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ON THE PROJECTIONS OF MUTUAL $L^{q,t}$-SPECTRUM

Abstract. In this paper we are interested in the mutual $L^{q,t}$-spectrum relatively to two Borel probability measures having the same compact support and also in the study of their behavior under orthogonal projections.

Key words: *orthogonal projection, dimension spectra, mutual multifractal analysis*

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1. Introduction. The notion of singularity exponents or spectrum and generalized dimensions are the major components of the multifractal analysis. They were introduced to characterize the geometry of measure and are linked with the multifractal spectrum. The multifractal spectrum is the map that affects the Hausdorff or packing dimension of the iso-Hölder set

$$E(\alpha,\beta)=\left\{x\in S_\mu\cap S_\nu;\ \lim_{r\to 0}\frac{\log\mu B_r(x)}{\log r}=\alpha \text{ and } \lim_{r\to 0}\frac{\log\nu B_r(x)}{\log r}=\beta\right\}$$

for a given $\alpha,\beta\geq 0$. Here S_μ is the topological support of the probability measure μ on $\mathbb{R}^n$ and $B_r(x)$ is the closed ball of center x and radius r. For $(q,t)\in\mathbb{R}^2$ the mutual $L^{q,t}$-spectrum of (μ,ν) is defined as the mapping

$$\tau_{\mu,\nu}(q,t)=\lim_{r\to 0}\frac{\log\left(\sup\left\{\sum\limits_i \mu(B_r(x_i))^q\nu(B_r(x_i))^t\right\}\right)}{\log r},$$

where the supremum is taken over all centered packing of $S_\mu\cap S_\nu$ by balls of radius r. It is easy to check that $\tau_{\mu,\nu}(q,t)$ is a concave function [24] of (q,t) over $\mathbb{R}^2$; for $q,t>1$ it has an integrand expression.

For $q,t\geq 1$ and for equal compact supports S_μ, S_ν we have

$$\tau_{\mu,\nu}(q,t)=\liminf_{r\to 0}\frac{1}{\log r}\,\log\iint\limits_{S_\mu\times S_\nu}\mu(B_r(x))^{q-1}\nu(B_r(y))^{t-1}d\mu(x)d\nu(y).$$

This equation unifies the mutual multifractal spectra to the mutual $L^{q,t}$-spectrum $\tau_{\mu,\nu}(q,t)$ via the Legendre transform [22, 23], i.e.,

$$\dim_H\big(E(\alpha,\beta)\big)=\inf_{q,t}\Big\{q\alpha+t\beta-\tau_{\mu,\nu}(q,t)\Big\}.$$

In this paper we provide the mutual $L^{q,t}$-spectrum relatively to two compactly supported Borel probability measures on $\mathbb{R}^n$. We write $\mathcal{P}_n$ for the set of compactly supported Borel probability measures on $\mathbb{R}^n$. Let $\mu,\nu\in\mathcal{P}_n$ be such that $S_\mu=S_\nu=K$. For $(q,t)\in\mathbb{R}^2$ define

$$\underline{I}_{\mu,\nu}(q,t)=\liminf_{r\to 0}\frac{1}{\log r}\log\iint\limits_{K^2\cap B_r(\mathbb{D}^2)}\mu(B_r(x))^q\nu(B_r(y))^t d\mu(x)d\nu(y),$$

and

$$\overline{I}_{\mu,\nu}(q,t)=\limsup_{r\to 0}\frac{1}{\log r}\log\iint\limits_{K^2\cap B_r(\mathbb{D}^2)}\mu(B_r(x))^q\nu(B_r(y))^t d\mu(x)d\nu(y),$$

where $\mathbb{D}^2=\big\{(x,x)\in\mathbb{R}^2: x\in\mathbb{R}\big\}$ is the diagonal ray in $\mathbb{R}^2$ and $B_r(\mathbb{D}^2)=$ $=\{x\in\mathbb{R}^n:\operatorname{dist}(x,\mathbb{D}^2)\leq r\}$ is the closed r-neighborhood of $\mathbb{D}^2$.

If $\underline{I}_{\mu,\nu}(q,t)=\overline{I}_{\mu,\nu}(q,t)$, then their common value at (q,t) is denoted by $I_{\mu,\nu}(q,t)$ and called the mutual $L^{q,t}$-spectrum of μ and ν. Note that these quantities are strictly related to the mutual multifractal analysis [22], [25]–[27] and the mixed multifractal analysis [18], introduced by Olsen.

