DOI: 10.15393/j3.art.2018.4370

UDC 519.58

Feng Qi

AN IMPROPER INTEGRAL, THE BETA FUNCTION, THE WALLIS RATIO, AND THE CATALAN NUMBERS

Abstract. In the paper we present closed and unified expressions for a sequence of improper integrals in terms of the beta function and the Wallis ratio. Hereafter, we derive integral representations for the Catalan numbers originating from combinatorics.

Key words: *improper integral, closed expression, unified expression, beta function, Wallis ratio, integral representation, Catalan number*

2010 Mathematical Subject Classification: *Primary 11B65; Secondary 11B75, 11B83, 26A39, 26A42, 33B15*

1. Introduction. In mathematics, a closed form is a mathematical expression that can be evaluated in a finite number of operations. It may contain constants, variables, four arithmetic operations, and elementary functions, but, usually, not limit.

Let a be a positive number. For $n \ge 0$, define

$$I_n = \int_{-a}^{a} x^n \sqrt{\frac{a+x}{a-x}}\, dx. \tag{1}$$

In [1, Section 3], Dana-Picard and Zeitoun computed $I_0 = a\pi$ and found a closed form of I_n for $n \in \mathbb{N}$ in three steps:

1) establishing a formula of recurrence between I_n and I_{n+1} in terms of

$$S_n = \int_{-\pi/2}^{\pi/2} \sin^n \theta\, d\theta; \tag{2}$$

2) establishing an equation for I_n in terms of S_n;

3) establishing different expressions for odd values and even values of n.

Consequently, they deduced an integral representation of the Catalan numbers which originate from combinatorics and number theory.

The aim of this note is to discuss again the sequence I_n, to present closed and unified expressions for the sequence I_n in terms of the beta function and the Wallis ratio, to derive integral representations for the Catalan numbers, and to correct some errors and typos found in [1, Section 3].

2. Closed and unified expressions for I_n. The sequence I_n can be computed by several methods shown below.

Theorem 1. *For $n \in \mathbb{N}$, the sequence I_n can be computed by*

$$I_n = a^{n+1}\pi\left[\frac{1+(-1)^n}{n}\frac{1}{B\left(\frac{1}{2},\frac{n}{2}\right)} + \frac{1+(-1)^{n+1}}{n+1}\frac{1}{B\left(\frac{1}{2},\frac{n+1}{2}\right)}\right], \tag{3}$$

where

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}\,dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \tag{4}$$

and

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$$

for $\operatorname{Re}(p), \operatorname{Re}(q) > 0$, *and* $\operatorname{Re}(z) > 0$ *denote the Euler integrals of the second kind* (*or, say, the classical beta and gamma functions*), *respectively.*

Proof. Using properties of definite integral we can write, by the straightforward computation:

$$I_n = \int_{-a}^0 x^n\sqrt{\frac{a+x}{a-x}}\,dx + \int_0^a x^n\sqrt{\frac{a+x}{a-x}}\,dx =$$

$$= \int_a^0 (-y)^n\sqrt{\frac{a+(-y)}{a-(-y)}}\,d(-y) + \int_0^a x^n\sqrt{\frac{a+x}{a-x}}\,dx =$$

$$= \int_0^a (-1)^n y^n \sqrt{\frac{a-y}{a+y}}\, dy + \int_0^a x^n \sqrt{\frac{a+x}{a-x}}\, dx =$$

$$= \int_0^a x^n \left[(-1)^n \sqrt{\frac{a-x}{a+x}} + \sqrt{\frac{a+x}{a-x}} \right] dx =$$

$$= \int_0^a x^n \frac{(a+x)+(-1)^n(a-x)}{\sqrt{a^2-x^2}}\, dx =$$

$$= \int_0^a x^n \frac{a[1+(-1)^n]+x[1-(-1)^n]}{\sqrt{a^2-x^2}}\, dx =$$

$$= a[1+(-1)^n] \int_0^a \frac{x^n}{\sqrt{a^2-x^2}}\, dx + [1-(-1)^n] \int_0^a \frac{x^{n+1}}{\sqrt{a^2-x^2}}\, dx.$$

