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MONOTONICITY AND CONVEXITY PROPERTIES OF THE NIELSEN'S β-FUNCTION

Abstract. The Nielsen's β-function provides a powerful tool for evaluating and estimating certain integrals, series and mathematical constants. It is related to other special functions such as the digamma function, the Euler's beta function and the Gauss' hypergeometric function. In this work, we prove some monotonicity and convexity properties of the function by employing largely the convolution theorem for Laplace transforms.

Key words: *Nielsen's β-function, Laplace transform for convolutions, completely monotonic function, convex function, GA-convex function, inequality*

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1. Introduction and Preliminaries. Throughout this paper we shall use the following notation: $\mathbb{N} = \{1, 2, 3, 4, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{R} = \{\text{all real numbers}\}$.
The Nielsen's β-function $\beta(x)$ was introduced in [15] and is defined by any of the following equivalent forms.

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t}\, dt, \quad x > 0, \tag{1}$$

$$= \int_0^\infty \frac{e^{-xt}}{1+e^{-t}}\, dt, \quad x > 0, \tag{2}$$

$$= \sum_{k=0}^\infty \frac{(-1)^k}{k+x}, \quad x > 0, \tag{3}$$

$$= \frac{1}{2}\left\{\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right\}, \quad x > 0, \tag{4}$$

where $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$ is the digamma or psi function and $\Gamma(x)$ is the Euler's Gamma function. It is known that the function $\beta(x)$ satisfies the following properties [4], [15].

$$\beta(x+1) = \frac{1}{x} - \beta(x), \tag{5}$$

$$\beta(x) + \beta(1-x) = \frac{\pi}{\sin \pi x}. \tag{6}$$

For additional information on the function refer to [4], [6], [8], [13], and [14].

Also, the function $\beta(x)$ is related to the classical Euler's beta function $B(x, y)$ and the Gauss' hypergeometric function ${}_2F_1(a, b; c; z)$ in the following ways.

$$\beta(x) = -\frac{d}{dx}\left\{\ln B\left(\frac{x}{2}, \frac{1}{2}\right)\right\},$$

$$\beta(x) + \beta(1-x) = B(x, 1-x),$$

$$\beta(x) = \frac{1}{x^2} F_1(1, x; x+1; -1).$$

Repeatedly differentiating (1), (2), (3), (4), and (5) obtain

$$\beta^{(m)}(x) = \int_0^1 \frac{(\ln t)^m t^{x-1}}{1+t}\, dt, \quad x > 0 \tag{7}$$

$$= (-1)^m \int_0^\infty \frac{t^m e^{-xt}}{1+e^{-t}}\, dt, \quad x > 0 \tag{8}$$

$$= (-1)^m m! \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+x)^{m+1}}, \quad x > 0 \tag{9}$$

$$= \frac{1}{2^{m+1}}\left\{\psi^{(m)}\left(\frac{x+1}{2}\right) - \psi^{(m)}\left(\frac{x}{2}\right)\right\}, \tag{10}$$

$$\beta^{(m)}(x+1) = \frac{(-1)^m m!}{x^{m+1}} - \beta^{(m)}(x), \tag{11}$$

where $m \in \mathbb{N}_0$ and $\beta^{(0)}(x) = \beta(x)$. Note that $\left|\beta^{(m)}(x)\right| = (-1)^m \beta^{(m)}(x)$ for $m \in \mathbb{N}_0$ and $x > 0$. Then, by multiplying the recurrence relation (11) by the factor $(-1)^m$, we obtain

$$\left|\beta^{(m)}(x+1)\right| = \frac{m!}{x^{m+1}} - \left|\beta^{(m)}(x)\right|, \tag{12}$$

and as an immediate consequence we obtain the upper bound

$$\left|\beta^{(m)}(x)\right| \leq \frac{m!}{x^{m+1}}. \tag{13}$$

Also, it can easily be shown from (8) that the function $\left|\beta^{(m)}(x)\right|$ is decreasing in terms of x.
Some special values of the function are given as follows.

