Probl. Anal. Issues Anal. Vol. 7 (25), No. 2, 2018, pp. 47–68

DOI: 10.15393/j3.art.2018.4650

UDC 514.76, 514.77

Dae Ho Jin

GENERIC LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE KAEHLER MANIFOLD WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION

Abstract. Recently, this author studied lightlike hypersurfaces of an indefinite Kaehler manifold endowed with a semi-symmetric non-metric connection in [7]. Further we study this subject. The object of study in this paper is generic lightlike submanifolds of an indefinite Kaehler manifold endowed with a semi-symmetric non-metric connection such that the induced structure tensor field on the submanifolds is recurrent or Lie recurrent.

Key words: *generic lightlike submanifold, semi-symmetric non-metric connection, indefinite Kaehler manifold*

2010 Mathematical Subject Classification: *53C25, 53C40, 53C50*

1. Introduction. A lightlike submanifold M of an indefinite almost complex manifold $\bar{M}$, with an indefinite almost complex structure J, is called *generic lightlike submanifold* if there exists a screen distribution $S(TM)$ of M such that

$$J(S(TM)^{\perp}) \subset S(TM), \tag{1}$$

where $S(TM)^{\perp}$ is the orthogonal complement of $S(TM)$ in the tangent bundle $T\bar{M}$ of $\bar{M}$, i. e., $T\bar{M} = S(TM) \oplus_{orth} S(TM)^{\perp}$. The notion of generic lightlike submanifold was introduced by Jin-Lee [8] at 2011 and later, studied by several authors (see [3–5]). The geometry of generic lightlike submanifold is an extension of that of lightlike hypersurface and half lightlike submanifold of codimension 2. Much of its theory will be immediately generalized in a formal way to general lightlike submanifolds.

A linear connection $\bar{\bar{\nabla}}$ on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called a *semi-symmetric non-metric connection* if it and its torsion $\bar{\bar{T}}$ satisfy

$$(\bar{\bar{\nabla}}_{\bar{X}}\bar{g})(\bar{Y}, \bar{Z}) = -\theta(\bar{Y})\bar{g}(\bar{X}, \bar{Z}) - \theta(\bar{Z})\bar{g}(\bar{X}, \bar{Y}), \tag{2}$$

$$\bar{T}(\overline{X}, \overline{Y}) = \theta(\overline{Y})\overline{X} - \theta(\overline{X})\overline{Y}, \tag{3}$$

where θ is a 1-form on $\overline{M}$ associated with a smooth unit vector field ζ by $\theta(\overline{X}) = \bar{g}(\overline{X}, \zeta)$. In the followings, we denote by $\overline{X}, \overline{Y}$ and $\bar{Z}$ the smooth vector fields on $\overline{M}$. The notion of semi-symmetric non-metric connection on a Riemannian manifold was introduced by Ageshe-Chafle [1].

Remark. *Denote by $\widetilde{\nabla}$ a Levi-Civita connection of a semi-Riemannian manifold $(\overline{M}, \bar{g})$. It is known [7] that a linear connection $\overline{\nabla}$ on $\overline{M}$ is a semi-symmetric non-metric connection if and only if $\overline{\nabla}$ satisfies*

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \widetilde{\nabla}_{\overline{X}}\overline{Y} + \theta(\overline{Y})\overline{X}. \tag{4}$$

The object of present study is generic lightlike submanifolds of an indefinite Kaehler manifold with a semi-symmetric non-metric connection. First, we study the geometry of two types of generic lightlike submanifolds, named by *recurrent* and *Lie recurrent*, of such an indefinite Kaehler manifold. Next, we characterize generic lightlike submanifolds of an indefinite complex space form with a semi-symmetric non-metric connection.

2. Semi-symmetric non-metric connections. Let $\overline{M} = (\overline{M}, \bar{g}, J)$ be an indedinite Kaeler manifold, where $\bar{g}$ is a semi-Riemannian metric and J is an indefinite almost complex structure:

$$J^2\overline{X} = -\overline{X}, \qquad \bar{g}(J\overline{X}, J\overline{Y}) = \bar{g}(\overline{X}, \overline{Y}), \qquad (\widetilde{\nabla}_{\bar{X}}J)\overline{Y} = 0. \tag{5}$$

Replacing the Levi-Civita connection $\widetilde{\nabla}$ by the semi-symmetric non-metric connection $\overline{\nabla}$ given by (4), the third equation of (5) is reduced to

$$(\overline{\nabla}_{\bar{X}}J)\overline{Y} = \theta(J\overline{Y})\overline{X} - \theta(\overline{Y})J\overline{X}. \tag{6}$$