In the recent decade there has been a great interest in understanding the fractal dimensions of projections of sets and measures. Recently, the projectional behavior of dimensions and multifractal spectra of sets and measures have generated a large interest in the mathematical literature [1]–[4], [7]–[13], [15, 19, 20]. The first significant work in this area was the result by Marstrand [15] who proved a well-known theorem: the Hausdorff dimension of a planar set is preserved under orthogonal projections.

Let us mention that Falconer and Mattila [8] and Falconer and Howroyd [7] have proved that the packing dimension of the projected set or measure will be the same for almost all projections. However, despite these substantial advances for fractal sets, only very little is known about the multifractal structure of projections of measures, except a paper by O'Neil [19] and some more recent papers by Barral and Bhouri [2]. The result of O'Neil was later generalized by Selmi et al. in [4] – [6], [21].

We continue of this research studying the behavior of the upper and lower mutual $L^{q,t}$-spectra under orthogonal projections onto a lower dimensional linear subspace. We employ theoretical methods first used in this context by Kaufman in [14] and later generalized in [16].

2. Preliminaries. Let m be an integer with $1 \leq m \leq n$ and $G_{n,m}$ represent the Grassmannian manifold of all m-dimensional linear subspaces of $\mathbb{R}^n$. By $\gamma_{n,m}$ denote the invariant Haar measure on $G_{n,m}$ such that $\gamma_{n,m}(G_{n,m}) = 1$. The projection map $\pi_V : \mathbb{R}^n \to V$ for $V \in G_{n,m}$ is the usual orthogonal projection onto V. Then $\{\pi_V,\ V \in G_{n,m}\}$ is compact in the space of all linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$, and identification of V with π_V induces a compact topology for $G_{n,m}$. Also, for a Borel probability measure μ with compact support $\operatorname{supp}\mu$ on $\mathbb{R}^n$ and for $V \in G_{n,m}$ define the projection μ_V of μ onto V by

$$\mu_V(A) = \mu(\pi_V^{-1}(A)) \quad \forall A \subseteq V.$$

Note that μ is compactly supported and $\operatorname{supp}\mu_V = \pi_V(\operatorname{supp}\mu)$ for all $V \in G_{n,m}$, then for any continuous function $f : V \longrightarrow \mathbb{R}$

$$\int f d\mu_V = \int f(\pi_V(x)) d\mu(x)$$

provided that these integrals exist (for more details see [9]). The convolution is defined for $1 \leq m \leq n$ and $r > 0$ by

$$\begin{array}{rccl} \overline{\phi}_r^m : & \mathbb{R}^n & \longrightarrow & \mathbb{R}, \\ & x & \longmapsto & \gamma_{n,m}\Big\{V \in G_{n,m} :\ |\pi_V(x)| \leq r\Big\}, \end{array}$$

where $\gamma_{n,m}$ is the rotation-invariant probability measure on $G_{n,m}$. Define

$$\begin{array}{rccl} \phi_r^m : & \mathbb{R}^n & \longrightarrow & \mathbb{R}, \\ & x & \longmapsto & \min\Big\{1\ ,\ r^m|x|^{-m}\Big\}. \end{array}$$

This $\phi_r^m(x)$ is equivalent to $\overline{\phi}_r^m(x)$. We write this equivalence as $\phi_r^m(x) \asymp \asymp \overline{\phi}_r^m(x)$. For a probability measure μ and $V \in G_{n,m}$ we have

$$\mu^{r,m_*}(x) = \mu * \phi_r^m(x) \asymp \mu * \overline{\phi}_r^m(x) = \int \mu_V B_r(x_V) dV,$$

and

$$\mu^{r,m_*}(x) = \int \min\left\{1\,,\ r^m |x-y|^{-m}\right\} d\mu(y).$$

So, integrating by parts and converting into spherical coordinates (see [9])

$$\mu^{r,m_*}(x) = mr^m \int_r^{+\infty} u^{-m-1} \mu B_u(x) du.$$

The following straightforward estimates concern the behaviour of $\mu^{r,m_*}(x)$ as $r \to 0$.

