DOI: 10.15393/j3.art.2017.3790

UDC 517.51

S. S. Dragomir

# INEQUALITIES OF HERMITE-HADAMARD TYPE FOR *HG*-CONVEX FUNCTIONS

**Abstract.** Some inequalities of Hermite-Hadamard type for $HG$-convex functions defined on positive intervals are given. Applications for special means are also provided.

**Key words:** *convex functions, integral inequalities, HG-convex functions*

**2010 Mathematical Subject Classification:** *26D15, 25D10*

**1. Introduction.** Following [4] (see also [26]) we say that the function $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is *HA-convex* if

$$f\left(\frac{xy}{tx+(1-t)\,y}\right) \leq (1-t)\,f(x) + tf(y) \tag{1}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1) is reversed, then $f$ is said to be *HA-concave.*

If $I \subset (0,\infty)$ and $f$ is convex and nondecreasing function then $f$ is $HA$-convex and if $f$ is $HA$-convex and nonincreasing function then $f$ is convex.

If $[a,b] \subset I \subset (0,\infty)$ and if we consider the function $g : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$, defined by $g(t) = f\left(\frac{1}{t}\right)$, then we can state the following fact [4]:

**Lemma 1**. *The function $f$ is HA-convex on $[a,b]$ if and only if $g$ is convex in the usual sense on $\left[\frac{1}{b}, \frac{1}{a}\right]$.*

Therefore, as examples of $HA$-convex functions we can take $f(t) = g\left(\frac{1}{t}\right)$, where $g$ is any convex function on $\left[\frac{1}{b}, \frac{1}{a}\right]$.

In the recent paper [16] we obtained the following characterization result as well:

**Lemma 2**. *Let $f, h : [a, b] \subset (0, \infty) \to \mathbb{R}$ be so that $h(t) = tf(t)$ for $t \in [a, b]$. Then $f$ is HA-convex on the interval $[a, b]$ if and only if $h$ is convex on $[a, b]$.*

Following [4] (see also [26]) we say that the function $f : I \subset \mathbb{R} \setminus \{0\} \to \to (0, \infty)$ is *HG-convex* if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \le [f(x)]^{1-t} [f(y)]^t \tag{2}$$

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (2) is reversed, then $f$ is said to be *HG-concave.*

By the geometric-mean - arithmetic mean inequality we have that any *HG*-convex function is *HA*-convex. The converse is obviously not true.

We observe that $f : I \subset \mathbb{R} \setminus \{0\} \to (0, \infty)$ is *HG*-convex *if and only if* the function $\ln f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is *HA*-convex on $I$.

Using Lemmas 1 and 2 we have:

**Theorem 1**. *Let $f : [a, b] \subset (0, \infty) \to (0, \infty)$ and define the associated functions $G_f : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$ defined by $G_f(t) = \ln f\left(\frac{1}{t}\right)$ and $H_f : [a, b] \subset \subset (0, \infty) \to \mathbb{R}$ defined by $H_f(t) = t \ln f(t)$. Then the following statements are equivalent:*

*(i) The function $f$ is HG-convex on $[a, b]$;*
*(ii) The function $G_f$ is convex on $\left[\frac{1}{b}, \frac{1}{a}\right]$;*
*(iii) The function $H_f$ is convex on $[a, b]$.*

For a convex function $h : [c, d] \to \mathbb{R}$, the following inequality is well known in the literature as the *Hermite-Hadamard inequality*

$$h\left(\frac{c+d}{2}\right) \le \frac{1}{d-c} \int_c^d h(t)\, dt \le \frac{h(c) + h(d)}{2}. \tag{3}$$

For related results, see [1] – [10], [12] – [28].

Motivated by the above results, we establish in this paper some inequalities of Hermite-Hadamard type for *HG*-convex functions defined on positive intervals. Applications for special means are also provided.

**2. Main Results.** The following result holds.

**Theorem 2**. *Let $f : [a, b] \subset (0, \infty) \to (0, \infty)$ be an HG-convex function on the interval $[a, b]$. Then for any $\lambda \in [0, 1]$ we have the inequalities*

$$f\left(\frac{2ab}{a+b}\right) \le \left[f\left(\frac{2ab}{(1-\lambda)a + (\lambda+1)b}\right)\right]^{1-\lambda} \left[f\left(\frac{2ab}{(2-\lambda)a + \lambda b}\right)\right]^{\lambda} \le$$

$$\leq \exp\left(\frac{ab}{b-a}\int_a^b \frac{\ln f(t)}{t^2}dt\right) \leq \tag{4}$$

$$\leq \sqrt{f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)[f(a)]^{1-\lambda}[f(b)]^{\lambda}} \leq \sqrt{f(a)f(b)}.$$