$$\beta(1) = \ln 2, \quad \beta\left(\frac{1}{2}\right) = \frac{\pi}{2}, \quad \beta\left(\frac{3}{2}\right) = 2 - \frac{\pi}{2}, \quad \beta(2) = 1 - \ln 2,$$

$$\beta'(1) = -\frac{1}{2}\zeta(2) = -\frac{\pi^2}{12}, \quad \beta'(2) = -1 + \frac{\pi^2}{12}, \quad \beta'(3) = \frac{3}{4} - \frac{\pi^2}{12},$$

$$\beta'\left(\frac{3}{2}\right) = 4(G-1), \quad \beta'\left(\frac{5}{2}\right) = \frac{40}{9} - 4G,$$

where $\zeta(x)$ is the Riemann zeta function and G is the Catalan's constant.

As shown in [4] and [8], the Nielsen's β-function is very useful in evaluating certain integrals.

In [1] it was established that the function $x^c \left|\psi^{(n)}(x)\right|$, where $n \in \mathbb{N}$, is strictly decreasing (increasing) on $(0, \infty)$ respectively if $c \leq n$ $(c \geq n+1)$.

The author of [3] established that $\psi(e^x)$ is strictly concave on $\mathbb{R}$, and that $\psi(x^c)$ is strictly concave (convex) respectively if $c > 0$ $(c \in [-1, 0))$. The author further established that $x^c \left|\psi^{(n)}(x)\right|$, $n \in \mathbb{N}$ is strictly convex if and only if $c \leq n$, $c = n+1$ or $c > n+2$.

The authors of [5] showed that $\psi^{(k)}(e^x)$ is strictly concave (convex) on $\mathbb{R}$ if, respectively, $k = 2n-2$ $(k = 2n-1)$, where $n \in \mathbb{N}$. They further showed that $\psi^{(k)}(x^c)$ is convex on $(0, \infty)$ if either $k = 2n-1$ and $c \in (-\infty, -\frac{1}{2n-1}] \cup (0, \infty)$ or $k = 2n-2$ and $c \in [-\frac{1}{2n-1}, 0)$.

Also, strict complete monotonicity of $x\left|\psi^{(m)}(x)\right|$, $m \in \mathbb{N}$ on $(0, \infty)$ was established in [9] among other things.

Then the author of [2] proved that $\Delta_n(x) = \frac{x^{n+1}}{n!}\left|\beta^{(n)}(x)\right|$, $n \in \mathbb{N}$ and $\Delta_n'(x)$ are strictly increasing on $(0,\infty)$, and that $\lim_{x\to 0} \Delta_n(x) = 1$ and $\lim_{x\to 0} \Delta_n'(x) = 0$.

On the account of these results, the natural question is whether similar results can be established for the Nielsen's β-function, since it satisfies some properties identical to those satisfied by the classical digamma function. This is what motivates the present work. We present our findings in the following section.

2. Main Results. We begin by recalling the following well-known definitions which pertain to our results.

Definition 1. *A function $f : (a,b) \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on the interval (a,b) if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y),$$

holds for all $x, y \in (a,b)$ and $\lambda \in [0,1]$. If f is twice differentiable, then it is said to be convex if and only if $f''(x) \geq 0$ for every $x \in (a,b)$. If the inequalities are strict, then f is said to be strictly convex. If the inequalities are reversed, then f is said to be concave.

Definition 2. *A function $f : (a,b) \subseteq (0,\infty) \to \mathbb{R}$ is said to be GA-convex on (a,b) if*

$$f(x^\lambda y^{1-\lambda}) \leq \lambda f(x) + (1-\lambda)f(y),$$

holds for all $x, y \in (a,b)$ and $\lambda \in [0,1]$ [16].

Definition 3. *A function $f : (0,\infty) \to \mathbb{R}$ is said to be completely monotonic if f has derivatives of all order and*

$$(-1)^k f^{(k)}(x) \geq 0,$$

holds for $x \in (0,\infty)$ and $k \in \mathbb{N}_0$.

Theorem 1. *Let F be defined as*

$$F(x) = x^a \left|\beta^{(m)}(x)\right|, \tag{14}$$

where $a \in \mathbb{R}$, $m \in \mathbb{N}$ and $x > 0$. Then $F(x)$ is decreasing if $a \leq m+1$ and increasing if $a \geq m+1+e^{-1}$.

