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ON SOLVABILITY OF ONE DIFFERENCE EQUATION

Abstract. We consider a system of difference equation similar to those that appear as a description of cumulative sums. Using Hamel bases, we construct pathological solutions to this system for constant right-hand sides. Also we show that bounded solutions do not exist for non-zero right-hand sides, while only constants can be solutions in the homogeneous case.

Key words: *difference equations, Hamel basis, pathological solutions, existence and uniqueness*

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1. Introduction. Let $z > 0$ and $b > z + 1$ be some real numbers. We consider functions $j_s = j(s)$ on $[0, b]$, s is the independent variable. Let $p \geq 0$, $q \geq 0$, $p + q = 1$ be real numbers. We consider the following system of difference equations:

$$j_s = j_{s+z} + r_s, \qquad\qquad s < 1, \qquad (1)$$
$$j_s = pj_{s-1} + qj_{s+z} + r_s, \qquad 1 \leq s \leq b - z, \qquad (2)$$
$$j_s = j_{s-1} + r_s, \qquad\qquad b - z < s \leq b. \qquad (3)$$

Here r_s is some function of s. Similar systems describe dynamics of cumulative sums [2, 4]. We studied a system of this kind in [3]. However, (1)–(3) is different: the factor at j in the right-hand side of (1) and (3) is 1, not p or q. This system describes special cases of random walk in a band when the walking particle is repelled from the boundaries.

We are going to study existence and uniqueness of a solution to this system. We show that there are multiple solutions provided that z is irrational, but these solutions are quite pathological and hardly can be used in any application.

2. Operator form of the system. Let us define operator T by

$$
\begin{aligned}
Tj_s &= j_{s+z}, & s &< 1, & (4)\\
Tj_s &= pj_{s-1} + qj_{s+z}, & 1 &\le s \le b - z, & (5)\\
Tj_s &= j_{s-1}, & b - z &< s \le b, & (6)
\end{aligned}
$$

and operator F as $E - T$ with E as the identity. Now the equation can be written as $Fj = r$ or $j - Tj = r$.

Now let us note that any constant function is an eigenfunction of T for the eigenvalue 1. Also any constant function belongs to the core of F. Therefore the solution j_s to the problem, if one exists, is not unique: $j_s + \text{const}$ is also a solution. However, the problem $Fj_s = r_s$ may lack solutions for some right-hand sides r_s.

Solutions to the problem with $r_s = \text{const}$, if exist, possess some kind of self-similarity:

Theorem 1. *If j_s is a solution to $Fj_s = c$, $c = \text{const}$, then Tj_s is also a solution.*

Proof. Operators T and $F = E - T$ commute; so, $Fj = c$ implies $TFj = Tc = c$ and, due to commuting, $FTj = c$. $\square$

3. Bounded solutions. Let us show that for positive right-hand sides there are no bounded solutions.

Theorem 2. *System (1)–(3) has no bounded solutions provided that $r_s \ge \bar{r} > 0$.*

Proof. First assume that $j^* = j(s^*)$ is the minimal value of j_s on $[0, b]$. If $1 \le s^* \le b - z$, then, due to (2),

$$
j^* = j_{s^*} = pj_{s^*-1} + qj_{s^*+z} + r_{s^*} \ge pj^* + qj^* + r^* = j^* + r_{s^*} > j^*,
$$

which is a contradiction. The similar contradiction appears if we assume that $s^* > b - z$ or $s^* < 1$. This means that a solution, if it exists, does not have the minimal value. Let us consider the case of a bounded solution with no minimum.

As the solution is bounded, $j_s > B$ for some B; let B be the infimum, so that for any $\varepsilon > 0$ $j_s < B + \varepsilon$ in at least one s.

Choose a positive $\epsilon < 1$ and find a $j^* = j_{s^*} < B + \epsilon$. Then in the similar way get $j^* \ge B + \bar{r}$ which contradicts $j^* < B + \epsilon < B + \bar{r}$. $\square$

The similar argument shows that in the homogeneous case $r_s = 0$ continuous solutions are only constants provided that z is irrational.

Theorem 3. *Let z be irrational and $r_s = 0$ for all $s \in [0, b]$. Then if j_s is a continuous solution to (1)–(3), $j_s = $ const.*

Proof. Indeed, a continuous function on a segment must have maximal and minimal values; if $j^* = j_{s^*}$ is the minimum, then $j_{s^*+z} = j_{s^*-1} = j^*$. Continue this argument to see that $j_s = j^*$ on a dense countable subset of $[0, b]$. It is dense because it consists of points $s^* + Qz - P$ with integers $P, Q \geq 0$ and Dirichlet's approximation theorem guarantees approximation of any point of $[0, b]$ with arbitrary precision. By continuity j_s is constant on $[0, b]$. $\square$

Obviously the problem can have solutions for specially chosen r_s; in fact for $r_s = Fj_s$ for any chosen j_s. The proved result yields a corollary:

Corollary. *For any function j_s its image $r_s = Fj_s$ cannot be "absolutely positive": for any $\epsilon > 0$ $r_s - \epsilon < 0$ in some s. Also it cannot be "absolutely negative".*

Here is another proof for constant right-hand sides.