If we take $\lambda = \frac{1}{2}$ in (4), then we get

$$f\left(\frac{2ab}{a+b}\right) \leq \sqrt{f\left(\frac{4ab}{a+3b}\right)f\left(\frac{4ab}{3a+b}\right)} \leq$$

$$\leq \exp\left(\frac{ab}{b-a}\int_a^b \frac{\ln f(t)}{t^2}dt\right) \leq \tag{5}$$

$$\leq \sqrt{f\left(\frac{2ab}{a+b}\right)\sqrt{f(a)f(b)}} \leq \sqrt{f(a)f(b)}.$$

The *identric mean* $I(a,b)$ for two distinct positive numbers $a, b$ is defined by

$$I(a,b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$$

while the *logarithmic mean* is defined by

$$L(a,b) := \frac{b-a}{\ln b - \ln a}.$$

**Theorem 3**. *Let* $f : [a,b] \subset (0,\infty) \to (0,\infty)$ *be an HG-convex function on the interval* $[a,b]$. *Then*

$$f(L(a,b)) \leq \exp\left(\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right) \leq [f(b)]^{\frac{(L(a,b)-a)b}{(b-a)L(a,b)}}[f(a)]^{\frac{(b-L(a,b))a}{(b-a)L(a,b)}}. \tag{6}$$

If we write the classical Hermite-Hadamard inequality for the function $H_f$ that is convex on $[a,b]$ when $f : [a,b] \subset (0,\infty) \to (0,\infty)$ is an *HG*-convex function on $[a,b]$ and perform the required calculations, we get:

**Theorem 4**. *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be an HG-convex function on the interval $[a,b]$. Then we have*

$$\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{a+b}{2}}\leq\exp\left(\frac{1}{b-a}\int_a^b t\ln f(t)\,dt\right)\leq\sqrt{[f(b)]^b[f(a)]^a}. \quad (7)$$

We have the reverse inequalities as well:

**Theorem 5**. *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be an HG-convex function on the interval $[a,b]$. Then we have*

$$1\leq\frac{\exp\left(\frac{ab}{b-a}\int_a^b\frac{\ln f(t)}{t^2}dt\right)}{f\left(\frac{2ab}{a+b}\right)}\leq \quad (8)$$

$$\leq\exp\left(\frac{1}{8}\left[\frac{f'_-(b)}{f(b)}b^2-\frac{f'_+(a)\,a^2}{f(a)}\right]\left(\frac{b-a}{ab}\right)\right)$$

*and*

$$1\leq\frac{\sqrt{f(a)\,f(b)}}{\exp\left(\frac{ab}{b-a}\int_a^b\frac{\ln f(t)}{t^2}dt\right)}\leq \quad (9)$$

$$\leq\exp\left(\frac{1}{8}\left[\frac{f'_-(b)}{f(b)}b^2-\frac{f'_+(a)\,a^2}{f(a)}\right]\left(\frac{b-a}{ab}\right)\right).$$

The following related result also holds:

**Theorem 6**. *Let $f:[a,b]\subset(0,\infty)\to(0,\infty)$ be an HG-convex function on the interval $[a,b]$. Then we have*

$$1\leq\frac{\sqrt{[f(a)]^a[f(b)]^b}}{\exp\left(\frac{1}{b-a}\int_a^b t\ln f(t)\,dt\right)}\leq$$

$$\leq\left(\frac{f(b)}{f(a)}\right)^{\frac{1}{8}(b-a)}\exp\left(\frac{1}{8}(b-a)\left(\frac{bf'_-(b)}{f(b)}-\frac{af'_+(a)}{f(a)}\right)\right) \quad (10)$$

*and*

$$1 \leq \frac{\exp\left(\frac{1}{b-a}\int_a^b t\ln f(t)\,dt\right)}{\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{a+b}{2}}} \leq$$

$$\leq \left(\frac{f(b)}{f(a)}\right)^{\frac{1}{8}(b-a)} \exp\left(\frac{1}{8}(b-a)\left(\frac{bf'_-(b)}{f(b)} - \frac{af'_+(a)}{f(a)}\right)\right). \tag{11}$$

From a different perspective we have:

**Theorem 7**. *Let $f : [a,b] \subset (0,\infty) \to (0,\infty)$ be an HG-convex function on the interval $[a,b]$. Then*

$$\exp\left(\frac{ab}{b-a}\int_a^b \frac{\ln f(t)}{t^2}dt\right) \leq \sqrt{f(x)\,[f(b)]^{\frac{a(b-x)}{x(b-a)}}\,[f(a)]^{\frac{b(x-a)}{x(b-a)}}} \tag{12}$$

