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EXTENSION OF THE REFINED GIBBS' INEQUALITY

Abstract. In this note, we give an extension of the refined Gibbs' inequality containing arithmetic and geometric means. As an application, we obtain converse and refinement of the arithmetic-geometric mean inequality.

Key words: *arithmetic-geometric mean inequality, Jensen's inequality, log-function, Gibbs' inequality*

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1. Introduction. Let $n \geq 2$ and $w_1, \ldots, w_n$ be non-negative real numbers such that $\sum_{j=1}^{n} w_j = 1$. Let A_n and G_n denote the weighted arithmetic and geometric means of the positive real numbers $x_1, \ldots, x_n$, that is,

$$A_n = \sum_{j=1}^{n} w_j x_j \quad \text{and} \quad G_n = \prod_{j=1}^{n} x_j^{w_j}.$$

The arithmetic-geometric mean inequality asserts that

$$A_n \geq G_n.$$

For more details about the arithmetic-geometric mean inequality the reader is referred to [3]–[8].

It is interesting that some classical inequalities such as arithmetic-geometric mean inequality (see [9]), the Jensen inequality (see [10]), the Hölder inequality (see [11]) play an important role in information sciences.

Let $p_j, q_j > 0$ $(j = 1, \ldots, n)$ and $\sum_{j=1}^{n} p_j = \sum_{j=1}^{n} q_j$. Then,

$$0 \leq \sum_{j=1}^{n} p_j \log \frac{p_j}{q_j}$$

with equality if and only if $p_j = q_j$ $(j = 1, \dots, n)$. This inequality is known in literature as the Gibbs' inequality (see [8, p. 382]). The Gibbs' inequality has many applications in information theory and also in mathematical statistics.

In 2004 Halliwell and Mercer [6] presented the following refinement of the Gibbs inequality:

Theorem 1. *Let p_j, q_j $(j = 1, \dots, n)$ be positive real numbers satisfying $\sum_{j=1}^n p_j = \sum_{j=1}^n q_j$. Then,*

$$\sum_{j=1}^n \frac{q_j(q_j - p_j)^2}{q_j^2 + \hat{M}_j} \le \sum_{j=1}^n p_j \log \frac{p_j}{q_j} \le \sum_{j=1}^n \frac{q_j(q_j - p_j)^2}{q_j^2 + \hat{m}_j} \tag{1}$$

where $\hat{m}_j = \min(p_j^2, q_j^2)$, $\hat{M}_j = \max(p_j^2, q_j^2)$ $(j = 1, \dots, n)$.

In 2014 H. Alzer [2] proved the following refinement of (1):

Theorem 2. *Let $\alpha, \beta \in \mathbb{R}$. Then, inequalities*

$$\sum_{j=1}^n \frac{q_j(q_j - p_j)^2}{q_j^2 + \hat{m}_j^\alpha \hat{M}_j^{1-\alpha}} \le \sum_{j=1}^n p_j \log \frac{p_j}{q_j} \le \sum_{j=1}^n \frac{q_j(q_j - p_j)^2}{q_j^2 + \hat{m}_j^\beta \hat{M}_j^{1-\beta}} \tag{2}$$

hold for positive real numbers p_j, q_j $(j = 1, \dots, n)$ with $\sum_{j=1}^n p_j = \sum_{j=1}^n q_j$ if and only if $\alpha \le 1/3$ and $\beta \ge 2/3$.

In this note we give an extension of (2) containing arithmetic and geometric means. As an application, we obtain refinement of the left-hand inequality in (1) and we also give a converse of the arithmetic-geometric mean inequality.

2. Main results. In order to prove our main results, we need the following lemmas.

Lemma 1. (see [2]) (i) *If $0 < x \le 1$, then*

$$x - 1 - \frac{(x-1)^2}{x + x^{1/3}} \le \log x \le x - 1 - \frac{(x-1)^2}{x+1} \tag{3}$$

with equality if and only if $x = 1$.

(ii) *If $x > 1$, then*

$$x - 1 - \frac{(x-1)^2}{x+1} < \log x < x - 1 - \frac{(x-1)^2}{x + x^{1/3}}. \tag{4}$$

Lemma 2. *Let* $f_a(x) := \dfrac{x(x-a)^2}{a(x^2+\max\{x^2,a^2\})} + \log x$, $a>0$. *Then* f_a *is a concave function on* $(0,+\infty)$.