In [6, Theorem 3.1], it was obtained that

$$\int_0^a \frac{x^n}{\sqrt{a^2-x^2}}\, dx = \sqrt{\pi}\, a^n \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)}$$

for $a > 0$ and $n \geq 0$. Accordingly, considering

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\,, \tag{5}$$

we acquire

$$I_n = a[1+(-1)^n]\sqrt{\pi}\, a^n \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)} +$$

$$+ [1-(-1)^n]\sqrt{\pi}\, a^{n+1} \frac{\Gamma\left(\frac{n+1}{2}+\frac{1}{2}\right)}{(n+1)\Gamma\left(\frac{n+1}{2}\right)} =$$

$$= \sqrt{\pi}\, a^{n+1} \left([1+(-1)^n] \frac{\Gamma\left(\frac{n+1}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)} + [1-(-1)^n] \frac{\Gamma\left(\frac{n}{2}+1\right)}{(n+1)\Gamma\left(\frac{n+1}{2}\right)} \right) =$$

$$= a^{n+1}\pi \left[\frac{1+(-1)^n}{n} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)} + \frac{1-(-1)^n}{n+1} \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)} \right] =$$

$$= a^{n+1}\pi\left[\frac{1+(-1)^n}{n}\frac{1}{B\left(\frac{1}{2},\frac{n}{2}\right)}+\frac{1+(-1)^{n+1}}{n+1}\frac{1}{B\left(\frac{1}{2},\frac{n+1}{2}\right)}\right].$$

The proof of Theorem 1 is complete. □

Theorem 2. *For $n \ge 0$, the sequence I_n can be computed by*

$$I_n = \frac{1}{2}a^{n+1}\left([1+(-1)^n]B\left(\frac{1}{2},\frac{n+1}{2}\right)+ \right.$$
$$\left. + \left[1+(-1)^{n+1}\right]B\left(\frac{1}{2},\frac{n+2}{2}\right)\right). \quad (6)$$

Proof. Changing the variable of integration by $x = at$ in (1) gives

$$I_n = \int_{-1}^{1}(at)^n\sqrt{\frac{a+at}{a-at}}\,a\,dt = a^{n+1}\int_{-1}^{1}t^n\sqrt{\frac{1+t}{1-t}}\,dt =$$

$$= a^{n+1}\left(\int_{-1}^{0}t^n\sqrt{\frac{1+t}{1-t}}\,dt+\int_{0}^{1}t^n\sqrt{\frac{1+t}{1-t}}\,dt\right) =$$

$$= a^{n+1}\left[\int_{0}^{1}(-s)^n\sqrt{\frac{1-s}{1+s}}\,ds+\int_{0}^{1}t^n\sqrt{\frac{1+t}{1-t}}\,dt\right] =$$

$$= a^{n+1}\int_{0}^{1}t^n\left[(-1)^n\sqrt{\frac{1-t}{1+t}}+\sqrt{\frac{1+t}{1-t}}\right]dt =$$

$$= a^{n+1}\int_{0}^{1}t^n\left[(-1)^n\frac{1-t}{\sqrt{1-t^2}}+\frac{1+t}{\sqrt{1-t^2}}\right]dt =$$

$$= a^{n+1}\left([1+(-1)^n]\int_{0}^{1}\frac{t^n}{\sqrt{1-t^2}}\,dt+[1-(-1)^n]\int_{0}^{1}\frac{t^{n+1}}{\sqrt{1-t^2}}\,dt\right) =$$

$$= a^{n+1}\left([1+(-1)^n]\int_{0}^{\pi/2}\frac{\sin^n s}{\sqrt{1-\sin^2 s}}\cos s\,ds+\right.$$

$$+\left[1-(-1)^n\right]\int_0^{\pi/2}\frac{\sin^{n+1}s}{\sqrt{1-\sin^2 s}}\cos s\,ds\right)=$$

$$=a^{n+1}\left(\left[1+(-1)^n\right]\int_0^{\pi/2}\sin^n s\,ds+\left[1-(-1)^n\right]\int_0^{\pi/2}\sin^{n+1}s\,ds\right).$$

Further making use of the formula

$$\int_0^{\pi/2}\sin^t x\,dx=\frac{1}{2}B\left(\frac{t+1}{2},\frac{1}{2}\right),\quad t>-1,$$

in [6, Remark 6.4] yields

$$I_n=a^{n+1}\left(\left[1+(-1)^n\right]\frac{1}{2}B\left(\frac{1}{2},\frac{n+1}{2}\right)+\left[1-(-1)^n\right]\frac{1}{2}B\left(\frac{1}{2},\frac{n+2}{2}\right)\right)=$$

$$=\frac{1}{2}a^{n+1}\left(\left[1+(-1)^n\right]B\left(\frac{1}{2},\frac{n+1}{2}\right)+\left[1+(-1)^{n+1}\right]B\left(\frac{1}{2},\frac{n+2}{2}\right)\right).$$