*for any $x \in [a,b]$.*

If we take in (12), $x = \frac{a+b}{2}$, then we get from (12) that

$$\exp\left(\frac{ab}{b-a}\int_a^b \frac{\ln f(t)}{t^2}dt\right) \leq \sqrt{f\left(\frac{a+b}{2}\right)[f(b)]^{\frac{a}{a+b}}\,[f(a)]^{\frac{b}{a+b}}}. \tag{13}$$

**3. Proofs.** In [15], in order to improve Işcan's inequality [26] for *HA*-convex functions $g : [a,b] \subset (0,\infty) \to \mathbb{R}$,

$$g\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{g(t)}{t^2}dt \leq \frac{g(a)+g(b)}{2}, \tag{14}$$

we obtained the following result:

$$g\left(\frac{2ab}{a+b}\right) \leq (1-\lambda)\,g\left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right) +$$

$$+\lambda g\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{g(t)}{t^2}dt \leq$$

$$\leq \frac{1}{2}\left[g\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)+(1-\lambda)g(a)+\lambda g(b)\right]\leq$$
$$\leq \frac{g(a)+g(b)}{2}, \tag{15}$$

where $\lambda \in [0,1]$.

Now, if $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is an *HG*-convex function on the interval $[a,b]$, then $g:=\ln f$ is *HA*-convex on $[a,b]$, and by (15) we get

$$\ln f\left(\frac{2ab}{a+b}\right)\leq(1-\lambda)\ln f\left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right)+$$
$$+\lambda\ln f\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right)\leq\frac{ab}{b-a}\int_a^b\frac{\ln f(t)}{t^2}dt\leq$$
$$\leq\frac{1}{2}\left[\ln f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)+(1-\lambda)\ln f(a)+\lambda\ln f(b)\right]\leq$$
$$\leq\frac{\ln f(a)+\ln f(b)}{2}, \tag{16}$$

that is equivalent to

$$\ln f\left(\frac{2ab}{a+b}\right)\leq$$
$$\leq\ln\left(\left[f\left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right)\right]^{1-\lambda}\left[f\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right)\right]^{\lambda}\right)\leq$$
$$\leq\frac{ab}{b-a}\int_a^b\frac{\ln f(t)}{t^2}dt\leq\ln\sqrt{f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)[f(a)]^{1-\lambda}[f(b)]^{\lambda}}\leq$$
$$\leq\ln\sqrt{f(a)f(b)},$$

and by taking the exponential we get the desired result (4).

We have the following result for *HA*-convex functions [15]:

**Lemma 3**. *Let $g:[a,b]\subset(0,\infty)\to\mathbb{R}$ be an HA-convex function on the interval $[a,b]$. Then*

$$g(L(a,b))\leq\frac{1}{b-a}\int_a^b g(x)\,dx\leq\frac{(L(a,b)-a)\,bg(b)+(b-L(a,b))\,ag(a)}{(b-a)\,L(a,b)}. \tag{17}$$

If $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is an $HG$-convex function on the interval $[a,b]$, then $g:=\ln f$ is $HA$-convex on $[a,b]$, and by (17) we have

$$\ln f\left(L\left(a,b\right)\right)\leq\frac{1}{b-a}\int\limits_{a}^{b}\ln f\left(x\right)dx\leq$$

$$\leq\frac{\left(L\left(a,b\right)-a\right)b\ln f\left(b\right)+\left(b-L\left(a,b\right)\right)a\ln f\left(a\right)}{\left(b-a\right)L\left(a,b\right)}=$$

$$=\ln\left(\left[f\left(b\right)\right]^{\frac{\left(L\left(a,b\right)-a\right)b}{\left(b-a\right)L\left(a,b\right)}}\left[f\left(a\right)\right]^{\frac{\left(b-L\left(a,b\right)\right)a}{\left(b-a\right)L\left(a,b\right)}}\right).\tag{18}$$

By taking the exponential in (18) we get the desired result (6).

We use the following results obtained by the author in [10] and [11]:

**Lemma 4**. *Let $h:[\alpha,\beta]\to\mathbb{R}$ be a convex function on $[\alpha,\beta]$. Then we have the inequalities*

$$0\leq\frac{h\left(\alpha\right)+h\left(\beta\right)}{2}-\frac{1}{\beta-\alpha}\int\limits_{\alpha}^{\beta}h\left(t\right)dt\leq\frac{1}{8}\left[h_{-}^{\prime}\left(\beta\right)-h_{+}^{\prime}\left(\alpha\right)\right]\left(\beta-\alpha\right)\tag{19}$$

*and*

$$0\leq\frac{1}{\beta-\alpha}\int\limits_{\alpha}^{\beta}h\left(t\right)dt-h\left(\frac{\alpha+\beta}{2}\right)\leq\frac{1}{8}\left[h_{-}^{\prime}\left(\beta\right)-h_{+}^{\prime}\left(\alpha\right)\right]\left(\beta-\alpha\right).\tag{20}$$

*The constant $\frac{1}{8}$ is best possible in (19) and (20).*

If $\ell:[a,b]\subset(0,\infty)\to\mathbb{R}$ is an $HA$-convex function on the interval $[a,b]$, then the function $g:\left[\frac{1}{b},\frac{1}{a}\right]\to\mathbb{R}$, $g\left(s\right)=\ell\left(\frac{1}{s}\right)$, is convex on $\left[\frac{1}{b},\frac{1}{a}\right]$.