Proof. If $x \geq a$, then $f_a(x) = \frac{(x-a)^2}{2ax} + \log x$, and consequently,

$$f_a''(x) = \frac{a-x}{x^3} \leq 0.$$

On the other hand, if $0<x<a$, then $f_a(x) = \dfrac{x(x-a)^2}{a(x^2+a^2)} + \log x$, which yields

$$f_a''(x) = -\frac{a^6+7a^4x^2-9a^2x^4+x^6}{x^2(x^2+a^2)^3} = -\frac{(a^2-x^2)(a^4+8a^2x^2-x^4)}{x^2(x^2+a^2)^3} < 0.$$

Therefore, the function $f_a(x)$ is concave for $x>0$. $\square$

Lemma 3. (see [1]) *Let* f_a *be as defined in Lemma 2,* $k \in \{2,\ldots,n-1\}$, *and*

$$s_k := \max_{1\leq\mu_1<\cdots<\mu_k\leq n}\left[\left(\sum_{j=1}^k w_{\mu_j}\right) f_{A_n}\left(\frac{\sum\limits_{j=1}^k w_{\mu_j}x_{\mu_j}}{\sum\limits_{j=1}^k w_{\mu_j}}\right) - \sum_{j=1}^k w_{\mu_j} f_{A_n}(x_{\mu_j})\right].$$

Then,

$$0 \leq s_2 \leq s_3 \leq \cdots \leq s_{n-1}.$$

First, we give an extension of (2) based on the corresponding result in [2].

Theorem 3. *Let* $\alpha,\beta\in\mathbb{R}$ *and* $m_j = \min(x_j^2, A_n^2)$, $M_j = \max(x_j^2, A_n^2)$ $(j=1,\ldots,n)$. *Then inequalities*

$$\frac{1}{A_n}\sum_{j=1}^n \frac{w_jx_j(x_j-A_n)^2}{x_j^2+m_j^\alpha M_j^{1-\alpha}} \leq \log A_n - \log G_n \leq \frac{1}{A_n}\sum_{j=1}^n \frac{w_jx_j(x_j-A_n)^2}{x_j^2+m_j^\beta M_j^{1-\beta}} \tag{5}$$

hold if and only if $\alpha \leq 1/3$ *and* $\beta \geq 2/3$.

Proof. We follow the method of proof given in [2].
(Necessity) Since the sums on the left-hand side and on the right-hand side of (5) are increasing with respect to α and β, respectively, it suffices

to prove (5) for $\alpha = 1/3$ and $\beta = 2/3$. Therefore, substituting x_j/A_n instead of x in (3) and (4), then multiplying by w_j, and summing, we obtain

$$\frac{1}{A_n}\sum_{x_j\le A_n}\left(w_jx_j - w_jA_n - \frac{w_jx_j(x_j-A_n)^2}{x_j^2+m_j^{2/3}M_j^{1/3}}\right) =$$

$$= \frac{1}{A_n}\sum_{x_j\le A_n}\left(w_jx_j - w_jA_n - \frac{w_j(x_j-A_n)^2}{x_j+x_j^{1/3}A_n^{2/3}}\right) \le$$

$$\le \sum_{x_j\le A_n} w_j\log\frac{x_j}{A_n} \le \frac{1}{A_n}\sum_{x_j\le A_n}\left(w_jx_j - w_jA_n - \frac{w_j(x_j-A_n)^2}{x_j+A_n}\right) =$$

$$= \frac{1}{A_n}\sum_{x_j\le A_n}\left(w_jx_j - w_jA_n - \frac{w_jx_j(x_j-A_n)^2}{x_j^2+m_j^{1/2}M_j^{1/2}}\right)$$

and

$$\frac{1}{A_n}\sum_{x_j> A_n}\left(w_jx_j - w_jA_n - \frac{w_jx_j(x_j-A_n)^2}{x_j^2+m_j^{1/2}M_j^{1/2}}\right) =$$

$$= \frac{1}{A_n}\sum_{x_j> A_n}\left(w_jx_j - w_jA_n - \frac{w_j(x_j-A_n)^2}{x_j+A_n}\right) < \sum_{x_j> A_n} w_j\log\frac{x_j}{A_n} <$$

$$< \frac{1}{A_n}\sum_{x_j> A_n}\left(w_jx_j - w_jA_n - \frac{w_j(x_j-A_n)^2}{x_j+x_j^{1/3}A_n^{2/3}}\right) =$$

$$= \frac{1}{A_n}\sum_{x_j> A_n}\left(w_jx_j - w_jA_n - \frac{w_jx_j(x_j-A_n)^2}{x_j^2+m_j^{1/3}M_j^{2/3}}\right).$$

