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G. G. Akniyev

DISCRETE LEAST SQUARES APPROXIMATION OF PIECEWISE-LINEAR FUNCTIONS BY TRIGONOMETRIC POLYNOMIALS

Abstract. Let N be a natural number greater than 1. Select N uniformly distributed points $t_k = 2\pi k/N$ $(0 \le k \le \le N-1)$ on $[0, 2\pi]$. Denote by $L_{n,N}(f) = L_{n,N}(f, x)$ $(1 \le \le n \le N/2)$ the trigonometric polynomial of order n possessing the least quadratic deviation from f with respect to the system $\{t_k\}_{k=0}^{N-1}$. In this article approximation of functions by the polynomials $L_{n,N}(f, x)$ is considered. Special attention is paid to approximation of 2π-periodic functions f_1 and f_2 by the polynomials $L_{n,N}(f, x)$, where $f_1(x) = |x|$ and $f_2(x) = \operatorname{sign} x$ for $x \in \in [-\pi, \pi]$. For the first function f_1 we show that instead of the estimation $|f_1(x) - L_{n,N}(f_1, x)| \le c \ln n/n$ which follows from the well-known Lebesgue inequality for the polynomials $L_{n,N}(f, x)$ we found an exact order estimation $|f_1(x) - L_{n,N}(f_1, x)| \le c/n$ $(x \in \mathbb{R})$ which is uniform with respect to $1 \le n \le N/2$. Moreover, we found a local estimation $|f_1(x) - L_{n,N}(f_1, x)| \le c(\varepsilon)/n^2$ $(|x - \pi k| \ge \varepsilon)$ which is also uniform with respect to $1 \le n \le \le N/2$. For the second function f_2 we found only a local estimation $|f_2(x) - L_{n,N}(f_2, x)| \le c(\varepsilon)/n$ $(|x - \pi k| \ge \varepsilon)$ which is uniform with respect to $1 \le n \le N/2$. The proofs of these estimations are based on comparing of approximating properties of discrete and continuous finite Fourier series.

Key words: *function approximation, trigonometric polynomials, Fourier series*

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1. Introduction. Let N be a natural number greater then 1. Let $\{t_k\}_{k=0}^{N-1}$ be a set of points on $[0, 2\pi]$ where $t_k = 2\pi k/N$. Denote by $L_{n,N}(f) = L_{n,N}(f, x)$ $(1 \le n \le \lfloor N/2 \rfloor)$ a trigonometric polynomial of order n possessing the least quadratic deviation from the function f with respect to the system $\{t_k\}_{k=0}^{N-1}$. In other words, the minimum of the sums $\sum_{k=0}^{N-1} |f(t_k) - T_n(t_k)|^2$ on the set of trigonometric polynomials T_n of order n is attained when $T_n = L_{n,N}(f)$. In particular, $L_{\lfloor N/2 \rfloor,N}(f,t_k) = f(t_k)$. It is easy to show (see [12]) that for $n < N/2$ the polynomial $L_{n,N}(f,x)$ can be represented as

$$L_{n,N}(f,x) = \sum_{\nu=-n}^{n} c_\nu^{(N)}(f) e^{i\nu x}, \quad c_\nu^{(N)}(f) = \frac{1}{N} \sum_{k=0}^{N-1} f(t_k) e^{-i\nu t_k},$$

and for $n = N/2$ (for an even N)

$$L_{N/2,N}(f,x) = L_{N/2-1,N}(f,x) + a_{N/2}^{(N)}(f) \cos N/2(x-u), \tag{1}$$

where

$$a_n^{(2n)}(f) = a_{N/2}^{(N)}(f) = \frac{1}{N} \sum_{k=0}^{N-1} f(t_k) \cos N/2(t_k - u). \tag{2}$$

To read more about function approximation by trigonometric polynomials see [1, 3, 5, 6], [8]–[11], [13]. In this article we obtain estimations for $|L_{n,N}(f_1,x) - f_1(x)|$ and $|L_{n,N}(f_2,x) - f_2(x)|$ as $n, N \to \infty$, where $f_1(x) = |x|$, $f_2(x) = \operatorname{sign} x$, $x \in [-\pi,\pi]$. The following theorems are proved:

Theorem 1. *Let $f_1(x) = |x|$, $x \in [-\pi,\pi]$ and $n \le \lfloor N/2 \rfloor$. The following estimations hold:*

$$|L_{n,N}(f_1,x) - f_1(x)| \le c/n, \quad x \in [-\pi,\pi],$$

$$|L_{n,N}(f_1,x) - f_1(x)| \le c(\varepsilon)/n^2, \quad x \in \Delta^I(\varepsilon).$$

Theorem 2. *Let $f_2(x) = \operatorname{sign} x$, $x \in [-\pi,\pi]$ and $n \le \lfloor N/2 \rfloor$. The following estimation holds:*

$$|L_{n,N}(f,x) - f(x)| \le c(\varepsilon)/n, \quad x \in \Delta^I(\varepsilon).$$

We begin with some notation. Denote by

$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-i\nu t} dt, \quad k \in \mathbb{Z},$$

the Fourier coefficients of a function f, and by

$$f(x) = \sum_{k\in\mathbb{Z}} c_k(f)e^{ikx}, \quad S_n(f,x) = \sum_{k=-n}^{n} c_k(f)e^{ikx}$$

the Fourier series of a function f and its partial sum of order n, respectively. Denote by $\Delta^I(\varepsilon)$ the set $[-\pi+\varepsilon, -\varepsilon]\cup[\varepsilon, \pi-\varepsilon]$, where $0 < \varepsilon < \pi/2$. By c and $c(\varepsilon)$ we denote some positive constants that depend only on specified parameters; these constants may be different in different places.

Lemma 1. [12] *If the Fourier series of f converges at the points $t_k = = u + 2k\pi/N$, then the representation*

$$L_{n,N}(f,x) = S_n(f,x) + R_{n,N}(f,x),$$

where

$$R_{n,N}(f,x) = \frac{2}{\pi}\sum_{\mu=1}^{\infty}\int_{-\pi}^{\pi} D_n(x-t)\cos\mu N(u-t)f(t)dt, \tag{3}$$

holds true when $2n < N$.