Now, by (19) and (20) we have

$$0\leq\frac{g\left(\frac{1}{a}\right)+g\left(\frac{1}{b}\right)}{2}-\frac{1}{\frac{1}{a}-\frac{1}{b}}\int\limits_{\frac{1}{b}}^{\frac{1}{a}}g\left(t\right)dt\leq$$

$$\leq\frac{1}{8}\left[g_{-}^{\prime}\left(\frac{1}{a}\right)-g_{+}^{\prime}\left(\frac{1}{b}\right)\right]\left(\frac{1}{a}-\frac{1}{b}\right)\tag{21}$$

and

$$0 \leq \frac{1}{\frac{1}{a}-\frac{1}{b}} \int_{\frac{1}{b}}^{\frac{1}{a}} g(t)\,dt - g\left(\frac{\frac{1}{a}+\frac{1}{b}}{2}\right) \leq$$

$$\leq \frac{1}{8}\left[g'_{-}\left(\frac{1}{a}\right) - g'_{+}\left(\frac{1}{b}\right)\right]\left(\frac{1}{a}-\frac{1}{b}\right). \tag{22}$$

We also have

$$g'_{\pm}(s) = \ell'_{\mp}\left(\frac{1}{s}\right)\left(-\frac{1}{s^2}\right)$$

and then

$$g'_{-}\left(\frac{1}{a}\right) = -\ell'_{+}(a)\,a^2 \quad \text{and} \quad g'_{+}\left(\frac{1}{b}\right) = -\ell'_{-}(b)\,b^2.$$

From (21) and (22) we have

$$0 \leq \frac{\ell(a)+\ell(b)}{2} - \frac{ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}} \ell\left(\frac{1}{s}\right) ds \leq$$

$$\leq \frac{1}{8}\left[\ell'_{-}(b)\,b^2 - \ell'_{+}(a)\,a^2\right]\left(\frac{b-a}{ab}\right) \tag{23}$$

and

$$0 \leq \frac{ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}} \ell\left(\frac{1}{s}\right) ds - \ell\left(\frac{2ab}{a+b}\right) \leq$$

$$\leq \frac{1}{8}\left[\ell'_{-}(b)\,b^2 - \ell'_{+}(a)\,a^2\right]\left(\frac{b-a}{ab}\right). \tag{24}$$

If we change the variable $\frac{1}{s} = u$, then $ds = -\frac{du}{u^2}$ and (23) and (24) can be written as

$$0 \leq \frac{\ell(a)+\ell(b)}{2} - \frac{ab}{b-a}\int_{a}^{b} \frac{\ell(t)}{t^2} dt \leq$$

$$\leq \frac{1}{8}\left[\ell'_{-}(b)\,b^2 - \ell'_{+}(a)\,a^2\right]\left(\frac{b-a}{ab}\right) \tag{25}$$

and

$$0 \le \frac{ab}{b-a} \int_a^b \frac{\ell(t)}{t^2} dt - \ell\left(\frac{2ab}{a+b}\right) \le$$

$$\le \frac{1}{8} \left[\ell'_-(b) b^2 - \ell'_+(a) a^2\right] \left(\frac{b-a}{ab}\right). \tag{26}$$

If $f : [a,b] \subset (0,\infty) \to (0,\infty)$ is an $HG$-convex function on the interval $[a,b]$, then $\ell := \ln f$ is $HA$-convex on $[a,b]$, and by (25) and (26) we have

$$0 \le \ln \sqrt{f(a) f(b)} - \frac{ab}{b-a} \int_a^b \frac{\ln f(t)}{t^2} dt \le$$

$$\le \frac{1}{8} \left[\frac{f'_-(b)}{f(b)} b^2 - \frac{f'_+(a) a^2}{f(a)}\right] \left(\frac{b-a}{ab}\right) \tag{27}$$

and

$$0 \le \frac{ab}{b-a} \int_a^b \frac{\ln f(t)}{t^2} dt - \ln f\left(\frac{2ab}{a+b}\right) \le$$

$$\le \frac{1}{8} \left[\frac{f'_-(b)}{f(b)} b^2 - \frac{f'_+(a) a^2}{f(a)}\right] \left(\frac{b-a}{ab}\right), \tag{28}$$

and the Theorem 5 is proved.