The proof of Theorem 2 is complete. □

Corollary. *For $m\geq 0$, the sequences I_{2m} and I_{2m+1} can be closedly computed by*

$$I_{2m}=\pi a^{2m+1}\frac{(2m-1)!!}{(2m)!!}$$

and

$$I_{2m+1}=\pi a^{2(m+1)}\frac{(2m+1)!!}{(2m+2)!!},$$

where the double factorial of negative odd integers $-2n-1$ is defined by

$$(-2n-1)!!=\frac{(-1)^n}{(2n-1)!!}=(-1)^n\frac{2^n n!}{(2n)!},\quad n\geq 0.$$

Proof. From the recurrence relation

$$\Gamma(x+1)=x\Gamma(x),\quad x>0 \tag{7}$$

and the identity (5), we obtain

$$\Gamma\left(m+\frac{1}{2}\right)=\frac{(2m-1)!!}{2^m}\Gamma\left(\frac{1}{2}\right)=\frac{(2m-1)!!}{2^m}\sqrt{\pi}\,.$$

By this equality and the last expression in (4), we derive

$$B\left(\frac{1}{2},\frac{n}{2}\right)=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}=\begin{cases}\dfrac{\Gamma\left(\frac{1}{2}\right)\Gamma(m)}{\Gamma\left(m+\dfrac{1}{2}\right)}, & n=2m\\ \dfrac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(m+\dfrac{1}{2}\right)}{\Gamma(m+1)}, & n=2m+1\end{cases}=$$

$$=\begin{cases}\dfrac{\sqrt{\pi}\,(m-1)!}{\dfrac{(2m-1)!!}{2^m}\sqrt{\pi}}, & n=2m\\ \dfrac{\sqrt{\pi}\,\dfrac{(2m-1)!!}{2^m}\sqrt{\pi}}{m!}, & n=2m+1\end{cases}=\begin{cases}2\dfrac{(2m-2)!!}{(2m-1)!!}, & n=2m;\\ \pi\dfrac{(2m-1)!!}{(2m)!!}, & n=2m+1.\end{cases}$$

Substituting this into (6) reveals

$$I_{2m}=\frac{1}{2}a^{2m+1}\left[2B\left(\frac{1}{2},\frac{2m+1}{2}\right)\right]=a^{2m+1}\pi\frac{(2m-1)!!}{(2m)!!},$$

$$I_{2m+1}=\frac{1}{2}a^{2(m+1)}\left[2B\left(\frac{1}{2},\frac{2m+3}{2}\right)\right]=a^{2(m+1)}\pi\frac{(2m+1)!!}{(2m+2)!!}.$$

The proof of Corollary is complete. □

3. Integral representations for the Catalan numbers. The Catalan numbers C_n for $n \geq 0$ form a sequence of natural numbers that occur in various counting problems in combinatorial mathematics. The nth Catalan number can be expressed in terms of the central binomial coefficients $\binom{2n}{n}$ by

$$C_n=\frac{1}{n+1}\binom{2n}{n}. \tag{8}$$

Theorem 3. *For $n \geq 0$ and $a > 0$, the Catalan numbers C_n can be represented by*

$$\begin{aligned}C_n&=\frac{1}{\pi}\frac{4^n}{n+1}\frac{1}{a^{2n+1}}\int_{-a}^{a}x^{2n}\sqrt{\frac{a+x}{a-x}}\,dx=\\&=\frac{1}{\pi}\frac{2^{2n+1}}{n+1}\frac{1}{a^{2n}}\int_0^a\frac{x^{2n}}{\sqrt{a^2-x^2}}\,dx=\frac{1}{\pi}\frac{2^{2n+1}}{n+1}\int_0^{\pi/2}\sin^{2n}x\,dx\end{aligned}\tag{9}$$

and

$$C_n = \frac{1}{\pi}\frac{2^{2n+1}}{2n+1}\frac{1}{a^{2n+2}}\int_{-a}^{a} x^{2n+1}\sqrt{\frac{a+x}{a-x}}\,dx = \\ = \frac{1}{\pi}\frac{2^{2n+2}}{2n+1}\frac{1}{a^{2n+2}}\int_{0}^{a}\frac{x^{2n+2}}{\sqrt{a^2-x^2}}\,dx = \frac{1}{\pi}\frac{2^{2n+2}}{2n+1}\int_0^{\pi/2}\sin^{2n+2}x\,dx. \tag{10}$$