Further, utilizing inequalities $m_j^{2/3}M_j^{1/3} \le m_j^{1/2}M_j^{1/2} \le m_j^{1/3}M_j^{2/3}$, we get

$$\frac{1}{A_n}\sum_{x_j\le A_n}\left(w_jx_j - w_jA_n - \frac{w_jx_j(x_j-A_n)^2}{x_j^2+m_j^{2/3}M_j^{1/3}}\right) \le \sum_{x_j\le A} w_j\log\frac{x_j}{A_n} \le$$

$$\le \frac{1}{A_n}\sum_{x_j\le A_n}\left(w_jx_j - w_jA_n - \frac{w_jx_j(x_j-A_n)^2}{x_j^2+m_j^{1/3}M_j^{2/3}}\right)$$

and

$$\frac{1}{A_n}\sum_{x_j>A_n}\left(w_jx_j - w_jA_n - \frac{w_jx_j(x_j-A_n)^2}{x_j^2+m_j^{2/3}M_j^{1/3}}\right) < \sum_{x_j>A_n} w_j\log\frac{x_j}{A_n} <$$

$$< \frac{1}{A_n}\sum_{x_j>A_n}\left(w_jx_j - w_jA_n - \frac{w_jx_j(x_j-A_n)^2}{x_j^2+m_j^{1/3}M_j^{2/3}}\right).$$

Combining this together, we obtain

$$\frac{1}{A_n}\sum_{j=1}^n \frac{w_jx_j(x_j-A_n)^2}{x_j^2+m_j^{2/3}M_j^{1/3}} \le \log A_n - \log G_n \le \frac{1}{A_n}\sum_{j=1}^n \frac{w_jx_j(x_j-A_n)^2}{x_j^2+m_j^{1/3}M_j^{2/3}},$$

as desired.
(Sufficiency) Let $s, t \in \mathbb{R}$ with $1 < t < s+1$. Set

$$x_1 = \frac{s+1-t}{s},\ x_2 = t,\ x_j = 1\ (j = 3, \dots, n);$$

$w_1 = \dfrac{s}{s+1+(n-2)t}$, $w_2 = \dfrac{1}{s+1+(n-2)t}$, $w_j = \dfrac{t}{s+1+(n-2)t}$ $(j = 3, \dots, n)$. Now, the same computation as in the proof of Theorem 2 (see [2]), provides that $\alpha \le 1/3$ and $\beta \ge 2/3$. $\square$

Remark. *Putting* $x_j = \frac{q_j}{p_j}, w_j = p_j/\sum_{j=1}^n q_j \quad (j = 1, \dots, n)$, *where* $\sum_{j=1}^n p_j = \sum_{j=1}^n q_j$, *in (5), we get (2).*

By using Theorem 3, we obtain the following consequence.

Theorem 4. *The inequality*

$$A_n - G_n \le \sum_{j=1}^n \frac{w_jx_j(x_j-A_n)^2}{x_j^2+m_j^{\beta}M_j^{1-\beta}} \tag{6}$$

holds for $\beta \ge 2/3$.

Proof. By the mean value theorem and Theorem 3, we have

$$A_n - G_n = \exp(\log A_n) - \exp(\log G_n) \le A_n(\log A_n - \log G_n) \le$$

$$\le \sum_{j=1}^n \frac{w_jx_j(x_j-A_n)^2}{x_j^2+m_j^{\beta}M_j^{1-\beta}},$$

which completes the proof. □

Our next result refines sign of the first inequality in (5) for $\alpha = 0$.

Theorem 5. *If* $C = \dfrac{1}{A_n}\displaystyle\sum_{j=1}^{n}\frac{w_j x_j (x_j - A_n)^2}{x_j^2 + M_j}$, *then*

$$C \le C + s_2 \le C + s_3 \le \cdots \le C + s_{n-1} \le \log(A_n) - \log(G_n). \quad (7)$$

Equality occurs if and only if all x_j are equal.

Proof. By Lemma 3 we have

$$C \le C + s_2 \le C + s_3 \le \cdots \le C + s_{n-1}.$$

Now, we have to prove the last inequality in (7). Let's choose an arbitrary $x_{\mu_j} \in \{x_1, \ldots, x_n\}$, $1 \le \mu_1 < \mu_2 < \cdots < \mu_{n-1} \le n$, with the corresponding weights $w_{\mu_j} \in \{w_1, \ldots, w_n\}$, and let $x_{\mu_n} = \{x_1, \ldots, x_n\} \setminus \{x_{\mu_1}, \ldots, x_{\mu_{n-1}}\}$. Now, utilizing the first inequality in (5) with $\alpha = 0$, we obtain