If $f : [a,b] \subset (0,\infty) \to (0,\infty)$ is an $HG$-convex function on the interval $[a,b]$, then $H_f$ is convex on $[a,b]$ and by (19) and (20) we have after appropriate calculations

$$0 \le \ln \sqrt{[f(a)]^a [f(b)]^b} - \frac{1}{b-a} \int_a^b t \ln f(t)\, dt \le$$

$$\le \frac{1}{8} \left[\ln f(b) + \frac{b f'_-(b)}{f(b)} - \ln f(a) - \frac{a f'_+(a)}{f(a)}\right] (b-a) =$$

$$= \ln \left(\frac{f(b)}{f(a)}\right)^{\frac{1}{8}(b-a)} + \frac{1}{8} (b-a) \left(\frac{b f'_-(b)}{f(b)} - \frac{a f'_+(a)}{f(a)}\right)$$

and

$$0 \le \frac{1}{b-a}\int_a^b t\ln f(t)\,dt - \ln\left(\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{a+b}{2}}\right) \le$$

$$\le \ln\left(\frac{f(b)}{f(a)}\right)^{\frac{1}{8}(b-a)} + \frac{1}{8}(b-a)\left(\frac{bf'_-(b)}{f(b)} - \frac{af'_+(a)}{f(a)}\right).$$

These inequalities are equivalent to

$$0 \le \ln\left(\frac{\sqrt{[f(a)]^a [f(b)]^b}}{\exp\left(\frac{1}{b-a}\int_a^b t\ln f(t)\,dt\right)}\right) \le$$

$$\le \ln\left[\left(\frac{f(b)}{f(a)}\right)^{\frac{1}{8}(b-a)} \exp\left(\frac{1}{8}(b-a)\left(\frac{bf'_-(b)}{f(b)} - \frac{af'_+(a)}{f(a)}\right)\right)\right]$$

and

$$0 \le \ln\left(\frac{\exp\left(\frac{1}{b-a}\int_a^b t\ln f(t)\,dt\right)}{\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{a+b}{2}}}\right) \le$$

$$\le \ln\left[\left(\frac{f(b)}{f(a)}\right)^{\frac{1}{8}(b-a)} \exp\left(\frac{1}{8}(b-a)\left(\frac{bf'_-(b)}{f(b)} - \frac{af'_+(a)}{f(a)}\right)\right)\right]$$

and by taking the exponential we get the desired results (10) and (11).

The following lemma is of interest in itself:

**Lemma 5**. *Let $g : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a HA-convex function on the interval $[a,b]$. Then*

$$\frac{1}{2x}\left(\frac{g(b)\,a\,(b-x) + g(a)\,b\,(x-a)}{b-a} + xg(x)\right) \ge \frac{ab}{b-a}\int_a^b \frac{g(y)}{y^2}dy \quad (29)$$

*for any $x \in [a,b]$.*

**Proof.** Since $h(t) = tg(t)$ for $t \in [a, b]$ is convex, then by the gradient inequality for convex functions we have

$$xg(x) - yg(y) \geq \left(g(y) + yg'_{-}(y)\right)(x - y)$$

for any $x, y \in (a, b)$.

This is equivalent to

$$xg(x) - xg(y) \geq yg'_{-}(y)(x - y) \tag{30}$$

for any $x, y \in (a, b)$.

From (30) we have, by division with $xy^2 > 0$, that

$$\frac{1}{y^2}g(x) - \frac{1}{y^2}g(y) \geq \frac{g'_{-}(y)}{y}\left(1 - \frac{y}{x}\right)$$

for any $x, y \in (a, b)$.

Taking the integral mean over $y$ we have

$$g(x)\frac{1}{b-a}\int_a^b \frac{1}{y^2}dy - \frac{1}{b-a}\int_a^b \frac{g(y)}{y^2}dy \geq$$

$$\geq \frac{1}{b-a}\int_a^b \frac{g'_{-}(y)}{y}dy - \frac{1}{x}\frac{1}{b-a}\int_a^b g'_{-}(y)\,dy$$

that is equivalent to

$$\frac{g(x)}{ab} - \frac{1}{b-a}\int_a^b \frac{g(y)}{y^2}dy \geq$$

$$\geq \frac{1}{b-a}\left[\frac{g(b)}{b} - \frac{g(a)}{a} + \int_a^b \frac{g(y)}{y^2}dy\right] - \frac{1}{x}\frac{g(b) - g(a)}{b-a} =$$

$$= \frac{1}{b-a}\left(\frac{g(b)}{b} - \frac{g(a)}{a}\right) + \frac{1}{b-a}\int_a^b \frac{g(y)}{y^2}dy - \frac{1}{x}\frac{g(b) - g(a)}{b-a},$$

for any $x \in (a,b)$. This can be written as

$$\frac{1}{x}\frac{g(b)-g(a)}{b-a}-\frac{1}{b-a}\left(\frac{g(b)}{b}-\frac{g(a)}{a}\right)\geq\frac{2}{b-a}\int_a^b\frac{g(y)}{y^2}dy-\frac{g(x)}{ab}$$

or as

$$\frac{1}{2}\left(\frac{1}{b-a}\left[g(b)\frac{b-x}{xb}+g(a)\frac{x-a}{ax}\right]+\frac{g(x)}{ab}\right)\geq\frac{1}{b-a}\int_a^b\frac{g(y)}{y^2}dy.$$