The following estimation follows from this lemma:

$$|L_{n,N}(f,x) - f(x)| \le |S_n(f,x) - f(x)| + |R_{n,N}(f,x)|, \quad n < N/2. \tag{4}$$

Let us consider the case $2n = N$. From (1) and (4)

$$|L_{n,2n}(f,x) - f(x)| \le$$
$$\le |S_{n-1}(f,x) - f(x)| + |R_{n-1,2n}(f,x)| + |a_n^{(2n)}(f)|, \quad n = N/2. \tag{5}$$

From (4) and (5) we can see that estimation of the values $|S_n(f_1,x) - -f_1(x)|$, $|S_n(f_2,x) - f_2(x)|$, $|R_{n,N}(f_1,x)|$, $|R_{n,N}(f_2,x)|$, $|a_n^{(2n)}(f_1)|$, and $|a_n^{(2n)}(f_2)|$ implies an estimation $|L_{n,N}(f,x) - f(x)|$ for the functions f_1 and f_2.

2. Estimations for $|S_n(f_1,x) - f_1(x)|$ and $|S_n(f_2,x) - f_2(x)|$.

Lemma 2. *For the value $|S_n(f_1,x) - f_1(x)|$, where $f_1(x) = |x|$, $x \in \in [-\pi,\pi]$, we have the following estimations:*

$$|S_n(f_1,x) - f_1(x)| \le c/n, \quad x \in [-\pi,\pi], \tag{6}$$

$$|S_n(f_1,x) - f_1(x)| \le c(\varepsilon)/n^2, \quad x \in \Delta^I(\varepsilon). \tag{7}$$

Proof. Using [2, p.443] or [4, p.690], we can get the following representation for $f_1(x)$ on $[-\pi,\pi]$:

$$f_1(x) = |x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)x}{(2k-1)^2}, \quad x \in [-\pi,\pi].$$

From the above we can obtain estimation (6) (using the fact that $|\cos(2k-1)x| \le 1$). Also, if we apply the Abel transformation to the above equation, we get (7). □

Lemma 3. *For the value $|S_n(f_2,x) - f_2(x)|$, where $f_2(x) = \operatorname{sign} x$, $x \in \in [-\pi,\pi]$ we have the following estimation:*

$$|S_n(f_2,x) - f_2(x)| \le c(\varepsilon)/n, \quad x \in \Delta^I(\varepsilon).$$

The proof of this lemma is obtained in [7].

3. An estimation for $R_{n,N}(f_1,x)$. The following lemma takes place.

Lemma 4. *For $R_{n,N}(f_1,x)$, $n < N/2$ the following estimations hold:*

$$|R_{n,N}(f_1,x)| \le \frac{\pi}{n}\left(4 + \frac{1}{2n}\right) \le \frac{c}{n}, \quad x \in [-\pi,\pi],$$

$$|R_{n,N}(f_1,x)| \le \frac{\pi}{n^2}\left(\frac{1}{6} + \frac{4}{|\sin\varepsilon|}\right) \le \frac{c(\varepsilon)}{n^2}, \quad x \in \Delta^I(\varepsilon).$$

Note that the estimation for remainder (3) for the function f_1 has the following form:

$$R_{n,N}(f_1,x) = \frac{2}{\pi}\sum_{\mu=1}^{\infty}\int_{-\pi}^{\pi}|t|\, D_n(x-t)\cos\mu N(u-t)dt,$$

where

$$D_n(x-t) = 1/2 + \sum_{k=1}^{n}\cos k(x-t).$$

From above we have

$$|R_{n,N}(f_1,x)| \le \left|R^1_{n,N}(f_1,x)\right| + \left|R^2_{n,N}(f_1,x)\right|, \tag{8}$$

where

$$R^1_{n,N}(f_1,x) = \frac{1}{\pi}\sum_{\mu=1}^{\infty}\int_{-\pi}^{\pi}|t|\cos\mu N(u-t)dt, \tag{9}$$

$$R^2_{n,N}(f_1,x) = \frac{2}{\pi}\sum_{\mu=1}^{\infty}\int_{-\pi}^{\pi}|t|\sum_{k=1}^{n}\cos k(x-t)\cos\mu N(u-t)dt. \tag{10}$$

Lemma 5. *The value* $\left|R^1_{n,N}(f_1,x)\right|$ *has the following estimation:*

$$\left|R^1_{n,N}(f_1,x)\right| \le c/n^2, \quad x\in[-\pi,\pi].$$

Proof. Using the formula

$$\cos(\mu Nu-\mu Nt) = \cos\mu Nu\cos\mu Nt + \sin\mu Nu\sin\mu Nt \tag{11}$$

we can rewrite (9) as follows:

$$R^1_{n,N}(f_1,x) = \frac{2}{\pi}\sum_{\mu=1}^{\infty}\cos\mu Nu\int_{0}^{\pi}t\cos\mu Nt dt.$$

Then we calculate the integral using the previous equation:

$$\int_0^{\pi}t\cos\mu Nt dt = \frac{(-1)^{\mu N}-1}{(\mu N)^2}.$$

From these equations we get

$$R^1_{n,N}(f_1,x) = \frac{2}{\pi N^2}\sum_{\mu=1}^{\infty}\frac{\cos\mu Nu}{\mu^2}\left((-1)^{\mu N}-1\right)$$

or

$$R^1_{n,N}(f_1,x) = \begin{cases} 0, & N=2l,\\ -\frac{4}{\pi N^2}\sum\limits_{\mu=1}^{\infty}\frac{\cos(2\mu-1)Nu}{(2\mu-1)^2}, & N=2l+1.\end{cases}$$

Now we can estimate $\left|R^1_{n,N}(f_1,x)\right|$:

$$\left|R^1_{n,N}(f_1,x)\right| \le \frac{4}{\pi N^2}\sum_{\mu=1}^{\infty}\frac{1}{(2\mu-1)^2}.$$

It is known that
$$\sum_{\mu=1}^{\infty} \frac{1}{(2\mu-1)^2} = \frac{\pi^2}{8}. \tag{12}$$
So, using (12) we obtain
$$\left|R^1_{n,N}(f_1,x)\right| \le \frac{\pi}{2N^2} \le \frac{\pi}{8n^2}.$$

□

Lemma 6. *The value* $\left|R^2_{n,N}(f_1,x)\right|$ *has the following estimation for* $x \in$ $\in [-\pi,\pi]:$
$$\left|R^2_{n,N}(f_1,x)\right| \le \frac{c}{n}, \quad x \in [-\pi,\pi].$$

Before proving Lemma 6, we prove another one:

Lemma 7. *The following estimations take place:*
$$\left|\sum_{i=0}^{k} \cos(2i-1)x\right| \le \frac{1}{2\left|\sin x\right|}, \quad \left|\sum_{i=0}^{k} \cos 2ix\right| \le \frac{1}{\left|\sin x\right|},$$
$$\left|\sum_{i=1}^{k} \sin(2i+1)x\right| \le \frac{1}{\left|\sin x\right|}, \quad \left|\sum_{i=0}^{k-1} \sin 2ix\right| \le \frac{1}{\left|\sin x\right|}.$$

Proof. After some simple transforms we obtain the following:
$$\sum_{i=1}^{k} \cos(2i-1)x = \frac{1}{\sin x}\sum_{i=1}^{k} \sin x \cos(2i-1)x =$$
$$= \frac{1}{2\sin x}\sum_{i=1}^{k} (\sin\left(x-(2i-1)x\right) + \sin(x+(2i-1)x) =$$
$$= \frac{1}{2\sin x}\sum_{i=1}^{k} (\sin 2ix - \sin 2\left(i-1\right)x) = \frac{\sin 2kx}{2\sin x}.$$

Likewise, obtain
$$\sum_{i=1}^{k} \cos 2ix = \frac{1}{\sin x}\sum_{i=1}^{k} \sin x \cos 2ix = \frac{\sin(2k+1)x - \sin x}{2\sin x},$$

$$\sum_{i=1}^{k} \sin(2i+1)x = \frac{\cos 2x - \cos(2k+2)x}{2\sin x},$$

$$\sum_{i=1}^{k} \sin 2ix = \frac{\cos x - \cos(2k+1)x}{2\sin x}.$$

Now the proof is complete. □

Proof. (Lemma 6) From (11) and

$$\cos k(x-t) = \cos kx \cos kt + \sin kx \sin kt \tag{13}$$

(10) can be rewritten as follows:

$$R_{n,N}^2(f_1,x) = \frac{4}{\pi}\sum_{\mu=1}^{\infty} \cos \mu N u \sum_{k=1}^{n} \cos kx \int_0^{\pi} t \cos kt \cos \mu N t dt +$$
$$+ \frac{4}{\pi}\sum_{\mu=1}^{\infty} \sin \mu N u \sum_{k=1}^{n} \sin kx \int_0^{\pi} t \sin kt \sin \mu N t dt.$$

From above and the formulas

$$\cos kt \cos \mu N t = 1/2 \left(\cos\left(\mu N - k\right) t + \cos\left(\mu N + k\right) t\right),$$

$$\sin kt \sin \mu N t = 1/2 \left(\cos\left(\mu N - k\right) t - \cos\left(\mu N + k\right) t\right)$$

we can obtain

$$R_{n,N}^2(f_1,x) = \frac{2}{\pi}\sum_{\mu=1}^{\infty} \cos \mu N u \times$$
$$\times \sum_{k=1}^{n} \cos kx \left(\int_0^{\pi} t \cos\left(\mu N - k\right) t dt + \int_0^{\pi} t \cos\left(\mu N + k\right) t dt \right) +$$
$$+ \frac{2}{\pi}\sum_{\mu=1}^{\infty} \sin \mu N u \times$$
$$\times \sum_{k=1}^{n} \sin kx \left(\int_0^{\pi} t \cos\left(\mu N - k\right) t dt - \int_0^{\pi} t \cos\left(\mu N + k\right) t dt \right).$$

After calculating the integrals, we have

$$R^2_{n,N}(f_1,x)=\frac{2}{\pi}\sum_{\mu=1}^{\infty}\cos\mu Nu\times$$
$$\times\sum_{k=1}^{n}\left((-1)^{\mu N+k}-1\right)\cos kx\left(\frac{1}{(\mu N-k)^2}+\frac{1}{(\mu N+k)^2}\right)+$$
$$+\frac{2}{\pi}\sum_{\mu=1}^{\infty}\sin\mu Nu\times$$
$$\times\sum_{k=1}^{n}\left((-1)^{\mu N+k}-1\right)\sin kx\left(\frac{1}{(\mu N-k)^2}-\frac{1}{(\mu N+k)^2}\right). \quad (14)$$

We can rewrite this as follows:

$$|R_{n,N}(f_1,x)|\le\frac{4}{\pi}\sum_{\mu=1}^{\infty}\sum_{k=1}^{n}\left(\frac{1}{(\mu N-k)^2}+\frac{1}{(\mu N+k)^2}\right)+$$
$$+\frac{4}{\pi}\sum_{\mu=1}^{\infty}\sum_{k=1}^{n}\left(\frac{1}{(\mu N-k)^2}-\frac{1}{(\mu N+k)^2}\right)\le$$
$$\le\frac{12}{\pi}\sum_{\mu=1}^{\infty}\sum_{k=1}^{n}\frac{1}{(\mu N-k)^2}\le\frac{12}{\pi N}\sum_{\mu=1}^{\infty}\frac{1}{\mu^2}\sum_{k=1}^{n}\frac{1}{N\left(1-\frac{n}{\mu N}\right)^2}\le\frac{4\pi}{N}\le\frac{2\pi}{n}.$$

□

Lemma 8. *The value* $\left|R^2_{n,N}(f_1,x)\right|$ *has the following estimation for* $x\in$ $\in\Delta^I(\varepsilon)$:

$$\left|R^2_{n,N}(f_1,x)\right|\le\frac{c}{n^2\sin\varepsilon}=\frac{c(\varepsilon)}{n^2},\quad x\in\Delta^I(\varepsilon).$$

Proof. To prove this lemma, we use (14). It is easy to show that for even N

$$R^2_{n,N}(f_1,x)=-\frac{4}{\pi}\sum_{\mu=1}^{\infty}\cos\mu NuA_n^{\mu,N}(x)-\frac{4}{\pi}\sum_{\mu=1}^{\infty}\sin\mu NuB_n^{\mu,N}(x), \quad (15)$$

where

$$A_n^{\mu,N}(x)=\sum_{k=1}^{\lceil n/2\rceil}\cos(2k-1)x\left(\frac{1}{(\mu N-(2k-1))^2}+\frac{1}{(\mu N+(2k-1))^2}\right),$$

$$B_n^{\mu,N}(x)=\sum_{k=1}^{\lceil n/2\rceil}\sin(2k-1)x\left(\frac{1}{(\mu N-(2k-1))^2}-\frac{1}{(\mu N+(2k-1))^2}\right).$$

And, therefore

$$\left|R_{n,N}^2(f_1,x)\right|=\frac{4}{\pi}\sum_{\mu=1}^{\infty}\left|A_n^{\mu,N}(x)\right|+\frac{4}{\pi}\sum_{\mu=1}^{\infty}\left|B_n^{\mu,N}(x)\right|.$$