Proof. By using (8) and the convolution theorem for Laplace transforms, we obtain the following.

$$F'(x) = ax^{a-1}\left|\beta^{(m)}(x)\right| - x^a\left|\beta^{(m+1)}(x)\right| =$$

$$= x^a\left[\frac{a}{x}\left|\beta^{(m)}(x)\right| - \left|\beta^{(m+1)}(x)\right|\right],$$

$$\frac{F'(x)}{x^a} = \frac{a}{x}\left|\beta^{(m)}(x)\right| - \left|\beta^{(m+1)}(x)\right| =$$

$$= a\int_0^\infty e^{-xt}\,dt\int_0^\infty \frac{t^m e^{-xt}}{1+e^{-t}}\,dt - \int_0^\infty \frac{t^{m+1}e^{-xt}}{1+e^{-t}}\,dt =$$

$$= a\int_0^\infty\left[\int_0^t \frac{s^m}{1+e^{-s}}\,ds\right]e^{-xt}\,dt - \int_0^\infty \frac{t^{m+1}e^{-xt}}{1+e^{-t}}\,dt = \int_0^\infty \phi_m(t)e^{-xt}\,dt,$$

where

$$\phi_m(t) = a\int_0^t \frac{s^m}{1+e^{-s}}\,ds - \frac{t^{m+1}}{1+e^{-t}}.$$

Then $\phi_m(0) = \lim\limits_{t\to 0^+}\phi_m(t) = 0$ and

$$\begin{aligned}\phi_m'(t) &= \frac{at^m}{1+e^{-t}} - \frac{(m+1)t^m}{1+e^{-t}} - \frac{t^{m+1}e^{-t}}{(1+e^{-t})^2} = \\ &= \frac{t^m}{1+e^{-t}}\left[a-(m+1)-\frac{te^{-t}}{1+e^{-t}}\right].\end{aligned} \tag{15}$$

If $a \le m+1$, then $\phi_m'(t) < 0$, which implies that $\phi_m(t)$ is decreasing. Then for $t > 0$ we obtain $\phi_m(t) < \phi_m(0) = 0$. Thus, $F'(x) < 0$ which gives the desired result. Likewise, if $a \ge m+1+e^{-1}$, then $\phi_m'(t) > 0$, which implies that $\phi_m(t)$ is increasing. Then for $t > 0$, we have $\phi_m(t) > \phi_m(0) = 0$. Hence, $F'(x) > 0$ and this completes the proof. □

Theorem 2. *Let $m \in \mathbb{N}$. Then the inequality*

$$\left|\beta^{(m)}(xy)\right| \le \left|\beta^{(m)}(x)\right| + \left|\beta^{(m)}(y)\right|, \tag{16}$$

holds for $x > 0$ and $y \ge 1$.

Proof. Let $G(x,y) = \left|\beta^{(m)}(xy)\right| - \left|\beta^{(m)}(x)\right| - \left|\beta^{(m)}(y)\right|$ for $m \in \mathbb{N}$, $x > 0$ and $y \geq 1$. Without loss of generality, let y be fixed. Then

$$G'(x,y) = -y\left|\beta^{(m+1)}(xy)\right| + \left|\beta^{(m+1)}(x)\right| = \\ = \frac{1}{x}\left[x\left|\beta^{(m+1)}(x)\right| - xy\left|\beta^{(m+1)}(xy)\right|\right].$$

Recall from Theorem 1 that the function $x\left|\beta^{(m)}(x)\right|$ is decreasing. Also, since $y \geq 1$, then $xy \geq x$. Hence, $G'(x,y) \geq 0$ and this implies that $G(x,y)$ is increasing. Then for $0 < x < \infty$, we obtain

$$G(x,y) \leq \lim_{x\to\infty} G(x,y) = -\left|\beta^{(m)}(y)\right| < 0,$$

which yields the result (16). □

Theorem 3. *Let $m \in \mathbb{N}$. Then the function*

$$H(x) = x\left|\beta^{(m)}(x)\right| \tag{17}$$

is strictly completely monotonic on $(0,\infty)$.