This is equivalent to the desired result (29). □

If $f:[a,b]\subset(0,\infty)\to(0,\infty)$ is an $HG$-convex function on the interval $[a,b]$, then $g:=\ln f$ is $HA$-convex on $[a,b]$, and by (29) we have

$$\frac{1}{2x}\left(\frac{a(b-x)\ln f(b)+b(x-a)\ln f(a)}{b-a}+x\ln f(x)\right)\geq$$

$$\geq\frac{ab}{b-a}\int_a^b\frac{\ln f(y)}{y^2}dy$$

for any $x\in[a,b]$.

This is clearly equivalent to

$$\ln\left(\sqrt{[f(b)]^{\frac{a(b-x)}{x(b-a)}}[f(a)]^{\frac{b(x-a)}{x(b-a)}}}\sqrt{f(x)}\right)\geq\frac{ab}{b-a}\int_a^b\frac{\ln f(y)}{y^2}dy \qquad (31)$$

for any $x\in[a,b]$.

If we take the exponential in (31), then we get the desired result (12).

**4. Applications.** Consider the function $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(t)=t$. Using the geometric mean - harmonic mean inequality, we have

$$f\left(\frac{xy}{tx+(1-t)y}\right)=\frac{xy}{tx+(1-t)y}\leq x^{1-t}y^t=[f(x)]^{1-t}[f(y)]^t,$$

which shows that $f$ is $HG$-convex on $[a,b]$.

We need the following integrals

$$\frac{1}{b-a}\int_a^b\ln f(t)\,dt=\frac{1}{b-a}\int_a^b\ln t\,dt=\ln I(a,b),$$

$$\frac{1}{b-a}\int_a^b t\ln f(t)\,dt = \frac{1}{b-a}\int_a^b t\ln t dt =$$

$$= \frac{1}{2}A(a,b)\ln I\left(a^2,b^2\right) = \ln\left[I\left(a^2,b^2\right)^{\frac{1}{2}A(a,b)}\right]$$

and

$$\int_a^b \frac{\ln f(t)}{t^2}dt = \int_a^b \frac{\ln t}{t^2}dt = \frac{b-a}{ab}\ln\left[I\left(a^{-1},b^{-1}\right)\right]^{-1}$$

giving that

$$\frac{ab}{b-a}\int_a^b \frac{\ln f(t)}{t^2}dt = \ln\left[I\left(a^{-1},b^{-1}\right)\right]^{-1}.$$

Now, if we write the inequality (4) for the function $f:[a,b]\subset(0,\infty)\to$ $\to(0,\infty)$, $f(t)=t$, we get

$$H(a,b) \le \left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right)^{1-\lambda}\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right)^{\lambda} \le$$

$$\le \left[I\left(a^{-1},b^{-1}\right)\right]^{-1} \le \sqrt{\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)a^{1-\lambda}b^{\lambda}} \le G(a,b), \tag{32}$$

where $H(a,b):=\frac{2ab}{a+b}$ is the *harmonic mean.*

If we use the inequality (6) for $f(t)=t$, then we have

$$(L(a,b)\le)\, I(a,b) \le b^{\frac{(L(a,b)-a)b}{(b-a)L(a,b)}}a^{\frac{(b-L(a,b))a}{(b-a)L(a,b)}}. \tag{33}$$

If we use the inequality (7) for $f(t)=t$, $t\in[a,b]$, then we also get

$$[A(a,b)]^{A(a,b)} \le I\left(a^2,b^2\right)^{\frac{1}{2}A(a,b)} \le G\left(b^b,a^a\right). \tag{34}$$

From (8) and (9) for $f(t)=t$ we have

$$1 \le \frac{\left[I\left(a^{-1},b^{-1}\right)\right]^{-1}}{H(a,b)} \le \exp\left(\frac{(b-a)^2}{8ab}\right) \tag{35}$$

and

$$1 \le \frac{G(a,b)}{\left[I\left(a^{-1},b^{-1}\right)\right]^{-1}} \le \exp\left(\frac{(b-a)^2}{8ab}\right). \tag{36}$$