Proof. From the recurrence relation (7) and the identity (5), it is not difficult to show that the Catalan numbers C_n can be expressed in terms of the gamma function Γ by

$$C_n = \frac{4^n\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+2)}, \quad n \geq 0.$$

This implies that

$$C_n = \frac{1}{\pi}\frac{4^n}{n+1}B\left(\frac{1}{2}, n+\frac{1}{2}\right). \tag{11}$$

Taking $n = 2p$ in (6) and utilizing (11) leads to

$$I_{2p} = a^{2p+1}B\left(\frac{1}{2}, \frac{2p+1}{2}\right) = a^{2p+1}\pi\frac{p+1}{4^p}C_p,$$

which is equivalent to

$$C_n = \frac{4^n}{n+1}\frac{1}{a^{2n+1}\pi}I_{2n} = \frac{1}{\pi}\frac{4^n}{n+1}\frac{1}{a^{2n+1}}\int_{-a}^{a} x^{2n}\sqrt{\frac{a+x}{a-x}}\,dx.$$

The first formula in (9) thus follows.

By a similar argument to the deduction of (11), we can discover

$$C_n = \frac{4^{n+1}}{(2n+1)(2n+2)}\frac{1}{B\left(\frac{1}{2}, n+1\right)}, \quad n \geq 0.$$

Employing this identity and setting $n = 2p+1$ in (3) figures out

$$I_{2p+1} = a^{2p+2}\frac{2\pi}{2p+2}\frac{1}{B\left(\frac{1}{2}, p+1\right)} = a^{2p+2}\frac{2\pi}{2p+2}\frac{(2p+1)(2p+2)}{4^{p+1}}C_p$$

which can be rearranged as

$$C_p = \frac{1}{a^{2p+2}} \frac{1}{\pi} \frac{2^{2p+1}}{2p+1} I_{2p+1} = \frac{1}{\pi} \frac{1}{a^{2p+2}} \frac{2^{2p+1}}{2p+1} \int_{-a}^{a} x^{2p+1} \sqrt{\frac{a+x}{a-x}}\, dx.$$

The first formula in (10) is thus proved.

The rest integral representations follow from techniques used in the proofs of Theorems (1) and (2) and from changing variable of integration. □

4. Remarks. Finally, we state several remarks on our main results.

Remark 1. *The expressions in Corollary and the integral representation* (9) *correct [1, Proposition 3.1 and Corollary 3.2], respectively.*

Remark 2. *Since*

$$B\left(\frac{1}{2}, \frac{t+1}{2}\right) B\left(\frac{1}{2}, \frac{t}{2}\right) = \frac{2\pi}{t}$$

for $t > 0$, *formulas* (3) *and* (6) *can be transferred to each other. However, formula* (6) *looks simpler.*

Remark 3. *Considering* (8), *we can rewrite the integral representations in* (9) *and* (10) *as*

$$\binom{2n}{n} = \frac{1}{\pi} \frac{4^n}{a^{2n+1}} \int_{-a}^{a} x^{2n} \sqrt{\frac{a+x}{a-x}}\, dx =$$

$$= \frac{1}{\pi} \frac{2^{2n+1}}{a^{2n}} \int_{0}^{a} \frac{x^{2n}}{\sqrt{a^2 - x^2}}\, dx = \frac{1}{\pi} 2^{2n+1} \int_{0}^{\pi/2} \sin^{2n} x\, dx$$

and

$$\binom{2n}{n} = \frac{1}{\pi} \frac{2^{2n+1}(n+1)}{2n+1} \frac{1}{a^{2n+2}} \int_{-a}^{a} x^{2n+1} \sqrt{\frac{a+x}{a-x}}\, dx =$$

$$= \frac{1}{\pi} \frac{2^{2n+2}(n+1)}{2n+1} \frac{1}{a^{2n+2}} \int_{0}^{a} \frac{x^{2n+2}}{\sqrt{a^2 - x^2}}\, dx =$$

$$= \frac{1}{\pi} \frac{2^{2n+2}(n+1)}{2n+1} \int_0^{\pi/2} \sin^{2n+2} x \, dx.$$

for $n \geq 0$.