Proof. Note that

$$H'(x) = \left|\beta^{(m)}(x)\right| - x\left|\beta^{(m+1)}(x)\right|,$$

$$H''(x) = -2\left|\beta^{(m+1)}(x)\right| + x\left|\beta^{(m+2)}(x)\right|,$$

$$H'''(x) = 3\left|\beta^{(m+2)}(x)\right| - x\left|\beta^{(m+3)}(x)\right|.$$

By continuing the process, we obtain

$$H^{(n)}(x) = (-1)^{n-1}n\left|\beta^{(m+n-1)}(x)\right| + (-1)^n x\left|\beta^{(m+n)}(x)\right|,$$

which implies

$$\frac{(-1)^n H^{(n)}(x)}{x} = -\frac{n}{x}\left|\beta^{(m+n-1)}(x)\right| + \left|\beta^{(m+n)}(x)\right|.$$

Then by the convolution theorem for Laplace transforms, we obtain

$$\frac{(-1)^n H^{(n)}(x)}{x} = -n\int_0^\infty e^{-xt}\,dt\int_0^\infty \frac{t^{m+n-1}e^{-xt}}{1+e^{-t}}\,dt + \int_0^\infty \frac{t^{m+n}e^{-xt}}{1+e^{-t}}\,dt =$$

$$= -n \int_0^\infty \left[\int_0^t \frac{s^{m+n-1}}{1+e^{-s}} \, ds \right] e^{-xt} \, dt + \int_0^\infty \frac{t^{m+n} e^{-xt}}{1+e^{-t}} \, dt = \int_0^\infty \delta_m(t) e^{-xt} \, dt,$$

where

$$\delta_m(t) = -n \int_0^t \frac{s^{m+n-1}}{1+e^{-s}} \, ds + \frac{t^{m+n}}{1+e^{-t}}.$$

Note that $\delta_m(0) = \lim_{t \to 0^+} \delta_m(t) = 0$ and also,

$$\delta_m'(t) = -n \frac{t^{m+n-1}}{1+e^{-t}} + \frac{(m+n)t^{m+n-1}}{1+e^{-t}} + \frac{t^{m+n}e^{-t}}{(1+e^{-t})^2} =$$

$$= \frac{t^{m+n-1}}{1+e^{-t}} \left[m + \frac{te^{-t}}{1+e^{-t}} \right] > 0.$$

Hence $\delta_m(t)$ is increasing. Then for $t > 0$, we have $\delta_m(t) > \delta_m(0) = 0$. Thus, $(-1)^n H^{(n)}(x) > 0$ which concludes the proof. □

Theorem 4. *Let $m \in \mathbb{N}$. Then the function*

$$Q(x) = \left| \beta^{(m)}(a^x) \right|, \quad a > 1, \tag{18}$$

is strictly convex on $(0, \infty)$.

Proof. Direct differentiation yields

$$Q''(x) = -(\ln a)^2 \left[a^x \left| \beta^{(m+1)}(a^x) \right| - a^{2x} \left| \beta^{(m+2)}(a^x) \right| \right].$$

Let $a^x = z$. Then

$$\frac{Q''(x)}{(\ln a)^2 z^2} = -\frac{1}{z} \left| \beta^{(m+1)}(z) \right| + \left| \beta^{(m+2)}(z) \right| =$$

$$= -\int_0^\infty e^{-zt} \, dt \int_0^\infty \frac{t^{m+1} e^{-zt}}{1+e^{-t}} \, dt + \int_0^\infty \frac{t^{m+2} e^{-zt}}{1+e^{-t}} \, dt = \int_0^\infty A_m(t) e^{-zt} \, dt,$$

where

$$A_m(t) = -\int_0^t \frac{s^{m+1}}{1+e^{-s}} \, ds + \frac{t^{m+2}}{1+e^{-t}}.$$