From (10) and (11) we also have

$$1 \le \frac{G\left(a^a, b^b\right)}{\left[I\left(a^2, b^2\right)\right]^{\frac{1}{2}A(a,b)}} \le \left(\frac{b}{a}\right)^{\frac{1}{8}(b-a)} \tag{37}$$

and

$$1 \le \frac{\left[I\left(a^2, b^2\right)\right]^{\frac{1}{2}A(a,b)}}{\left[A\left(a,b\right)\right]^{A(a,b)}} \le \left(\frac{b}{a}\right)^{\frac{1}{8}(b-a)}. \tag{38}$$

Finally, from (13) we obtain

$$\left[I\left(a^{-1}, b^{-1}\right)\right]^{-1} \le \sqrt{A\left(a,b\right) b^{\frac{a}{a+b}} a^{\frac{b}{a+b}}}. \tag{39}$$

Now consider the function $f : [a, b] \subset (0, \infty) \to (0, \infty)$, $f(t) = \exp(t)$. Using the harmonic mean-arithmetic mean inequality we have

$$f\left(\frac{xy}{tx + (1-t)y}\right) = \exp\left(\frac{xy}{tx + (1-t)y}\right) \le \exp\left((1-t)x + ty\right) =$$

$$= \left[\exp(x)\right]^{1-t}\left[\exp(y)\right]^t = \left[f(x)\right]^{1-t}\left[f(y)\right]^t$$

for any $x, y \in [a, b]$ and $t \in [0, 1]$.

Now, if we use the inequality (4) for the $HG$-convex function $f : [a, b] \subset \subset (0, \infty) \to (0, \infty)$, $f(t) = \exp(t)$, then we get, after suitable calculations, that

$$H\left(a,b\right) \le \frac{2(1-\lambda)ab}{(1-\lambda)a + (\lambda+1)b} + \frac{2\lambda ab}{(2-\lambda)a + \lambda b} \le$$

$$\le \frac{G^2\left(a,b\right)}{L\left(a,b\right)} \le \frac{1}{2}\left(\frac{ab}{(1-\lambda)a + \lambda b} + (1-\lambda)a + \lambda b\right) \le A\left(a,b\right),$$

for any $\lambda \in [0, 1]$.

If we use the inequalities (8) and (9) for the $HG$-convex function $f : [a, b] \subset (0, \infty) \to (0, \infty)$, $f(t) = \exp(t)$, then, by performing the required calculations, we get

$$0 \le \frac{G^2\left(a,b\right)}{L\left(a,b\right)} - H\left(a,b\right) \le \frac{1}{4}\frac{A\left(a,b\right)}{G^2\left(a,b\right)}\left(b-a\right)^2 \tag{40}$$

and

$$0 \le A\left(a,b\right) - \frac{G^2\left(a,b\right)}{L\left(a,b\right)} \le \frac{1}{4}\frac{A\left(a,b\right)}{G^2\left(a,b\right)}\left(b-a\right)^2. \tag{41}$$

From the inequality (13) we also have

$$\frac{G^2(a,b)}{L(a,b)} \leq \frac{1}{2}\left(A(a,b) + b^{\frac{a}{a+b}} a^{\frac{b}{a+b}}\right). \tag{42}$$

**Acknowledgement.** The author would like to thank the anonymous referees for valuable comments that have been implemented in the final version of the paper.

## References

[1] Alomari M., Darus M. *The Hadamard's inequality for s-convex function.* Int. J. Math. Anal., 2008, vol. 2, no. 13-16, pp. 639–646.

[2] Alomari M., Darus M. *Hadamard-type inequalities for s-convex functions.* Int. Math. Forum, 2008, vol. 3, no. 37-40, pp. 1965–1975.

[3] Anastassiou G. A. *Univariate Ostrowski inequalities, revisited.* Monatsh. Math., 2002, vol. 135, no. 3, pp. 175–189.

[4] Anderson G. D., Vamanamurthy M. K., Vuorinen M. *Generalized convexity and inequalities.* J. Math. Anal. Appl., 2007, vol. 335, pp. 1294–1308.

[5] Barnett N. S., Cerone P., Dragomir S. S., Pinheiro M. R., Sofo A. *Ostrowski type inequalities for functions whose modulus of the derivatives are convex and applications.* Inequality Theory and Applications, vol. 2 (Chinju/Masan, 2001), 19–32, Nova Sci. Publ., Hauppauge, NY, 2003. Preprint: RGMIA Res. Rep. Coll., 2002, vol. 5, no. 2, art. 1 [Online http://rgmia.org/papers/v5n2/Paperwapp2q.pdf].

[6] Beckenbach E. F. *Convex functions.* Bull. Amer. Math. Soc., 1948, vol. 54, pp. 439–460.

[7] Bombardelli M., Varošanec S. *Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities.* Comput. Math. Appl., 2009, vol. 58, no. 9, pp. 1869–1877.