Lemma 1. [9] *Let $1 \le m \le n$ and $\mu \in \mathcal{P}_n$. For all $x \in \mathbb{R}^n$*

$$cr^m \le \mu^{r,m_*}(x)$$

for all sufficiently small r, where $c > 0$ is independent of r.

Lemma 2. [9] *Let $\mu \in \mathcal{P}_n$.*

1) *For all $x \in \mathbb{R}^n$ and $r > 0$*

$$\mu B_r(x) \le \mu^{r,n_*}(x).$$

2) *Let $\varepsilon > 0$. For μ-almost all x*

$$r^{-\varepsilon} \mu B_r(x) \ge \mu^{r,n_*}(x),$$

if r is sufficiently small.

We use the properties of $\mu^{r,m_*}(x)$ to obtain a relationship between the kernels and projected measures.

Lemma 3. [9] *Let $1 \le m \le n$, $\mu \in \mathcal{P}_n$, $\varepsilon > 0$, and r be sufficiently small.*

1) *For all $V \in G_{n,m}$ and for μ-almost all $x \in \mathbb{R}^n$*

$$r^\varepsilon \mu^{r,m_*}(x) \le \mu_V B_r(x_V).$$

2) *For $\gamma_{n,m}$-almost all $V \in G_{n,m}$ and all $x \in \mathbb{R}^n$*

$$r^{-\varepsilon}\mu^{r,m_*}(x) \geq \mu_V B_r(x_V).$$

3. Projection results. In this section we need an alternative characterization of the upper and lower mutual $L^{q,t}$-spectra in terms of convolution. We specify this to the mutual (q,t)-dimensions relatively to μ and ν using appropriate definitions in terms of kernels.

From now on $1 \leq m \leq n$ are two integers and the measures $\mu, \nu \in \mathcal{P}_n$ are such that $S_\mu = S_\nu = K$. For $q, t > 0$ we define

$$\underline{I}^m_{\mu,\nu}(q,t) = \liminf_{r\to 0} \frac{1}{\log r} \log \iint\limits_{K^2 \cap B_r(\mathbb{D}^2)} (\mu^{r,m_*}(x))^q \left(\nu^{r,m_*}(y)\right)^t d\mu(x)d\nu(y),$$

$$\overline{I}^m_{\mu,\nu}(q,t) = \limsup_{r\to 0} \frac{1}{\log r} \log \iint\limits_{K^2 \cap B_r(\mathbb{D}^2)} (\mu^{r,m_*}(x))^q \left(\nu^{r,m_*}(y)\right)^t d\mu(x)d\nu(y).$$

Note that for all $x, y \in K$ and $r > 0$

$$\mu^{r,m_*}(x) \geq \mu B_r(x) \quad \text{and} \quad \nu^{r,m_*}(y) \geq \nu B_r(y).$$

It is clear that for $q > 0$ and $t > 0$ and for a sufficiently small r

$$\underline{I}^m_{\mu,\nu}(q,t) \leq \underline{I}_{\mu,\nu}(q,t) \qquad \text{and} \qquad \overline{I}^m_{\mu,\nu}(q,t) \leq \overline{I}_{\mu,\nu}(q,t). \tag{1}$$

From Lemma 1 we see that for all $x, y \in \mathbb{R}^n$ and for any sufficiently small r

$$cr^m \leq \mu^{r,m_*}(x) \quad \text{and} \quad c'r^m \leq \nu^{r,m_*}(y),$$

where $c, c' > 0$ are independent of r. This leads to

$$\underline{I}^m_{\mu,\nu}(q,t) \leq \overline{I}^m_{\mu,\nu}(q,t) \leq m(q+t). \tag{2}$$

Proposition 1. *Let $\varepsilon > 0$, $\mu, \nu \in \mathcal{P}_n$.*

1) *Let $q, t > 0$. For all $V \in G_{n,m}$ we have*

$$\iint \mu_V(B_r(x_V))^q \nu_V(B_r(y_V))^t d\mu_V(x_V)d\nu_V(y_V) \geq$$

$$\geq r^{\varepsilon(q+t)} \iint \left(\mu^{r,m_*}(x)\right)^q \left(\nu^{r,m_*}(y)\right)^t d\mu(x)d\nu(y)$$

for all sufficiently small r.