Then $A_m(0) = \lim_{t\to 0^+} A_m(t) = 0$ and also,

$$A'_m(t) = -\frac{t^{m+1}}{1+e^{-t}} + \frac{(m+2)t^{m+1}}{1+e^{-t}} + \frac{t^{m+2}e^{-t}}{(1+e^{-t})^2} = \\ = \frac{t^{m+1}}{1+e^{-t}}\left[m+1+\frac{te^{-t}}{1+e^{-t}}\right] > 0.$$

Thus $A_m(t)$ is increasing. Then for $t > 0$, we have $A_m(t) > A_m(0) = 0$. Therefore, $Q''(x) > 0$ and this completes the proof. □

With regard to Theorem 4, there is no a such that $Q(x)$ is concave. Also, $Q(x)$ is increasing if $0 < a < 1$ and decreasing if $a > 1$. Furthermore, the convexity of $Q(x)$ implies that for $r, s > 0$, $u > 1$, $\frac{1}{u} + \frac{1}{v} = 1$ we have

$$\left|\beta^{(m)}\left(a^{\frac{r}{u}+\frac{s}{v}}\right)\right| \leq \frac{\left|\beta^{(m)}(a^r)\right|}{u} + \frac{\left|\beta^{(m)}(a^s)\right|}{v}.$$

By letting $x = a^r$ and $y = a^s$ we obtain

$$\left|\beta^{(m)}\left(x^{\frac{1}{u}}y^{\frac{1}{v}}\right)\right| \leq \frac{\left|\beta^{(m)}(x)\right|}{u} + \frac{\left|\beta^{(m)}(y)\right|}{v}, \tag{19}$$

which implies that the function $\left|\beta^{(m)}(x)\right|$ is GA-convex. Moreover, since the exponential function is convex, we have

$$a^{\frac{r}{u}+\frac{s}{v}} \leq \frac{a^r}{u} + \frac{a^s}{v},$$

for $u > 1$, $\frac{1}{u} + \frac{1}{v} = 1$. Then by the decreasing property of $\left|\beta^{(m)}(x)\right|$ we obtain

$$\left|\beta^{(m)}\left(\frac{a^r}{u} + \frac{a^s}{v}\right)\right| \leq \left|\beta^{(m)}\left(a^{\frac{r}{u}+\frac{s}{v}}\right)\right|,$$

which upon letting $x = a^r$ and $y = a^s$, yields

$$\left|\beta^{(m)}\left(\frac{x}{u} + \frac{y}{v}\right)\right| \leq \left|\beta^{(m)}\left(x^{\frac{1}{u}}y^{\frac{1}{v}}\right)\right|. \tag{20}$$

Now, combining (19) and (20) gives the double-inequality

$$\left|\beta^{(m)}\left(\frac{x}{u} + \frac{y}{v}\right)\right| \leq \left|\beta^{(m)}\left(x^{\frac{1}{u}}y^{\frac{1}{v}}\right)\right| \leq \frac{\left|\beta^{(m)}(x)\right|}{u} + \frac{\left|\beta^{(m)}(y)\right|}{v}.$$

Theorem 5. *Let $m \in \mathbb{N}$. Then the function*

$$T(x) = \left|\beta^{(m)}(x^c)\right|, \tag{21}$$

is strictly convex on $(0, \infty)$ if $c \leq -1$ or $c > 0$.

Proof. Similarly, direct differentiation gives

$$T''(x) = -c\left[(c-1)x^{c-2}\left|\beta^{(m+1)}(x^c)\right| - cx^{2c-2}\left|\beta^{(m+2)}(x^c)\right|\right] =$$

$$= (cx^{c-1})^2\left[\frac{1-c}{c}\frac{1}{x^c}\left|\beta^{(m+1)}(x^c)\right| + \left|\beta^{(m+2)}(x^c)\right|\right].$$