[8] Cristescu G. *Hadamard type inequalities for convolution of h-convex functions.* Ann. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convexity, 2010, vol. 8, pp. 3–11.

[9] Dragomir S. S. *On the Ostrowski's inequality for mappings of bounded variation and applications.* Math. Ineq. & Appl., 2001, vol. 4, no. 1, pp. 33–40.

[10] Dragomir S. S. *An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products.* J. Inequal. Pure Appl. Math., 2002, vol. 3, no. 2, art. 31, 8 p.

[11] Dragomir S. S. *An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products.* J. Inequal. Pure Appl. Math., 2002, vol. 3, no. 3, art. 35.

[12] Dragomir S. S. *An Ostrowski like inequality for convex functions and applications.* Revista Math. Complutense, 2003, vol. 16, no. 2, pp. 373–382.

[13] Dragomir S. S. *Operator Inequalities of Ostrowski and Trapezoidal Type.* Springer Briefs in Mathematics. Springer, New York, 2012. x+112 pp. ISBN: 978-1-4614-1778-1

[14] Dragomir S. S. *Some new inequalities of Hermite-Hadamard type for GA-convex functions.* Preprint RGMIA Res. Rep. Coll., 2015, vol. 18, art. 30. [http://rgmia.org/papers/v18/v18a30.pdf].

[15] Dragomir S. S. *Inequalities of Hermite-Hadamard type for HA-convex functions.* Preprint RGMIA Res. Rep. Coll., 2015, vol. 18, art. 38. [http://rgmia.org/papers/v18/v18a38.pdf].

[16] Dragomir S. S. *New inequalities of Hermite-Hadamard type for HA-convex functions.* Preprint RGMIA Res. Rep. Coll., 2015, vol. 18, art. 41. [http://rgmia.org/papers/v18/v18a41.pdf].

[17] Dragomir S. S., Cerone P., Roumeliotis J., Wang S. *A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis.* Bull. Math. Soc. Sci. Math. Romanie, 1999, vol. 42(90), no. 4, pp. 301–314.

[18] Dragomir S. S., Fitzpatrick S. *The Hadamard inequalities for s-convex functions in the second sense.* Demonstratio Math., 1999, vol. 32, no. 4, pp. 687–696.

[19] Dragomir S. S., Fitzpatrick S. *The Jensen inequality for s-Breckner convex functions in linear spaces.* Demonstratio Math., 2000, vol. 33, no. 1, pp. 43–49.

[20] Dragomir S. S., Mond B. *On Hadamard's inequality for a class of functions of Godunova and Levin.* Indian J. Math., 1997, vol. 39, no. 1, pp. 1–9.

[21] Dragomir S. S., Pearce C. E. M. *On Jensen's inequality for a class of functions of Godunova and Levin.* Period. Math. Hungar., 1996, vol. 33, no. 2, pp. 93–100.

[22] Dragomir S. S., Pearce C. E. M. *Quasi-convex functions and Hadamard's inequality.* Bull. Austral. Math. Soc., 1998, vol. 57, pp. 377–385.

[23] Dragomir S. S., Pečarić J., Persson L.-E. *Some inequalities of Hadamard type.* Soochow J. Math., 1995, vol. 21, no. 3, pp. 335–341.

[24] Dragomir S. S., Rassias Th. M. (Eds) *Ostrowski Type Inequalities and Applications in Numerical Integration.* Kluwer Academic Publisher, 2002.

[25] El Farissi A. *Simple proof and refinement of Hermite-Hadamard inequality.* J. Math. Ineq., 2010, vol. 4, no. 3, pp. 365–369.

[26] Işcan I. *Hermite-Hadamard type inequalities for harmonically convex functions.* Hacettepe Journal of Mathematics and Statistics, 2014, vol. 43, no. 6, pp. 935–942.

[27] Kikianty E., Dragomir S. S. *Hermite-Hadamard's inequality and the p-HH-norm on the Cartesian product of two copies of a normed space*. Math. Inequal. Appl., 2010, vol. 13, no. 1, pp. 1–32.

[28] Pečarić J. E., Dragomir S. S. *On an inequality of Godunova-Levin and some refinements of Jensen integral inequality.* Itinerant Seminar on Functional Equations, Approximation and Convexity (Cluj-Napoca, 1989), pp. 263–268, Preprint, 89-6, Univ. "Babeş-Bolyai", Cluj-Napoca, 1989.

*Received April 19, 2017.*
*In revised form, July 11, 2017.*
*Accepted July 11, 2017.*
*Published online August 16, 2017.*

Mathematics, College of Engineering & Science
Victoria University, PO Box 14428
Melbourne City, MC 8001, Australia;
DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences,
School of Computer Science and Applied Mathematics
University of the Witwatersrand,
Private Bag 3, Johannesburg 2050, South Africa
E-mail: sever.dragomir@vu.edu.au