Consider the values $\left|A_n^{\mu,N}(x)\right|$ and $\left|B_n^{\mu,N}(x)\right|$. Apply the Abel transformation for $A_n^{\mu,N}(x)$:

$$A_n^{\mu,N}(x)=\left(\frac{1}{(\mu N-(2\lceil n/2\rceil-1))^2}+\frac{1}{(\mu N+(2\lceil n/2\rceil-1))^2}\right)\times$$
$$\times\sum_{i=1}^{\lceil n/2\rceil}\cos(2i-1)x-\sum_{k=1}^{\lceil n/2\rceil-1}\left[\left(\frac{1}{(\mu N-(2k+1))^2}+\frac{1}{(\mu N+(2k+1))^2}\right)-\right.$$
$$\left.-\left(\frac{1}{(\mu N-(2k-1))^2}+\frac{1}{(\mu N+(2k-1))^2}\right)\right]\sum_{i=1}^{k}\cos(2i-1)x.$$

From this we get the following estimation:

$$\left|A_n^{\mu,N}(x)\right|\le\left(\frac{1}{(\mu N-(2\lceil n/2\rceil-1))^2}+\frac{1}{(\mu N+(2\lceil n/2\rceil-1))^2}\right)\times$$
$$\times\left|\sum_{i=1}^{\lceil n/2\rceil}\cos(2i-1)x\right|+\sum_{k=1}^{\lceil n/2\rceil-1}\left[\left(\frac{1}{(\mu N-(2k+1))^2}+\frac{1}{(\mu N+(2k+1))^2}\right)-\right.$$
$$\left.-\left(\frac{1}{(\mu N-(2k-1))^2}+\frac{1}{(\mu N+(2k-1))^2}\right)\right]\left|\sum_{i=0}^{k}\cos(2i-1)x\right|.$$

Using Lemma 7, we can rewrite the obtained estimation:

$$\left|A_n^{\mu,N}(x)\right|\le\frac{2}{|\sin x|}\left(\frac{1}{(\mu N-(2\lceil n/2\rceil-1))^2}+\right.$$
$$\left.+\frac{1}{(\mu N+(2\lceil n/2\rceil-1))^2}\right)\le\frac{10}{(\mu N)^2|\sin x|}. \quad (16)$$

Using the similar approach, we get an estimation for $\left|B_n^{\mu,N}(x)\right|$:

$$\left|B_n^{\mu,N}(x)\right| \le \frac{8}{3\mu^3 N^2 \left|\sin x\right|}. \tag{17}$$

Using (15), (16), and (17) we can write

$$\left|R_{n,N}^2(f_1,x)\right| \le \frac{4}{\pi}\sum_{\mu=1}^{\infty}\left|A_n^{\mu,N}(x)\right| + \frac{4}{\pi}\sum_{\mu=1}^{\infty}\left|B_n^{\mu,N}(x)\right| \le$$
$$\le \frac{4}{\pi}\sum_{\mu=1}^{\infty}\frac{10}{\left(\mu N\right)^2\left|\sin x\right|} + \frac{4}{\pi}\sum_{\mu=1}^{\infty}\frac{8}{3\mu^3 N^2\left|\sin x\right|} \le \frac{26}{N^2\left|\sin x\right|}, \quad N = 2l. \tag{18}$$

If N is odd, then we can rewrite (14) as follows:

$$R_{n,N}^2(f_1,x) =$$
$$= -\frac{4}{\pi}\sum_{\mu=1}^{\infty} C_n^{\mu,N}(x)\cos(2\mu-1)Nu - \frac{4}{\pi}\sum_{\mu=1}^{\infty} D_n^{\mu,N}(x)\sin(2\mu-1)Nu -$$
$$- \frac{4}{\pi}\sum_{\mu=1}^{\infty} E_n^{\mu,N}(x)\cos 2\mu Nu - \frac{4}{\pi}\sum_{\mu=1}^{\infty} F_n^{\mu,N}(x)\sin 2\mu Nu,$$

where

$$C_n^{\mu,N}(x) = \sum_{k=1}^{\lfloor n/2\rfloor}\cos 2kx\left(\frac{1}{\left((2\mu-1)N-2k\right)^2} + \frac{1}{\left((2\mu-1)N+2k\right)^2}\right),$$

$$D_n^{\mu,N}(x) = \sum_{k=1}^{\lfloor n/2\rfloor}\sin 2kx\left(\frac{1}{\left((2\mu-1)N-2k\right)^2} - \frac{1}{\left((2\mu-1)N+2k\right)^2}\right),$$

$$E_n^{\mu,N}(x) = \sum_{k=1}^{\lceil n/2\rceil}\cos(2k-1)x\left(\frac{1}{\left(2\mu N-(2k-1)\right)^2} + \frac{1}{\left(2\mu N+(2k-1)\right)^2}\right),$$

$$F_n^{\mu,N}(x) = \sum_{k=1}^{\lceil n/2\rceil}\sin(2k-1)x\left(\frac{1}{\left(2\mu N-(2k-1)\right)^2} - \frac{1}{\left(2\mu N+(2k-1)\right)^2}\right).$$

$$\text{And } \left|R^2_{n,N}(f_1,x)\right| \leq \frac{4}{\pi}\sum_{\mu=1}^{\infty}\left|C_n^{\mu,N}(x)\right| + \frac{4}{\pi}\sum_{\mu=1}^{\infty}\left|D_n^{\mu,N}(x)\right| +$$

$$+\frac{4}{\pi}\sum_{\mu=1}^{\infty}\left|E_n^{\mu,N}(x)\right| + \frac{4}{\pi}\sum_{\mu=1}^{\infty}\left|F_n^{\mu,N}(x)\right|.$$

Using the Abel transformation, we can easily obtain these estimations:

$$\left|C_n^{\mu,N}(x)\right| \leq \frac{10}{|\sin x|\,\mu^2 N^2}, \quad \left|D_n^{\mu,N}(x)\right| \leq \frac{8}{|\sin x|\,\mu^2 N^2},$$

$$\left|E_n^{\mu,N}(x)\right| \leq \frac{2}{|\sin x|\,\mu^2 N^2}, \quad \left|F_n^{\mu,N}(x)\right| \leq \frac{8}{9\,|\sin x|\,\mu^2 N^2}.$$