Remark 4. *It is well known that the Wallis ratio is defined by*

$$W_n = \frac{(2n-1)!!}{(2n)!!} = \frac{(2n)!}{2^{2n}(n!)^2} = \frac{1}{\sqrt{\pi}} \frac{\Gamma\big(n+1/2\big)}{\Gamma(n+1)}, \quad n \geq 0.$$

As a result, we have

$$I_{2m} = \pi a^{2m+1} W_m, \quad m \geq 0,$$

$$I_{2m+1} = \pi a^{2m+2} W_{m+1}, \quad m \geq 0.$$

The Wallis ratio has been studied and applied by many mathematicians. For more information, please refer to the survey article [5] and the paper [11], for example, and plenty of literature therein.

Remark 5. *In [2] the formula*

$$\int_0^{\pi/2} \sin^t x \, dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma\big(\frac{t+1}{2}\big)}{\Gamma\big(\frac{t+2}{2}\big)}, \quad t > -1 \tag{12}$$

was stated. See also [5, p. 16, Eq. (2.18)]. In [3, p. 142, Eq. 5.12.2], the formula

$$\int_0^{\pi/2} \sin^{2a-1} \theta \cos^{2b-1} \theta \, d\theta = \frac{1}{2} B(a,b), \quad \operatorname{Re}(a), \operatorname{Re}(b) > 0 \tag{13}$$

was listed. By (12) *or* (13), *we find that the quantity* S_n *defined in* (2) *is*

$$S_n = \int_{-\pi/2}^{0} \sin^n x \, dx + \int_0^{\pi/2} \sin^n x \, dx =$$

$$= \int_0^{\pi/2} (-1)^n \sin^n x \, dx + \int_0^{\pi/2} \sin^n x \, dx = \frac{1+(-1)^n}{2} B\left(\frac{n+1}{2}, \frac{1}{2}\right).$$

Remark 6. *In [9, Theorem 2.3] the integral formulas*

$$\int_a^b \left(\frac{b-t}{t-a}\right)^\lambda dt = (b-a)\frac{\lambda\pi}{\sin(\lambda\pi)},$$

$$\int_a^b \left(\frac{b-t}{t-a}\right)^\lambda \frac{1}{t} dt = \frac{\pi}{\sin(\lambda\pi)}\left[\left(\frac{b}{a}\right)^\lambda - 1\right],$$

$$\int_a^b \left(\frac{b-t}{t-a}\right)^\lambda \frac{1}{t^{k+1}} dt = \frac{\pi}{\sin(\lambda\pi)}\left(\frac{b}{a}\right)^\lambda \frac{1}{a^k}\sum_{\ell=0}^{k}\frac{\langle\lambda\rangle_\ell}{\ell!}\binom{k-1}{\ell-1}\left(1-\frac{a}{b}\right)^\ell$$

for $b > a > 0$, $k \in \mathbb{N}$, and $\lambda \in (-1,1) \setminus \{0\}$ were derived, where

$$\langle x\rangle_n = \begin{cases} \prod_{k=0}^{n-1}(x-k), & n \ge 1 \\ 1, & n = 0 \end{cases}$$

is called the falling factorial. In [9, Remark 4.4], the integral formula

$$\int_a^b \left(\frac{b-t}{t-a}\right)^\lambda \frac{1}{t}\ln\frac{b-t}{t-a} dt = \\ = \begin{cases} \dfrac{\pi}{\sin(\lambda\pi)}\left\{\left(\dfrac{b}{a}\right)^\lambda \ln\dfrac{b}{a} - \pi\cot(\lambda\pi)\left[\left(\dfrac{b}{a}\right)^\lambda - 1\right]\right\}, & \lambda \ne 0 \\ \dfrac{1}{2}\left(\ln\dfrac{b}{a}\right)^2, & \lambda = 0 \end{cases}$$

was concluded from [9, Theorem 2.3]. By comparing the forms of these integrals and I_n, we naturally propose a problem: can one closedly compute the integrals

$$\int_a^b \left(\frac{b-t}{t-a}\right)^\lambda t^\alpha\, dt \quad \text{and} \quad \int_a^b t^\alpha \left(\frac{b-t}{t-a}\right)^\lambda \ln\frac{b-t}{t-a}\, dt$$

for $\lambda \in (-1,1)$ and

$$\alpha \in \begin{cases} \mathbb{R}, & b > a > 0 \\ \mathbb{N}, & b > 0 > a \end{cases} \quad ?$$

