta}(S_\mu)$.

The case where $mq \leq \big(1+\underline{\Delta}(S_\mu)\big)^{-1}\underline{T}_{\mu,\nu}(q)$ is similar.

Computing the (s,q)-energy of μ_V relatively of ν_V, we have

$$I_{s,q}(\mu_V,\nu_V) = \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\frac{d\mu_V(u)}{\mid u-v\mid^{\frac{s}{q}}}\right)^q d\nu_V(v) =$$

$$= \int\left(\int\frac{d\mu(y)}{\mid \pi_V(x-y)\mid^{\frac{s}{q}}}\right)^q d\nu(x).$$

We integrate the energy over $V \in G_{n,m}$. Thanks to the fact that $0 < q \leq 1$ and by the Fubini-Tonelli's theorem, we have

$$\int_V I_{s,q}(\mu_V,\nu_V)d\gamma_{n,m}(V) =$$

$$= \int_V\int\left(\int\frac{d\mu(y)}{\mid \pi_V(x-y)\mid^{\frac{s}{q}}}\right)^q d\nu(x)d\gamma_{n,m}(V) =$$

$$= \int\int_V\left(\int\frac{d\mu(y)}{\mid \pi_V(x-y)\mid^{\frac{s}{q}}}\right)^q d\gamma_{n,m}(V)d\nu(x) \leq$$

$$\leq \int\left(\int_V\int\frac{d\mu(y)}{\mid \pi_V(x-y)\mid^{\frac{s}{q}}}d\gamma_{n,m}(V)\right)^q d\nu(x) =$$

$$= \int\left(\int\left[\int_V\frac{d\gamma_{n,m}(V)}{\mid \pi_V(x-y)\mid^{\frac{s}{q}}}\right]d\mu(y)\right)^q d\nu(x).$$

Now, applying Lemma 4 to the preceding inequality, we get

$$\int_V I_{s,q}(\mu_V,\nu_V)d\gamma_{n,m}(V) \leq$$

$$\leq \int \Big(\int \Big[\frac{C}{\min\big\{\mid x-y\mid, 1\big\}^{\frac{s(1+\rho)}{q}}} \Big] d\mu(y) \Big)^q d\nu(x) < +\infty.$$

Thus,

$$I_{s(1+\rho),q}(\mu,\nu) < \infty \quad \Rightarrow \quad I_{s,q}(\mu_V,\nu_V) < \infty,$$

for $\gamma_{n,m}$-a.e. $V \in G_{n,m}$.
2) Fix $q > 1$ and choose $\rho > \underline{\Delta}(S_\mu)$. We will show that, under the assumption $0 \leq s < \big(1+\rho\big)^{-1} \underline{T}_{\mu,\nu}(q)$, we have

$$I_{s(1+\rho),q}(\mu,\nu) < \infty \quad \Rightarrow \quad I_{s,q}(\mu_V,\nu_V) < \infty,$$

for $\gamma_{n,m}$-a.e. $V \in G_{n,m}$.

By Fubini-Tonelli's theorem and Minkowski's inequality as well as the fact that $q > 1$, we have

$$\begin{aligned} &\int_V I_{s,q}(\mu_V,\nu_V) d\gamma_{n,m}(V) = \\ &= \int \int_V \Big(\int \frac{d\mu(y)}{\mid \pi_V(x-y)\mid^{\frac{s}{q}}} \Big)^q d\gamma_{n,m}(V) d\nu(x) \leq \\ &\leq \int \Big(\int \Big[\int_V \frac{d\gamma_{n,m}(V)}{\mid \pi_V(x-y)\mid^s} \Big]^{\frac{1}{q}} d\mu(y) \Big)^q d\nu(x). \end{aligned} \tag{3}$$

Applying lemma 4 to the inequality (3), we get

$$\int_V I_{s,q}(\mu_V,\nu_V) d\gamma_{n,m}(V) \leq$$

$$\leq \int \Big(\int \Big[\frac{C}{\min\big\{\mid x-y\mid, 1\big\}^{\frac{s(1+\rho)}{q}}} \Big] d\mu(y) \Big)^q d\nu(x) < +\infty,$$

which shows that $I_{s,q}(\mu_V,\nu_V)$ is finite for $\gamma_{n,m}$-a.e. $V \in G_{n,m}$. This proves the result. □

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