2) *Let* $0 < q, t \le 1$. *For* $\gamma^2_{n,m}$*-almost all* $V \times W \in G^2_{n,m}$ *we have*

$$\iint \mu_V(B_r(x_V))^q \nu_W(B_r(y_W))^t d\mu_V(x_V) d\nu_W(y_W) \le$$

$$\le C \iint \left(\mu^{r,m_*}(x)\right)^q \left(\nu^{r,m_*}(y)\right)^t d\mu(x) d\nu(y)$$

for all sufficiently small r *and* $C > 0$ *independent of* r.

Proof. 1) For all $V \in G_{n,m}$ and $x, y \in K$

$$\mu^{r,m_*}(x) \le \mu_V^{r,m_*}(x_V) \quad \text{and} \quad \nu^{r,m_*}(y) \le \nu_V^{r,m_*}(y_V).$$

Take $\varepsilon > 0$ and $r > 0$. From Lemma 3 we see that for all $V \in G_{n,m}$ and μ_V-almost all $x_V \in V$

$$\mu_V B_r(x_V) \ge r^\varepsilon \mu_V^{r,m_*}(x_V),$$

and for ν_V-almost all $y_V \in V$

$$\nu_V B_r(y_V) \ge r^\varepsilon \nu_V^{r,m_*}(y_V).$$

This means that

$$\iint (\mu^{r,m_*}(x))^q (\nu^{r,m_*}(y))^t d\mu(x) d\nu(y) \le$$

$$\le \iint (\mu_V^{r,m_*}(x_V))^q (\nu_V^{r,m_*}(y_V))^t d\mu_V(x_V) d\nu_V(y_V) \le$$

$$\le r^{-\varepsilon(q+t)} \iint \mu_V(B_r(x_V))^q \nu_V(B_r(y_V))^t d\mu_V(x_V) d\nu_V(y_V).$$

2) For $0 < q, t \le 1$, using Lemma 3.11 in [17], we obtain

$$I = \iint\limits_{K^2 \cap B_r(\mathbb{D}^2)} (\mu^{r,m_*}(x))^q (\nu^{r,m_*}(y))^t d\mu(x) d\nu(y) =$$

$$= \iint\limits_{K^2 \cap B_r(\mathbb{D}^2)} \left(\int \min\left\{1,\, r^m |x-u|^{-m}\right\} d\mu(u) \right)^q \times$$

$$\times \left(\int \min\left\{1,\, r^m |y-v|^{-m}\right\} d\nu(v) \right)^t d\mu(x) d\nu(y) \ge$$

$$\geq c \iint\limits_{K^2 \cap B_r(\mathbb{D}^2)} \left(\int \gamma_{n,m} \left\{ V \in G_{n,m} : |\pi_V(x) - \pi_V(u)| \leq r \right\} d\mu(u) \right)^q \times$$

$$\times \left(\int \gamma_{n,m} \left\{ W \in G_{n,m} : |\pi_W(y) - \pi_W(v)| \leq r \right\} d\nu(v) \right)^t d\mu(x) d\nu(y) \geq$$

$$\geq c \iint\limits_{K^2 \cap B_r(\mathbb{D}^2)} \left(\int \mu \left\{ u \in \mathbb{R}^n : |\pi_V(x) - \pi_V(u)| \leq r \right\} d\gamma_{n,m}(V) \right)^q \times$$

$$\times \left(\int \nu \left\{ v \in \mathbb{R}^n : |\pi_W(y) - \pi_W(v)| \leq r \right\} d\gamma_{n,m}(W) \right)^t d\mu(x)\, d\nu(y).$$

The Jensen inequality and the Fubini Theorem imply

$$I \geq c_1 \iint \mu_V(B_r(x_V))^q \nu_W(B_r(y_W))^t d\mu_V(x_V) d\nu_W(y_W)$$

for some c and c_1 independent of r. $\square$

Corollary 1. For all $q, t > 0$ and $V \in G_{n,m}$ we have

$$\liminf_{r \longrightarrow 0} \frac{1}{\log r} \left(\log \left[\frac{\iint (\mu^{r,m_*}(x))^q (\nu^{r,m_*}(y))^t d\mu(x) d\nu(y)}{\iint \mu_V(B_r(x_V))^q \nu_V(B_r(y_V))^t d\mu_V(x_V) d\nu_V(y_V)} \right] \right) \geq 0.$$