Remark 7. *An anonymous reviewer points out that the Catalan numbers C_n emerge frequently in probability, for example, in the closed distribution of the first return to zero of the symmetric coin tossing random walk, where*

$$T_0 = \inf\{k > 0 : S_k = 0\}, \quad S_k = \sum_{j=1}^{k} X_j, \quad \text{and} \quad X_j = \begin{cases} 1, & p = \frac{1}{2} \\ -1, & p = \frac{1}{2} \end{cases}$$

has distribution

$$P(T_0 = 2n + 2) = \binom{2n}{n} \frac{1}{n+1} \frac{1}{2^{n+1}}, \quad n = 0, 1, 2, \dots$$

Remark 8. *In recent years, the Catalan numbers C_n have been analytically generalized and studied in the papers [10, 12]. For more information, please refer to the survey articles [7, 8] and closely-related references therein.*

Remark 9. *This paper is a slightly revised version of the preprint [4].*

Acknowledgement. The author thanks Mr. Qing Zou at University of Iowa and other three anonymous reviewers for their careful corrections to and valuable comments on the original version of this paper.

References

[1] Dana-Picard T., Zeitoun D. G. *Parametric improper integrals, Wallis formula and Catalan numbers.* Internat. J. Math. Ed. Sci. Tech., 2012, vol. 43, no. 4, pp. 515–520. DOI: 10.1080/0020739X.2011.599877.

[2] Kazarinoff D. K. *On Wallis' formula.* Edinburgh Math. Notes, 1956, vol. 1956, no. 40, pp. 19–21.

[3] Olver F. W. J., Lozier D. W., Boisvert R. F., Clark C. W. (eds.) *NIST Handbook of Mathematical Functions.* Cambridge University Press, New York, 2010.

[4] Qi F. *An improper integral with a square root.* Preprints, 2016, Article ID 2016100089, 8 pp. DOI: 10.20944/preprints201610.0089.v1.

[5] Qi F. *Bounds for the ratio of two gamma functions.* J. Inequal. Appl., 2010, Article ID 493058, 84 pp. DOI: 10.1155/2010/493058.

[6] Qi F. *Parametric integrals, the Catalan numbers, and the beta function.* Elem. Math., 2017, vol. 72, no. 3, pp. 103–110. DOI: 10.4171/EM/332.

[7] Qi F., Guo B.-N. *Integral representations of Catalan numbers and their applications.* Mathematics, 2017, vol. 5, no. 3, Article 40, 31 pp. DOI: 10.3390/math5030040.

[8] Qi F., Guo B.-N. *Some properties and generalizations of the Catalan, Fuss, and Fuss–Catalan numbers.* Chapter 5 in *Mathematical Analysis and Applications: Selected Topics*, First Edition, pp. 101–133; Edited by Michael Ruzhansky, Hemen Dutta, and Ravi P. Agarwal; Published by John Wiley & Sons, Inc. 2018. DOI: 10.1002/9781119414421.ch5.

[9] Qi F., Guo B.-N. *The reciprocal of the weighted geometric mean is a Stieltjes function.* Bol. Soc. Mat. Mex. (3), 2018, vol. 24, no. 1, pp. 181–202. DOI: 10.1007/s40590-016-0151-5.

[10] Qi F., Mahmoud M., Shi X.-T., Liu F.-F. *Some properties of the Catalan–Qi function related to the Catalan numbers.* SpringerPlus, 2016, vol. 5, Article No. 1126, 20 pp. DOI: 10.1186/s40064-016-2793-1.

[11] Qi F., Mortici C. *Some best approximation formulas and inequalities for the Wallis ratio.* Appl. Math. Comput., 2015, vol. 253, pp. 363–368. DOI: 10.1016/j.amc.2014.12.039.

[12] Qi F., Shi X.-T., Mahmoud M., Liu F.-F. *The Catalan numbers: a generalization, an exponential representation, and some properties.* J. Comput. Anal. Appl., 2017, vol. 23, no. 5, pp. 937–944.

Received January 22, 2018.
In revised form, April 28, 2018.
Accepted April 30, 2018.
Published online June 07, 2018.

Institute of Mathematics, Henan Polytechnic University
Jiaozuo, Henan, 454010, China;
College of Mathematics, Inner Mongolia University for Nationalities
Tongliao, Inner Mongolia, 028043, China;
Department of Mathematics, College of Science
Tianjin Polytechnic University
Tianjin, 300387, China
E-mail: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com
URL: `https://qifeng618.wordpress.com`