From this we have

$$\left|R^2_{n,N}(f_1,x)\right| \leq \frac{14\pi}{N^2\,|\sin x|}, \quad N = 2l+1. \tag{19}$$

From (18) and (19) we have (for every $N \geq 2$)

$$\left|R^2_{n,N}(f_1,x)\right| \leq \frac{14\pi}{N^2\,|\sin x|} \leq \frac{7\pi}{2n^2\,|\sin x|} \leq \frac{7\pi}{2n^2\,|\sin \varepsilon|}, \quad x \in \Delta^I(\varepsilon).$$

□

4. An estimation for $a_n^{(2n)}(f_1)$. When $2n = N$ from (2) we have the following:

$$a_n^{(2n)}(f) = \frac{1}{2n}\sum_{k=0}^{2n-1} f(t_k)\cos\pi k = \frac{1}{2n}\sum_{k=0}^{2n-1}(-1)^k f\left(u+\frac{\pi k}{n}\right). \tag{20}$$

Subtract $\frac{\pi}{n}$ from u to get (provided that f is 2π periodic function)

$$\frac{1}{2n}\sum_{k=0}^{2n-1}(-1)^k f\left(\left(u-\frac{\pi}{n}\right)+\frac{\pi k}{n}\right) = \frac{1}{2n}\sum_{k=0}^{2n-1}(-1)^k f\left(u+\frac{\pi\,(k-1)}{n}\right) =$$

$$= \frac{1}{2n}\sum_{k=1}^{2n}(-1)^{k-1} f\left(u+\frac{\pi k}{n}\right) = -\frac{1}{2n}\sum_{k=1}^{2n}(-1)^k f\left(u+\frac{\pi k}{n}\right) =$$

$$= -\frac{1}{2n}\sum_{k=1}^{2n-1}(-1)^k f\left(u+\frac{\pi k}{n}\right) + (-1)^{2n} f(u+2\pi) =$$

$$= -\frac{1}{2n}\sum_{k=0}^{2n-1}(-1)^k f\left(u+\frac{\pi k}{n}\right) = -a_n^{(2n)}(f).$$

Therefore, if we subtract or add $\frac{\pi l}{n}$ from u, we get

$$a_n^{(2n)}(f) = \frac{(-1)^l}{2n} \sum_{k=0}^{2n-1} (-1)^k f\left(\left(u + \frac{\pi l}{n}\right) + \frac{\pi k}{n}\right).$$

In other words, adding $\frac{\pi l}{n}$ $(l \in \mathbb{Z})$ to u does not change the value of $\left|a_n^{(2n)}(f)\right|$, so we can assume, without loss of generality, that $0 \le u < \frac{\pi}{n}$. It holds for both $\left|a_n^{(2n)}(f_1)\right|$ and $\left|a_n^{(2n)}(f_2)\right|$.

Lemma 9. *The following estimation takes place:*

$$\left|a_n^{(2n)}(f_1)\right| \le \frac{c}{n^2}, \quad x \in [-\pi, \pi].$$

Proof. It is easy to show that

$$a_n^{(2n)}(f_1) = \frac{1}{2n} \sum_{k=-n}^{n-1} (-1)^k \left|\frac{\pi k}{n} + u\right|, \quad 0 \le u < \frac{\pi}{n}.$$

$$a_n^{(2n)}(f_1) = \frac{1}{2n} \sum_{k=-n}^{n-1} (-1)^k \left|\frac{\pi k}{n} + u\right| =$$

$$= \frac{1}{2n} \sum_{k=-n}^{-1} (-1)^k \left(-\frac{\pi k}{n} - u\right) + \frac{1}{2n} \sum_{k=0}^{n-1} (-1)^k \left(\frac{\pi k}{n} + u\right) =$$

$$= \frac{1}{2n} \sum_{k=1}^{n} (-1)^k \left(\frac{\pi k}{n} - u\right) + \frac{1}{2n} \sum_{k=0}^{n-1} (-1)^k \left(\frac{\pi k}{n} + u\right) =$$

$$= \frac{1}{2n} \sum_{k=1}^{n} (-1)^k \left(\frac{\pi k}{n} - u\right) - \frac{1}{2n} \sum_{k=1}^{n} (-1)^k \left(\frac{\pi (k-1)}{n} + u\right) =$$

$$= \frac{1}{2n} \sum_{k=1}^{n} (-1)^k \left(\frac{\pi k}{n} - \frac{\pi(k-1)}{n} - 2u\right) = \frac{1}{2n} \sum_{k=1}^{n} (-1)^k \left(\frac{\pi}{n} - 2u\right).$$

Get the following estimation:

$$\left|a_n^{(2n)}(f_1)\right| \le \left|\frac{1}{2n} \sum_{k=1}^{n} (-1)^k \left(\frac{\pi}{n} - 2u\right)\right| \le \frac{1}{2n} \left|\frac{\pi}{n} - 2u\right| \le \frac{\pi}{2n^2}.$$

□

5. An estimation for $R_{n,N}(f_2,x)$. We prove the following lemma.

Lemma 10. *The following takes place:*

$$|R_{n,N}(f_2,x)| \leq \frac{c(\varepsilon)}{n}, \quad x \in \Delta^I(\varepsilon).$$

To prove this lemma we estimate $|R_{n,N}(f_1,x)|$ as follows:

$$|R_{n,N}(f_2,x)| \leq \left|R^1_{n,N}(f_2,x)\right| + \left|R^2_{n,N}(f_2,x)\right|, \tag{21}$$

where

$$R^1_{n,N}(f_2,x) = \frac{1}{\pi}\sum_{\mu=1}^{\infty}\int_{-\pi}^{\pi} \cos\mu N(u-t)\operatorname{sign} t dt, \tag{22}$$

$$R^2_{n,N}(f_2,x) = \frac{2}{\pi}\sum_{\mu=1}^{\infty}\int_{-\pi}^{\pi}\sum_{k=1}^{n} \cos k(x-t)\cos\mu N(u-t)\operatorname{sign} t dt.$$

First we prove the following lemmas:

Lemma 11. *The following takes place:*

$$\left|R^1_{n,N}(f_2,x)\right| \leq \frac{c}{n}.$$

Proof. Using (11) for (22) we have

$$R^1_{n,N}(f_2,x) = \frac{1}{\pi}\sum_{\mu=1}^{\infty}\int_{-\pi}^{\pi} \cos\mu N(u-t)\operatorname{sign} t dt =$$

$$= \frac{1}{\pi}\sum_{\mu=1}^{\infty}\int_{-\pi}^{\pi} (\cos\mu N u\cos\mu N t + \sin\mu N u \sin\mu N t)\operatorname{sign} t dt =$$

$$= \frac{1}{\pi}\sum_{\mu=1}^{\infty}\left(\cos\mu N u\int_{-\pi}^{\pi}\cos\mu N t \operatorname{sign} t dt + \sin\mu N u\int_{-\pi}^{\pi}\sin\mu N t\operatorname{sign} t dt\right) =$$

$$= \frac{1}{\pi}\sum_{\mu=1}^{\infty}\sin\mu N u\int_{-\pi}^{\pi}\sin\mu N t\operatorname{sign} t dt = \frac{2}{\pi}\sum_{\mu=1}^{\infty}\sin\mu N u\int_{0}^{\pi}\sin\mu N t dt.$$

From above and

$$\int_0^\pi \sin\mu N t dt = -\frac{\cos\mu N t}{\mu N}\Bigg|_0^\pi = \frac{1-(-1)^{\mu N}}{\mu N}$$

we have

$$R^1_{n,N}(f_2,x) = \frac{2}{\pi N}\sum_{\mu=1}^{\infty}\left(1-(-1)^{\mu N}\right)\frac{\sin\mu N u}{\mu}.$$

If N is even, then $R^1_{n,N}(f_2,x)\equiv 0$, otherwise

$$R^1_{n,N}(f_2,x) = -\frac{4}{\pi N}\sum_{\mu=1}^{\infty}\frac{\sin(2\mu-1)Nu}{2\mu-1},$$

where

$$\left|\frac{4}{\pi}\sum_{\mu=1}^{\infty}\frac{\sin(2\mu-1)x}{2\mu-1}\right| = |f_2(x)|\le 1.$$

Therefore

$$\left|R^1_{n,N}(f_2,x)\right| \le \frac{1}{N}\le\frac{1}{2n}.$$

□

Lemma 12. *Denote* $\gamma_n(x) = \sum\limits_{k=1}^{n}(1-(-1)^{k+m})\alpha_k\cos kx$, *where all* $\alpha_k>0$, *the sequence* $\{\alpha_k\}_{k=1}^n$ *is monotone, and* $m=0,1$. *Then*

$$|\gamma_n(x)|\le\frac{c}{|\sin x|}.$$

Proof. Assume that $\alpha_k>\alpha_{k+1}$. After applying the Abel transformation we get

$$\gamma_n(x) = \sum_{k=1}^{n}(1-(-1)^{k+m})\alpha_k\cos kx =$$

$$= \alpha_n\sum_{i=1}^{n}(1-(-1)^{i+m})\cos ix - \sum_{k=1}^{n-1}(\alpha_{k+1}-\alpha_k)\sum_{i=1}^{k}(1-(-1)^{i+m})\cos ix.$$

From the above equation and Lemma 7 we have

$$\begin{aligned}|\gamma_n(x)| \le \alpha_n \left|\sum_{i=1}^{n}(1-(-1)^{i+m})\cos ix\right| + \\ + \sum_{k=1}^{n-1}(\alpha_{k+1}-\alpha_k)\left|\sum_{i=1}^{k}(1-(-1)^{i+m})\cos ix\right| \le \\ \le \frac{1}{|\sin x|}(\alpha_n+\alpha_n-\alpha_0) \le \frac{2\alpha_n}{|\sin x|} = \frac{c}{|\sin x|}.\end{aligned}$$

□

Lemma 13. *If* $\nu_n(x) = \sum\limits_{k=1}^{n}(1-(-1)^{k+m})\alpha_k \sin kx$, *where all* $\alpha_k > 0$, *the sequence* α_k *is monotone, and* $m = 0, 1$. *Then*

$$|\nu_n(x)| \le \frac{c}{|\sin x|}.$$

The proof of this lemma is analogous to the proof of Lemma 12.

Lemma 14.

$$\left|\sum_{k=1}^{n}\sin kx\right| \le \frac{1}{\left|\sin\frac{x}{2}\right|}, \quad \forall n \in \mathbb{N}.$$

Proof. From

$$\sum_{k=1}^{n}\sin kx = \frac{1}{\sin\frac{x}{2}}\sum_{k=1}^{n}\sin kx \sin\frac{x}{2}$$

we can easily obtain the equality

$$\sum_{k=1}^{n}\sin kx = \frac{\sin\frac{n+1}{2}x \sin\frac{n}{2}x}{\sin\frac{x}{2}},$$

which gives us the desired estimation $|\sum_{k=1}^{n}\sin kx| \le \frac{1}{|\sin\frac{x}{2}|}$. □

Lemma 15. *The following takes place:*

$$\left|R^2_{n,N}(f_2,x)\right| \le \frac{c(\varepsilon)}{n}, \quad x \in \Delta^I(\varepsilon). \tag{23}$$

Proof. From (11) and (13) we have

$$R^2_{n,N}(f_2,x)=\frac{2}{\pi}\sum_{\mu=1}^{\infty}\int\limits_{-\pi}^{\pi}\sum_{k=1}^{n}\cos k(x-t)\cos\mu N(u-t)\operatorname{sign}t dt=$$

$$=\frac{2}{\pi}\sum_{\mu=1}^{\infty}\sin\mu Nu\sum_{k=1}^{n}\cos kx\int\limits_{-\pi}^{\pi}(\cos kt\sin\mu Nt)\operatorname{sign}t dt+$$

$$+\frac{2}{\pi}\sum_{\mu=1}^{\infty}\cos\mu Nu\sum_{k=1}^{n}\sin kx\int\limits_{-\pi}^{\pi}(\sin kt\cos\mu Nt)\operatorname{sign}t dt=$$

$$=\frac{4}{\pi}\sum_{\mu=1}^{\infty}\sin\mu Nu\sum_{k=1}^{n}\cos kx\int\limits_{0}^{\pi}\cos kt\sin\mu Nt dt+$$

$$+\frac{4}{\pi}\sum_{\mu=1}^{\infty}\cos\mu Nu\sum_{k=1}^{n}\sin kx\int\limits_{0}^{\pi}\sin kt\cos\mu Nt dt.$$