For $0 < q, t \leq 1$, $\gamma^2_{n,m}$-almost all $V \times W \in G^2_{n,m}$ and sufficiently small $r > 0$

$$\lim_{r \longrightarrow 0} \frac{1}{\log r} \left(\log \left[\frac{\iint (\mu^{r,m_*}(x))^q (\nu^{r,m_*}(y))^t d\mu(x) d\nu(y)}{\iint \mu_V(B_r(x_V))^q \nu_W(B_r(y_W))^t d\mu_V(x_V) d\nu_W(y_W)} \right] \right) = 0.$$

Proof. Follows directly from Proposition 1. $\square$

Theorem 1. *Let $\mu, \nu \in \mathcal{P}_n$.*

1) *For all $q, t > 0$ and $V \in G_{n,m}$*
   (i) $\underline{I}_{\mu_V,\nu_V}(q,t) \leq \underline{I}^m_{\mu,\nu}(q,t) \leq \min\left(m(q+t), \underline{I}_{\mu,\nu}(q,t)\right)$,
   (ii) $\overline{I}_{\mu_V,\nu_V}(q,t) \leq \overline{I}^m_{\mu,\nu}(q,t) \leq \min\left(m(q+t), \overline{I}_{\mu,\nu}(q,t)\right)$.

2) *For all $0 < q, t \leq 1$ and $\gamma^2_{n,m}$-almost all $V \times W \in G^2_{n,m}$*
   (i) $\underline{I}_{\mu_V,\nu_W}(q,t) = \underline{I}^m_{\mu,\nu}(q,t)$,
   (ii) $\overline{I}_{\mu_V,\nu_W}(q,t) = \overline{I}^m_{\mu,\nu}(q,t)$.

Proof. This follows from (1), (2) and Corollary 1. □

Take an $r > 0$ and denote the set of r-mesh cubes in $\mathbb{R}^n$ by $\mathfrak{C}_r$. These cubes have the form

$$\prod_{j=1}^{n} [k_j r, (k_j+1)r[\,,$$

where $k_j \in \mathbb{Z}$.

Proposition 2. *If $q > -1$ and $t > -1$, then*

$$\underline{I}_{\mu,\nu}(q,t) = \liminf_{r\to 0} \frac{1}{\log r} \log \sum_{C\in\mathfrak{C}_r} \mu(C)^{q+1}\nu(C)^{t+1},$$

$$\overline{I}_{\mu,\nu}(q,t) = \limsup_{r\to 0} \frac{1}{\log r} \log \sum_{C\in\mathfrak{C}_r} \mu(C)^{q+1}\nu(C)^{t+1}.$$

Proposition 2 is a consequence from the following lemma.

Lemma 4. *Let $\mu, \nu \in \mathcal{P}_n$. If $q \geq 0$ and $t \geq 0$, then*

$$\iint\limits_{K^2\cap B_r(\mathbb{D}^2)} \mu(B_{\sqrt{n}r}(x))^q \nu(B_{\sqrt{n}r}(y))^t d\mu(x)d\nu(y) \geq \sum_{C\in\mathfrak{C}_r} \mu(C)^{q+1}\nu(C)^{t+1} \geq$$
$$\geq C(n,q,t) \iint\limits_{K^2\cap B_r(\mathbb{D}^2)} \mu(B_r(x))^q \nu(B_r(y))^t d\mu(x)d\nu(y).$$

Proof. Let $q \geq 0$ and $t \geq 0$, then

$$\sum_{C\in\mathfrak{C}_r} \mu(C)^{q+1}\nu(C)^{t+1} = \sum_{C\in\mathfrak{C}_r} \mu(C)^q\nu(C)^t \iint\limits_{C^2} d\mu(x)d\nu(y) =$$

$$= \sum_{C\in\mathfrak{C}_r} \iint\limits_{C^2} \mu(C)^q\nu(C)^t d\mu(x)d\nu(y) \leq$$

$$\leq \sum_{C\in\mathfrak{C}_r} \iint\limits_{C^2\cap B_r(\mathbb{D}^2)} \mu(B_{\sqrt{n}r}(x))^q (\nu B_{\sqrt{n}r}(y))^t d\mu(x)d\nu(y) =$$

$$= \iint\limits_{K^2\cap B_r(\mathbb{D}^2)} \mu(B_{\sqrt{n}r}(x))^q \nu(B_{\sqrt{n}r}(y))^t d\mu(x)d\nu(y).$$