$$\log(A_n) = \log\left(w_{\mu_n} x_{\mu_n} + \left(\sum_{j=1}^{n-1} w_{\mu_j}\right) \frac{\sum_{j=1}^{n-1} w_{\mu_j} x_{\mu_j}}{\sum_{j=1}^{n-1} w_{\mu_j}} \right) \ge$$

$$\ge \frac{1}{A_n} \frac{w_{\mu_n} x_{\mu_n} (x_{\mu_n} - A_n)^2}{x_{\mu_n}^2 + \max(x_{\mu_n}^2, A_n^2)} +$$

$$+ \frac{1}{A_n} \frac{\left(\sum_{j=1}^{n-1} w_{\mu_j}\right) \frac{\sum_{j=1}^{n-1} w_{\mu_j} x_{\mu_j}}{\sum_{j=1}^{n-1} w_{\mu_j}} \left(\frac{\sum_{j=1}^{n-1} w_{\mu_j} x_{\mu_j}}{\sum_{j=1}^{n-1} w_{\mu_j}} - A_n \right)^2}{\left(\frac{\sum_{j=1}^{n-1} w_{\mu_j} x_{\mu_j}}{\sum_{j=1}^{n-1} w_{\mu_j}} \right)^2 + \max\left(\left(\frac{\sum_{j=1}^{n-1} w_{\mu_j} x_{\mu_j}}{\sum_{j=1}^{n-1} w_{\mu_j}} \right)^2, A_n^2 \right)} +$$

$$+ \log\left(x_{\mu_n}^{w_{\mu_n}} \left(\frac{\sum_{j=1}^{n-1} w_{\mu_j} x_{\mu_j}}{\sum_{j=1}^{n-1} w_{\mu_j}} \right)^{\sum_{j=1}^{n-1} w_{\mu_j}} \right) =$$

$$= \frac{1}{A_n} \sum_{j=1}^{n} \frac{w_j x_j (x_j - A_n)^2}{x_j^2 + \max(x_j^2, A_n^2)} - \frac{1}{A_n} \sum_{j=1}^{n-1} \frac{w_{\mu_j} x_{\mu_j} (x_{\mu_j} - A_n)^2}{x_{\mu_j}^2 + \max(x_{\mu_j}^2, A_n^2)} +$$

$$+\log(G_n) - \sum_{j=1}^{n-1} w_{\mu_j} \log x_{\mu_j} + \left(\sum_{j=1}^{n-1} w_{\mu_j}\right) f_{A_n}\left(\frac{\sum_{j=1}^{n-1} w_{\mu_j} x_{\mu_j}}{\sum_{j=1}^{n-1} w_{\mu_j}}\right) =$$

$$= \log(G_n) + C + \left(\sum_{j=1}^{n-1} w_{\mu_j}\right) f_{A_n}\left(\frac{\sum_{j=1}^{n-1} w_{\mu_j} x_{\mu_j}}{\sum_{j=1}^{n-1} w_{\mu_j}}\right) - \sum_{j=1}^{n-1} w_{\mu_j} f_{A_n}(x_{\mu_j}).$$

Since $x_{\mu_j},\ i = \{1,\dots,k\}$ are arbitrary, the last inequality in (7) holds. The theorem is proved. □

Putting $x_j = \frac{q_j}{p_j}, w_j = p_j/\sum_{j=1}^n q_j$ $(j = 1,\dots,n)$, where $\sum_{j=1}^n p_j = \sum_{j=1}^n q_j$, in Theorem 5, we obtain the following refinement of the first inequality in (1).

Corollary 1. *Let* p_j, q_j $(j = 1,\dots,n)$ *be positive real numbers satisfying* $\sum_{j=1}^n p_j = \sum_{j=1}^n q_j$. *Then*

$$\sum_{j=1}^{n} \frac{q_j(q_j-p_j)^2}{q_j^2+\hat{M}_j} = \hat{C} \le \hat{C} + \hat{s}_2 \le \hat{C} + \hat{s}_3 \le \cdots \le \hat{C} + \hat{s}_{n-1} \le \sum_{j=1}^{n} p_j \log\frac{p_j}{q_j} \tag{8}$$

where

$$\hat{s}_k = \max_{1\le\mu_1<\mu_2<\cdots<\mu_k\le n}\left[\left(\sum_{j=1}^{k} p_{\mu_j}\right) f_1\left(\frac{\sum_{j=1}^k q_{\mu_j}}{\sum_{j=1}^k q_{\mu_j}}\right) - \sum_{j=1}^{k} p_{\mu_j} f_1\left(\frac{q_{\mu_j}}{p_{\mu_j}}\right)\right],$$

and $\hat{M}_j = \max(p_j^2, q_j^2)$ $(j = 1,\dots,n)$.

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