Let $x^c = z$. Then

$$\frac{T''(x)}{(cz^{\frac{c-1}{c}})^2} = \frac{1-c}{c}\frac{1}{z}\left|\beta^{(m+1)}(z)\right| + \left|\beta^{(m+2)}(z)\right| =$$

$$= \frac{1-c}{c}\int_0^\infty e^{-zt}\,dt\int_0^\infty \frac{t^{m+1}e^{-zt}}{1+e^{-t}}\,dt + \int_0^\infty \frac{t^{m+2}e^{-zt}}{1+e^{-t}}\,dt = \int_0^\infty K_m(t)e^{-zt}\,dt,$$

where

$$K_m(t) = \frac{1-c}{c}\int_0^t \frac{s^{m+1}}{1+e^{-s}}\,ds + \frac{t^{m+2}}{1+e^{-t}}.$$

Then, $K_m(0) = \lim_{t\to 0^+} K_m(t) = 0$ and

$$K_m'(t) = \frac{t^{m+1}}{1+e^{-t}}\left[\frac{1}{c} + m + 1 + \frac{te^{-t}}{1+e^{-t}}\right] > 0$$

for $c \leq -1$ or $c > 0$. Thus $K_m(t)$ is increasing. Then for $t > 0$ we have $K_m(t) > K_m(0) = 0$. Therefore, $T''(x) > 0$ which yields the desired result. □

Theorem 6. *Let a function D_m be defined as*

$$D_m(x) = \frac{x^{m+1}}{m!}\left|\beta^{(m)}(x)\right|, \tag{22}$$

where $x > 0$ and $m \in \mathbb{N}$. Then $D_m(x)$ is strictly decreasing, strictly concave, and the limits

$$\lim_{x\to 0} D_m(x) = 1 \quad \textit{and} \quad \lim_{x\to 0} D_m'(x) = 0 \quad \textit{hold.} \tag{23}$$

Proof. By direct differentiation we obtain

$$D'_m(x) = \frac{(m+1)x^m}{m!}\left|\beta^{(m)}(x)\right| - \frac{x^{m+1}}{m!}\left|\beta^{(m+1)}(x)\right| = \tag{24}$$
$$= \frac{x^{m+1}}{m!}\left[\frac{m+1}{x}\left|\beta^{(m)}(x)\right| - \left|\beta^{(m+1)}(x)\right|\right].$$

That is,

$$\frac{m!}{x^{m+1}}D'_m(x) = \frac{m+1}{x}\left|\beta^{(m)}(x)\right| - \left|\beta^{(m+1)}(x)\right| =$$

$$= (m+1)\int_0^\infty e^{-xt}\,dt\int_0^\infty \frac{t^m e^{-xt}}{1+e^{-t}}\,dt - \int_0^\infty \frac{t^{m+1}e^{-zt}}{1+e^{-t}}\,dt = \int_0^\infty \chi_m(t)e^{-zt}\,dt,$$

where

$$\chi_m(t) = (m+1)\int_0^t \frac{s^m}{1+e^{-s}}\,ds - \frac{t^{m+1}}{1+e^{-t}}.$$

Then $\chi_m(0) = \lim_{t\to 0^+}\chi_m(t) = 0$ and also,

$$\chi'_m(t) = -\frac{t^{m+1}}{(1+e^{-t})^2} < 0.$$

Hence, $\chi_m(t)$ is decreasing. Then for $t > 0$ we have $\chi_m(t) < \chi_m(0) = 0$. Thus, $D'_m(x) < 0$, which implies that $D_m(x)$ is strictly decreasing. Next, we have $D''_m(x) =$

$$= \frac{x^{m+1}}{m!}\left[\frac{m(m+1)}{x^2}\left|\beta^{(m)}(x)\right| - \frac{2(m+1)}{x}\left|\beta^{(m+1)}(x)\right| + \left|\beta^{(m+2)}(x)\right|\right].$$

Then,

$$\frac{m!}{x^{m+1}}D''_m(x) = \frac{m(m+1)}{x^2}\left|\beta^{(m)}(x)\right| - \frac{2(m+1)}{x}\left|\beta^{(m+1)}(x)\right| +$$

$$+\left|\beta^{(m+2)}(x)\right| = m(m+1)\int_0^\infty te^{-xt}\,dt\int_0^\infty \frac{t^m e^{-xt}}{1+e^{-t}}\,dt -$$

$$-2(m+1)\int_0^\infty e^{-xt}\,dt\int_0^\infty \frac{t^{m+1}e^{-xt}}{1+e^{-t}}\,dt+\int_0^\infty \frac{t^{m+2}e^{-zt}}{1+e^{-t}}\,dt=\int_0^\infty \Omega_m(t)e^{-zt}\,dt,$$

where

$$\Omega_m(t)=m(m+1)\int_0^t \frac{(t-s)s^m}{1+e^{-s}}\,ds-2(m+1)\int_0^t \frac{s^{m+1}}{1+e^{-s}}\,ds+\frac{t^{m+2}}{1+e^{-t}}.$$