For the other inequality we have

$$\iint\limits_{K^2\cap B_r(\mathbb{D}^2)} \mu(B_r(x))^q\nu(B_r(y))^t d\mu(x)d\nu(y) \leq$$

$$\leq \sum_{C\in\mathfrak{C}_r} \iint\limits_{C^2\cap B_r(\mathbb{D}^2)} \mu(B_r(x))^q\nu(B_r(y))^t d\mu(x)d\nu(y) \leq$$

$$\leq \sum_{C\in\mathfrak{C}_r} \iint\limits_{C^2\cap B_r(\mathbb{D}^2)} \mu(\widehat{C})^q\nu(\widehat{C})^t d\mu(x)d\nu(y) =$$

$$= \sum_{C\in\mathfrak{C}_r} \mu(\widehat{C})^q\nu(\widehat{C})^t \iint\limits_{C^2\cap B_r(\mathbb{D}^2)} d\mu(x)d\nu(y) \leq \sum_{C\in\mathfrak{C}_r} \mu(\widehat{C})^{q+1}\nu(\widehat{C})^{t+1} \leq$$

$$\leq \sum_{C\in\mathfrak{C}_r} \left(\sum_{i=1}^{3^n}\mu(C_i)\right)^{q+1} \left(\sum_{j=1}^{3^n}\nu(C_j)\right)^{t+1} \leq$$

$$\leq 3^{n(q+t)} \sum_{C\in\mathfrak{C}_r}\sum_{i,j=1}^{3^n} \mu(C_i)^{q+1}\nu(C_j)^{t+1} \leq c(n,q,t)\sum_{C\in\mathfrak{C}_r} \mu(C)^{q+1}\nu(C)^{t+1},$$

where $\widehat{C}$ is the cube of side $3r$, concentric with C. $\square$

Proposition 3. *Let $\mu,\nu\in\mathcal{P}_n$. For all $q,t>0$ and $V\in G_{n,m}$ we have*

1) $\underline{I}_{\mu_V,\nu_V}(q,t) \leq \underline{I}_{\mu,\nu}(q,t)$,

2) $\overline{I}_{\mu_V,\nu_V}(q,t) \leq \overline{I}_{\mu,\nu}(q,t)$.

Proof. Assume that $S_\mu = S_\nu \subseteq B_R(0)$ with $R>0$. Let $V\in G_{n,m}$ be given, $0<r\leq 1$ and $\big(C_i\big)_i$ be a set of r-mesh sub-cubes of V that cover the unit cube in V with center at the origin. For each i let $\big(C_{i,j}\big)_j$ be a column of cubes of side r above C_i, so that $\big(C_{i,j}\big)_{i,j}$ are a set of r-mesh cubes which cover the unit cube with center at the origin. Let $q>0$ and $t>0$; then

$$\sum_i\sum_j \mu(C_{i,j})^{q+1}\nu(C_{i,j})^{t+1} \leq \sum_i \left(\sum_j \mu(C_{i,j})^{q+1}\right)\left(\sum_j \nu(C_{i,j})^{t+1}\right) \leq$$

$$\leq \sum_i \left(\sum_j \mu(C_{i,j}) \right)^{q+1} \left(\sum_j \nu(C_{i,j}) \right)^{t+1}.$$

So,

$$\sum_i \sum_j \mu(C_{i,j})^{q+1} \nu(C_{i,j})^{t+1} \leq \sum_i \mu_V(C_i)^{q+1} \nu_V(C_i)^{t+1}.$$

Taking the lower and upper limits gives the desired result. □

Definition. *For a measure μ on $\mathbb{R}^n$ and for $p \geq 1$ we say that $\mu \in L^p(\mathbb{R}^n)$ if there is a function $f \in L^p(\mathbb{R}^n)$ such that f is the Radon-Nikodym derivative of μ with respect to $\mathcal{L}^n$ for μ-a.e. x.*