Using the formula $\sin\alpha\cos\beta=\frac{1}{2}\left(\sin(\alpha-\beta)+\sin(\alpha+\beta)\right)$ and calculating the integrals from the above equation, we get $\left|R^2_{n,N}(f_2,x)\right|\le$ $\le\left|R^{2.1}_{n,N}(f_2,x)\right|+\left|R^{2.2}_{n,N}(f_2,x)\right|$, where

$$R^{2.1}_{n,N}(f_2,x)=\frac{2}{\pi}\sum_{\mu=1}^{\infty}\sin\mu Nu\sum_{k=1}^{n}\left(1-(-1)^{\mu N+k}\right)\cos kx\frac{\mu N}{(\mu N)^2-k^2},$$

$$R^{2.2}_{n,N}(f_2,x)=\frac{4}{\pi}\sum_{\mu=1}^{\infty}\cos\mu Nu\sum_{k=1}^{n}\left(1-(-1)^{\mu N+k}\right)\sin kx\frac{k}{k^2-(\mu N)^2}.$$

We estimate values $\left|R^{2.1}_{n,N}(f_2,x)\right|$ and $\left|R^{2.2}_{n,N}(f_2,x)\right|$ separately. Begin with $\left|R^{2.1}_{n,N}(f_2,x)\right|$.

$$R^{2.1}_{n,N}(f_2,x)=\frac{2}{\pi N}\sum_{\mu=1}^{\infty}\frac{\sin\mu Nu}{\mu}\sum_{k=1}^{n}(1-(-1)^{\mu N+k})\cos kx\frac{1}{1-\frac{k^2}{(\mu N)^2-k^2}}=$$

$$=\frac{2}{\pi N}\sum_{\mu=1}^{\infty}\frac{\sin\mu Nu}{\mu}\sum_{k=1}^{n}(1-(-1)^{\mu N+k})\cos kx\left(1+\frac{k^2}{(\mu N)^2-k^2}\right).$$

We can rewrite it as follows:

$$\begin{aligned}R_{n,N}^{2.1}(f_2,x) &= \frac{2}{\pi N}\sum_{\mu=1}^{\infty}\frac{\sin\mu N u}{\mu}\sum_{k=1}^{n}\cos kx+\\ &+\frac{2}{\pi N}\sum_{\mu=1}^{\infty}\frac{(-1)^{\mu N}\sin\mu N u}{\mu}\sum_{k=1}^{n}(-1)^k\cos kx+\\ &+\frac{2}{\pi N}\sum_{\mu=1}^{\infty}\frac{\sin\mu N u}{\mu}\sum_{k=1}^{n}\cos kx\frac{k^2}{(\mu N)^2-k^2}+\\ &+\frac{2}{\pi N}\sum_{\mu=1}^{\infty}\frac{(-1)^{\mu N}\sin\mu N u}{\mu}\sum_{k=1}^{n}\cos kx\frac{(-1)^k k^2}{(\mu N)^2-k^2}.\end{aligned}$$

Now we have

$$\begin{aligned}\left|R_{n,N}^{2.1}(f_2,x)\right| &\le \frac{2}{\pi N}\left|\sum_{\mu=1}^{\infty}\frac{\sin\mu N u}{\mu}\right|\cdot\left|\sum_{k=1}^{n}\cos kx\right|+\\ &+\frac{2}{\pi N}\left|\sum_{\mu=1}^{\infty}\frac{(-1)^{\mu N}\sin\mu N u}{\mu}\right|\cdot\left|\sum_{k=1}^{n}(-1)^k\cos kx\right|+\\ &+\left|\frac{2}{\pi N}\sum_{\mu=1}^{\infty}\frac{\sin\mu N u}{\mu}\sum_{k=1}^{n}\cos kx\frac{k^2}{(\mu N)^2-k^2}\right|+\\ &+\left|\frac{2}{\pi N}\sum_{\mu=1}^{\infty}\frac{(-1)^{\mu N}\sin\mu N u}{\mu}\sum_{k=1}^{n}\cos kx\frac{(-1)^k k^2}{(\mu N)^2-k^2}\right|.\end{aligned}$$

From [4, p. 448] we get

$$\frac{2}{\pi}\left|\sum_{\mu=1}^{\infty}\frac{\sin\mu N u}{\mu}\right|\le 1 \text{ and } \frac{2}{\pi}\left|\sum_{\mu=1}^{\infty}\frac{(-1)^{\mu N}\sin\mu N u}{\mu}\right|\le 1.$$

Also, from Lemma 7

$$\left|\sum_{k=1}^{n}\cos kx\right|\le\frac{1}{|\sin x|},\quad \left|\sum_{k=1}^{n}(-1)^k\cos kx\right|\le\frac{1}{|\sin x|}.$$

Using the above estimations we can write

$$\left|R^{2.1}_{n,N}(f_2,x)\right| \le \frac{2}{N\left|\sin x\right|} + \frac{2}{\pi N}\left|\sum_{\mu=1}^{\infty}\frac{\sin\mu N u}{\mu}\sum_{k=1}^{n}\cos kx\frac{k^2}{(\mu N)^2-k^2}\right| +$$
$$+\frac{2}{\pi N}\left|\sum_{\mu=1}^{\infty}\frac{(-1)^{\mu N}\sin\mu N u}{\mu}\sum_{k=1}^{n}\cos kx\frac{(-1)^k k^2}{(\mu N)^2-k^2}\right|. \quad (24)$$

To estimate the value

$$\frac{2}{\pi N}\left|\sum_{\mu=1}^{\infty}\frac{\sin\mu N u}{\mu}\sum_{k=1}^{n}\cos kx\frac{k^2}{(\mu N)^2-k^2}\right|$$

make some transformations:

$$\frac{2}{\pi N}\left|\sum_{\mu=1}^{\infty}\frac{\sin\mu N u}{\mu}\sum_{k=1}^{n}\cos kx\frac{k^2}{(\mu N)^2-k^2}\right| =$$

$$= \frac{2}{\pi N}\left|\sum_{\mu=1}^{\infty}\frac{\sin\mu N u}{\mu^3}\sum_{k=1}^{n}\cos kx\frac{k^2}{N^2-\frac{k^2}{\mu^2}}\right| \le$$

$$\le \frac{2}{\pi N}\sum_{\mu=1}^{\infty}\frac{1}{\mu^3}\left|\sum_{k=1}^{n}\cos kx\frac{k^2}{N^2-\frac{k^2}{\mu^2}}\right| = \frac{2}{\pi N}\sum_{\mu=1}^{\infty}\frac{1}{\mu^3}\left|\frac{n^2}{N^2-\frac{k^2}{\mu^2}}\sum_{j=1}^{n}\cos jx - \right.$$
$$\left. -\sum_{k=1}^{n-1}\left(\frac{(k+1)^2}{N^2-\frac{(k+1)^2}{\mu^2}}-\frac{k^2}{N^2-\frac{k^2}{\mu^2}}\right)\sum_{j=1}^{k}\cos jx\right| \le$$