Clearly, $\Omega_m(0)=\lim_{t\to 0^+}\Omega_m(t)=0$. In addition,

$$\begin{aligned}\Omega'_m(t)&=\frac{t^{m+1}e^{-t}}{(1+e^{-t})^2}\left[-m-me^t+t\right]=\\&=-\frac{t^{m+1}e^{-t}}{(1+e^{-t})^2}\left[2m+(m-1)t+\sum_{k=2}^\infty \frac{mt^k}{k!}\right]<0\end{aligned}$$

Hence, $\Omega_m(t)$ is decreasing. Then for $t>0$ we have $\Omega_m(t)<\Omega_m(0)=0$. Thus, $D''_m(x)<0$, which implies that $D_m(x)$ is strictly concave. Finally, the limits (23) are deduced from (12), (22), and (24). □

3. Concluding Remarks.

In this work we have shown that:

(a) $F(x)=x^a\left|\beta^{(m)}(x)\right|$, $m\in\mathbb{N}$ is decreasing if $a\le m+1$ and increasing if $a\ge m+1+e^{-1}$.

(b) $\left|\beta^{(m)}(xy)\right|\le\left|\beta^{(m)}(x)\right|+\left|\beta^{(m)}(y)\right|$, for $x>0$, $y\ge 1$ and $m\in\mathbb{N}$.

(c) $H(x)=x\left|\beta^{(m)}(x)\right|$, $m\in\mathbb{N}$ is strictly completely monotonic.

(d) $Q(x)=\left|\beta^{(m)}(a^x)\right|$, $m\in\mathbb{N}$, $a>1$ is strictly convex.

(e) $T(x)=\left|\beta^{(m)}(x^c)\right|$, $m\in\mathbb{N}$ is strictly convex if $c\le -1$ or $c>0$.

(f) $\left|\beta^{(m)}\left(\frac{x}{u}+\frac{y}{v}\right)\right|\le\left|\beta^{(m)}\left(x^{\frac{1}{u}}y^{\frac{1}{v}}\right)\right|\le\frac{\left|\beta^{(m)}(x)\right|}{u}+\frac{\left|\beta^{(m)}(y)\right|}{v}$, where $u>1$, $\frac{1}{u}+\frac{1}{v}=1$ and $m\in\mathbb{N}$.

(g) $D_m(x)=\frac{x^{m+1}}{m!}\left|\beta^{(m)}(x)\right|$, $m\in\mathbb{N}$ is strictly decreasing, strictly concave, $\lim_{x\to 0}D_m(x)=1$, and $\lim_{x\to 0}D'_m(x)=0$.

The function $\beta(x)$ studied in this paper satisfies the equality

$$\beta(x)=\frac{1}{2}G(x),$$

where $G(x)$ is the function referred to as the Bateman's G-function in the works [10], [11], and [12]. In these works their authors made reference to the work of Erdelyi et al. [7] where, probably, the name "Bateman's G-function" originates. However, before the work [7] the function $\beta(x)$ appeared in Nielsen's work [15]. Also, it is worth noting that page 54 of the work [7] actually captures Nielsen's work in the reference list. As a result of this, we prefer to call either of the functions the Nielsen's β-function.

4. Open Problems.

1. In relation to Theorem 3, find all values of $a \in \mathbb{R}$ such that the function $H_a(x) = x^a \left|\beta^{(m)}(x)\right|$ is completely monotonic.

2. With regard to Theorem 5, is the function $T(x) = \left|\beta^{(m)}(x^c)\right|$ concave on $(0, \infty)$ if $c \in (-1, 0)$?

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