Now let us obtain conditions that projections of a measure belong to L^p for some $p \geq 1$. Consider a compactly supported Borel probability measure μ on $\mathbb{R}^n$. For s, $m \leq s < n$ define the s-energy of μ by

$$I_s(\mu) = \iint |x - y|^{-s} d\mu(x) d\mu(y).$$

Mattila [17] proved that if $I_s(\mu)$ is finite for $m \leq s < n$, then for almost all $V \in G_{n,m}$ the measure μ_V is absolutely continuous with respect to the m-dimensional Lebesgue measure $\mathcal{L}^m_V$ on V (denoted by $\mathbb{R}^m$), where $\mathcal{L}^m_V(E) = \mathcal{L}^m(E \cap V)$ for $E \subset \mathbb{R}^n$, and $\mu_V \in L^2(V)$.

Proposition 4. [9] *Let μ be a compactly supported Radon measure on $\mathbb{R}^n$. Let $m \leq s < n$. Suppose that $I_s(\mu) < \infty$. Then μ_V is absolutely continuous with respect to $\mathcal{L}^m_V$, with μ_V in $L^2(V)$ for $\gamma_{n,m}$-almost all $V \in G_{n,m}$. Moreover, for $\gamma_{n,m}$-almost all $V \in G_{n,m}$*

1) *if $m < s \leq 2m$ then $\mu_V \in L^p(V)$ for all p satisfying $1 < p \leq \dfrac{2m}{2m - s}$,*
2) *if $2m < s < n$ then the Radon-Nikodym derivative of μ_V with respect to $\mathcal{L}^m_V$ is bounded and essentially continuous.*

Theorem 2. *Let $m \leq s < n$, $I_s(\mu) < \infty$ and $I_s(\nu) < \infty$. Then for $\gamma_{n,m}$-almost all $V \in G_{n,m}$*
1) if $2m < s < n$ then

$$\underline{I}_{\mu_V,\nu_V}(q,t) = \overline{I}_{\mu_V,\nu_V}(q,t) = m(q+t) \text{ for } 0 < q, t < \infty;$$

2) if $m < s < 2m$ then

(i) for $0 < q,t \le \frac{2m}{2m-s} - 1$

$$\underline{I}_{\mu_V,\nu_V}(q,t) = \overline{I}_{\mu_V,\nu_V}(q,t) = m(q+t),$$

(ii) for $q,t > \frac{2m}{2m-s} - 1$

$$\frac{s(q+t+2)}{2} \le \underline{I}_{\mu_V,\nu_V}(q,t) \le \overline{I}_{\mu_V,\nu_V}(q,t) \le m(q+t),$$

(iii) for $q > \frac{2m}{2m-s} - 1 > t$

$$m\left(q+t+1-\frac{(2m-s)(q+1)}{2m}\right) \le \underline{I}_{\mu_V,\nu_V}(q,t) \le \overline{I}_{\mu_V,\nu_V}(q,t) \le m(q+t),$$

(iv) for $t > \frac{2m}{2m-s} - 1 > q$,

$$m\left(q+t+1-\frac{(2m-s)(t+1)}{2m}\right) \le \underline{I}_{\mu_V,\nu_V}(q,t) \le \overline{I}_{\mu_V,\nu_V}(q,t) \le m(q+t).$$

Proof. It is a simple consequence of Theorem 1, Proposition 4, and the following Lemma. □

Lemma 5. *Fix a $p \ge 1$. Suppose that μ, $\nu \in L^p(\mathbb{R}^n)$, $q,t > 0$. Then*

$$\underline{I}_{\mu,\nu}(q,t) \ge \begin{cases} n(q+t+2)\left(1-\frac{1}{p}\right), & \text{if } q+1 \ge p \text{ and } t+1 \ge p; \\ n\left(q+t+1-\frac{q+1}{p}\right), & \text{if } q+1 \ge p \text{ and } t+1 < p; \\ n\left(q+t+1-\frac{t+1}{p}\right), & \text{if } q+1 < p \text{ and } t+1 \ge p; \\ n(q+t), & \text{if } p > q+1 \text{ and } p > t+1. \end{cases}$$