$$\le \frac{2}{\pi N}\sum_{\mu=1}^{\infty}\frac{1}{\mu^3}\left(\frac{2n^2}{N^2-\frac{k^2}{\mu^2}}\left|\sum_{j=1}^{n}\cos jx\right| + \right.$$
$$\left. +\sum_{k=1}^{n-1}\left(\frac{(k+1)^2}{N^2-\frac{(k+1)^2}{\mu^2}}-\frac{k^2}{N^2-\frac{k^2}{\mu^2}}\right)\left|\sum_{j=1}^{k}\cos jx\right|\right) \le$$

$$\le \frac{2}{\pi N\left|\sin x\right|}\sum_{\mu=1}^{\infty}\frac{1}{\mu^3}\left(\frac{2n^2}{N^2-\frac{k^2}{\mu^2}}-\frac{1}{N^2-\frac{1}{\mu^2}}\right) \le$$

$$\leq \frac{8}{3\pi N\,|\sin x|}\sum_{\mu=1}^{\infty}\frac{1}{\mu^3} \leq \frac{4}{\pi N\,|\sin x|}.$$

Using the similar approach, prove

$$\frac{2}{\pi N}\left|\sum_{\mu=1}^{\infty}\frac{(-1)^{\mu N}\sin\mu N u}{\mu}\sum_{k=1}^{n}\cos kx\frac{(-1)^k k^2}{(\mu N)^2-k^2}\right| \leq \frac{4}{\pi N\,|\sin x|}.$$

Now we can rewrite (24) as follows:

$$\left|R_{n,N}^{2.1}(f_2,x)\right| \leq \frac{1}{N\,|\sin x|}\left(2+\frac{8}{\pi}\right) = \frac{c}{N\,|\sin x|}. \tag{25}$$

Now consider $\left|R_{n,N}^{2.2}(f_2,x)\right|$.

$$R_{n,N}^{2.2}(f_2,x) = \frac{4}{\pi}\sum_{\mu=1}^{\infty}\cos\mu N u\sum_{k=1}^{n}\left(1-(-1)^{\mu N+k}\right)\sin kx\frac{k}{k^2-(\mu N)^2} =$$

$$= \frac{4}{\pi N}\sum_{\mu=1}^{\infty}\frac{\cos\mu N u}{\mu^2}\sum_{k=1}^{n}(1-(-1)^{\mu N+k})\sin kx\frac{k}{\frac{k^2}{\mu^2 N}-1}.$$

$$\left|R_{n,N}^{2.2}(f_2,x)\right| \leq \frac{4}{\pi N}\sum_{\mu=1}^{\infty}\frac{1}{\mu^2}\left|\sum_{k=1}^{n}(1-(-1)^{\mu N+k})\sin kx\frac{k}{\frac{k^2}{\mu^2 N}-1}\right|.$$

Using the approach we used for estimating $\left|R_{n,N}^{2.1}(f_2,x)\right|$, prove that

$$\left|R_{n,N}^{2.2}(f_2,x)\right| \leq \frac{c}{N\,|\sin x|}. \tag{26}$$

From the obtained estimations (25) and (26), and the inequalities $n \leq N/2$ and $|\sin\varepsilon| \leq |\sin x|$ for $x \in \Delta^I(\varepsilon)$, we have (23). $\square$

From (21) and Lemmas 11 and 15 we have

$$|R_{n,N}(f_2,x)| \leq \frac{1}{n}\left(c+c(\varepsilon)\right) = \frac{c(\varepsilon)}{n}, \quad x \in \Delta^I(\varepsilon).$$

So, Lemma 10 is proved.

6. An estimation for $a_n^{(2n)}(f_2)$. We prove a lemma:

Lemma 16. *The following estimation takes place:*

$$\left|a_n^{(2n)}(f_2)\right| \le \frac{c}{n}.$$

Proof. Using formula (20) we get (for a 2π-periodic function f)

$$a_n^{(2n)}(f) = \frac{1}{2n}\sum_{k=0}^{2n-1} f(t_k)\cos\pi k = \frac{1}{2n}\sum_{k=0}^{2n-1}(-1)^k f\left(u+\pi k/n\right).$$

As has been mentioned earlier, we can safely assume that $0 \le u < \pi/n$; so we write

$$a_n^{(2n)}(f_2) = 1/2n\sum_{k=-n}^{n-1}(-1)^k f_2\left(\pi k/n + u\right), \quad 0 \le u < \pi/n.$$

$$a_n^{(2n)}(f_2) = \frac{1}{2n}\left(\sum_{k=-n+1}^{-1}(-1)^{k+1} + \sum_{k=1}^{n-1}(-1)^k + \right.$$
$$\left. + (-1)^n f_2\left(u-\pi\right) + f_2\left(u\right)\right) = \frac{1}{2n}\left(f_2(u) - (-1)^n f_2(u)\right).$$

From this we get

$$\left|a_n^{(2n)}(f_2)\right| \le 1/n.$$

□

7. Proofs of Theorems 1 and 2. From (4) and (5) we have

$$|L_{n,N}(f_1,x) - f_1(x)| \le |S_n(f_1,x) - f_1(x)| + |R_{n,N}(f_1,x)|, \quad n < N/2,$$

$$|L_{n,2n}(f_1,x) - f_1(x)| \le$$
$$\le |S_{n-1}(f_1,x) - f_1(x)| + |R_{n-1,2n}(f_1,x)| + |a_n^{(2n)}(f_1)|, \quad n = N/2.$$

From Lemmas 2, 4, 9 we easily get

$$|L_{n,N}(f_1,x) - f_1(x)| \le c/n, \quad x \in [-\pi,\pi],$$

$$|L_{n,N}(f_1,x) - f_1(x)| \le c(\varepsilon)/n^2, \quad x \in \Delta^I(\varepsilon).$$

Theorem 1 is proved.

Using the same technique, inequalities (4) and (5), and Lemmas 3, 10 and 16, we get

$$|L_{n,N}(f,x) - f(x)| \le c(\varepsilon)/n, \quad x \in \Delta^I(\varepsilon).$$

So, theorem 2 is also proved.

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Dagestan scientific center of RAS
45, Gadzhieva st., Makhachkala 367025, Russia
E-mail: hasan.akniyev@gmail.com