Theorem 4. *Bounded solutions to $Fj = 1$ do not exist.*

Proof. Note that T does not decrease the upper bound of its argument: if $j_s > M$, $Tj_s > M$. This is easily checked directly. Now, let j_s be a bounded solution to $Fj = 1$: $Tj_s = j_s - 1$. This decreases the upper bound of j_s and thus provides a contradiction. $\square$

Let us consider the case of a rational z. Choose any $s \in [0, b]$ and consider the set U of all points $s + mz + n \in [0, b]$ with integer n and m. The smallest possible distance between these points is bounded from below by N^{-1} where N is the smallest natural in $z = M/N$ (M is also natural). So the set U is finite. Operators T and F transform U into U and therefore are linear finite dimensional mappings. Matrix T is degenerate. The results proved above can be made stronger:

Theorem 5. *Let z be rational. System (1)–(3) has no bounded solutions provided that $r_s \geq 0$ (or $r_s \leq 0$). For $r_s = 0$ solutions are constant at any U.*

Proof. Proof of theorem 2 remains valid for the considered case. Minimal value is obtained in some point of any U. The only difference is possibility

of minimal value at some s^* with $r_{s^*} = 0$. But then j_{s^*-1} and j_{s^*+z} are also minimal (provided that the arguments belong to $[0, b]$). Then $r_s = 0$ also in these points. Continue to prove that $r_s = 0$ in all $s \in U$ for any U, so that $r_s = 0$ everywhere. In this case solution exists, but is constant on any U; though, it can be different for different sets U with no common points. $\square$

4. The Hamel solutions. Now let us construct a solution for the special case $r_s = 1$. Choose any Hamel basis on R^1 that contains 1 and z. This is a set H of real numbers, $(1, z \in H)$ such that any $x \in R^1$ is a finite linear combination of numbers from H with rational coefficients. It is known [1] that such basis exists if the lemma of choice is accepted. Let us denote the coefficient at $h \in H$ for expansion of x by $x_{\langle h \rangle}$.

Consider the function $j_s = s_{\langle 1 \rangle} - s_{\langle z \rangle}$. Substitute it to (2):

$$s_{\langle 1 \rangle} - s_{\langle z \rangle} = s_{\langle 1 \rangle} - (s_{\langle z \rangle} + 1) + 1, \quad s < 1,$$
$$s_{\langle 1 \rangle} - s_{\langle z \rangle} = p(s_{\langle 1 \rangle} - 1 - s_{\langle z \rangle}) + q(s_{\langle 1 \rangle} - (s_{\langle z \rangle} + 1)) + 1, \quad 1 \le s \le b - z,$$
$$s_{\langle 1 \rangle} - s_{\langle z \rangle} = s_{\langle 1 \rangle} - 1 - s_{\langle z \rangle} + 1, \quad b - z < s \le b,$$

which is true. So the constructed function indeed is a soluton, together with any $j_s + \text{const}$.

In the homogeneous case $r_s = 0$ any $j_s = s_{\langle h \rangle}$ is a solution for any $h \in H$ except 1 and z.

Let us note a few points as a conclusion.

- Although we have constructed numerous solutions to the system with special right-hand sides, they are pathological and thus hardly useful, being discontinuous in every point, unbounded in any neighbourhood of each point, and having only rational values in all points. It is hardly possible to evaluate the solution, except for special points.

- In rational points s the constructed solution $j_s = s$; however, $j_s = s$ for any s is not a solution.

- For practical purposes it is sufficient to evaluate the solution in points $Pz - Q$ for integer $P, Q \ge 0$: $j_{Pz-Q} = Q - P$.

- For different Hamel bases we get the same solutions; however, there are different solutions (up to adding a constant) for the same Hamel basis: $j_s + s_{\langle h \rangle}$ is obviously a solution for any $h \in H$ except 1 and z, provided that j_s is a solution.

- Choosing any rational number instead of 1 in H provides different solutions; this can be used to provide desired properties of solutions. The same is true for replacing z by az for a rational a.

- Existence of such solutions in the case of non-constant right-hand sides is still an open question.

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