Proof. Let $f = \dfrac{d\mu}{d\mathcal{L}^n} \in L^p(\mathbb{R}^n)$ and $g = \dfrac{d\nu}{d\mathcal{L}^n} \in L^p(\mathbb{R}^n)$. The Hölder inequality gives

$$\sum_{C\in\mathfrak{C}_r}\mu(C)^{q+1}\nu(C)^{t+1}=\sum_{C\in\mathfrak{C}_r}\left(\int\limits_C fd\mathcal{L}^n\right)^{q+1}\left(\int\limits_C gd\mathcal{L}^n\right)^{t+1}\le$$

$$\le r^{n(q+1)\left(1-\frac{1}{p}\right)}r^{n(t+1)\left(1-\frac{1}{p}\right)}\sum_{C\in\mathfrak{C}_r}\left(\int\limits_C f^p d\mathcal{L}^n\right)^{\frac{q+1}{p}}\left(\int\limits_C g^p d\mathcal{L}^n\right)^{\frac{t+1}{p}}\le$$

$$\le r^{n(q+t+2)\left(1-\frac{1}{p}\right)}\sum_{C\in\mathfrak{C}_r}\left(\int\limits_C f^p d\mathcal{L}^n\right)^{\frac{q+1}{p}}\sum_{C\in\mathfrak{C}_r}\left(\int\limits_C g^p d\mathcal{L}^n\right)^{\frac{t+1}{p}}\le$$

$$\le\begin{cases}\begin{cases}r^{n(q+t+2)(1-\frac{1}{p})}\times\\ \times\left(\sum\limits_{C\in\mathfrak{C}_r}\int\limits_C f^p d\mathcal{L}^n\right)^{\frac{q+1}{p}}\left(\sum\limits_{C\in\mathfrak{C}_r}\int\limits_C g^p d\mathcal{L}^n\right)^{\frac{t+1}{p}},\end{cases} & \text{if } q+1\ge p,\ t+1\ge p;\\ \begin{cases}c_1 r^{n(q+t+2)\left(1-\frac{1}{p}\right)}r^{-n\left(1-\frac{t+1}{p}\right)}\times\\ \times\left(\sum\limits_{C\in\mathfrak{C}_r}\int\limits_C f^p d\mathcal{L}^n\right)^{\frac{q+1}{p}}\left(\sum\limits_{C\in\mathfrak{C}_r}\int\limits_C g^p d\mathcal{L}^n\right)^{\frac{t+1}{p}},\end{cases} & \text{if } q+1\ge p,\ t+1<p;\\ \begin{cases}c_2 r^{n(q+t+2)\left(1-\frac{1}{p}\right)}r^{-n\left(1-\frac{q+1}{p}\right)}\times\\ \times\left(\sum\limits_{C\in\mathfrak{C}_r}\int\limits_C f^p d\mathcal{L}^n\right)^{\frac{q+1}{p}}\left(\sum\limits_{C\in\mathfrak{C}_r}\int\limits_C g^p d\mathcal{L}^n\right)^{\frac{t+1}{p}},\end{cases} & \text{if } q+1<p,\ t+1\ge p;\\ \begin{cases}c_3 r^{n(q+t+2)\left(1-\frac{1}{p}\right)}r^{-n\left(1-\frac{q+1}{p}\right)}r^{-n\left(1-\frac{t+1}{p}\right)}\times\\ \times\left(\sum\limits_{C\in\mathfrak{C}_r}\int\limits_C f^p d\mathcal{L}^n\right)^{\frac{q+1}{p}}\left(\sum\limits_{C\in\mathfrak{C}_r}\int\limits_C g^p d\mathcal{L}^n\right)^{\frac{t+1}{p}},\end{cases} & \text{if } p>q+1,\ p>t+1\end{cases}$$

$$\leq \begin{cases} C_1 r^{n(q+t+2)\left(1-\frac{1}{p}\right)}, & \text{if } q+1 \geq p,\ t+1 \geq p; \\ C_2 r^{n\left(q+t+1-\frac{q+1}{p}\right)}, & \text{if } q+1 \geq p,\ t+1 < p; \\ C_3 r^{n\left(q+t+1-\frac{t+1}{p}\right)}, & \text{if } q+1 < p,\ t+1 \geq p; \\ C_4 r^{n(q+t)}, & \text{if } p > q+1,\ p > t+1, \end{cases}$$

where C_1, C_2, C_3, and C_4 are independent of r. Since r is sufficiently small, we obtain the required inequality taking the lower limit. □

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