Let (M, g) be an m-dimensional lightlike submanifold of an indefinite Kaehler manifold $(\overline{M}, \bar{g})$ of dimension $(m+n)$. Then the radical distribution $Rad(TM)$, defined by $Rad(TM) = TM \cap TM^{\perp}$, of M is a subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank $r\,(1 \leqslant r \leqslant \min\{m, n\})$. In general, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$ respectively, which are called the *screen distribution* and the *co-screen distribution* of M [2], such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \quad TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Also denote by $(5)_i$ the i-th equation of (5). We use the same notations for any others. Let X, Y, Z and W be the vector fields on M, unless otherwise specified. We use the following range of indices:

$$i, j, k, \ldots \in \{1, \ldots, r\}, \qquad a, b, c, \ldots \in \{r+1, \ldots, n\}.$$

Let $tr(TM)$ and $ltr(TM)$ be complementary vector bundles to TM in $T\bar{M}_{|M}$ and $TM^{\perp}$ in $S(TM)^{\perp}$ respectively and let $\{N_1, \ldots, N_r\}$ be a lightlike basis of $ltr(TM)_{|\mathcal{U}}$, where $\mathcal{U}$ is a neighborhood of M, such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0,$$

where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $Rad(TM)_{|\mathcal{U}}$. Then we have

$$\begin{aligned} T\overline{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) = \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \end{aligned}$$

A lightlike submanifold $M = (M, g, S(TM), S(TM^{\perp}))$ of $\overline{M}$ is called

(1) *r-lightlike submanifold* if $1 \leqslant r < \min\{m, n\}$;

(2) *co-isotropic submanifold* if $1 \leqslant r = n < m$;

(3) *isotropic submanifold* if $1 \leqslant r = m < n$;

(4) *totally lightlike submanifold* if $1 \leqslant r = m = n$.

The above three classes $(2) - (4)$ are particular cases of (1) as follows:

$$S(TM^{\perp}) = \{0\}, \qquad S(TM) = \{0\}, \qquad S(TM) = S(TM^{\perp}) = \{0\}$$

respectively. The geometry of r-lightlike submanifolds is more general than that of the other three types. Thus we consider only r-lightlike submanifolds M, with following quasi-orthonormal field of frames of $\overline{M}$:

$$\{\xi_1, \ldots, \xi_r, N_1, \ldots, N_r, F_{r+1}, \ldots, F_m, E_{r+1}, \ldots, E_n\},$$

where $\{F_A\}$ and $\{E_a\}$ are orthonormal bases of $S(TM)$ and $S(TM^{\perp})$, respectively. Denote $\epsilon_a = \bar{g}(E_a, E_a)$. Then $\epsilon_a \delta_{ab} = \bar{g}(E_a, E_b)$.

Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss-Weingarten formulae of M and $S(TM)$ are given respectively by

$$\bar{\bar{\nabla}}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^{\ell}(X,Y)N_i + \sum_{a=r+1}^{n} h_a^s(X,Y)E_a, \tag{7}$$

$$\bar{\bar{\nabla}}_X N_i = -A_{N_i}X + \sum_{j=1}^{r} \tau_{ij}(X)N_j + \sum_{a=r+1}^{n} \rho_{ia}(X)E_a, \tag{8}$$

$$\bar{\bar{\nabla}}_X E_a = -A_{E_a}X + \sum_{i=1}^{r} \lambda_{ai}(X)N_i + \sum_{b=r+1}^{n} \mu_{ab}(X)E_b; \tag{9}$$

$$\nabla_X PY = \nabla^*_X PY + \sum_{i=1}^{r} h_i^*(X,PY)\xi_i, \tag{10}$$

$$\nabla_X \xi_i = -A^*_{\xi_i}X - \sum_{j=1}^{r} \sigma_{ji}(X)\xi_j, \tag{11}$$

where ∇ and ∇^* are induced linear connections on M and $S(TM)$ respectively, h_i^{ℓ} and h_a^s are called the *local second fundamental forms* on M, h_i^* are called the *local second fundamental forms* on $S(TM)$. A_{N_i}, A_{E_a} and $A^*_{\xi_i}$ are called the *shape operators*, and $\tau_{ij}, \rho_{ia}, \lambda_{ai}$, μ_{ab} and σ_{ji} are 1-forms on M. Using (2), (3) and (7), we see that

$$\begin{aligned}(\nabla_X g)(Y,Z) = \sum_{i=1}^{r}\{h_i^{\ell}(X,Y)\eta_i(Z) + h_i^{\ell}(X,Z)\eta_i(Y)\} -& \\ -\theta(Y)g(X,Z) - \theta(Z)g(X,Y),& \quad (12)\\ T(X,Y) = \theta(Y)X - \theta(X)Y,& \quad (13)\end{aligned}$$

and both h_i^{ℓ} and h_a^s are symmetric, where η_i's are 1-forms such that

$$\eta_i(X) = \bar{g}(X, N_i).$$

In the sequel, denote by α_i, β_i and γ_a the functions given by

$$\alpha_i = \theta(\xi_i), \qquad \beta_i = \theta(N_i), \qquad \gamma_a = \theta(E_a).$$

As $h_i^{\ell}(X,Y) = \bar{g}(\bar{\bar{\nabla}}_X Y, \xi_i)$ and $\epsilon_a h_a^s(X,Y) = \bar{g}(\bar{\bar{\nabla}}_X Y, E_a)$, we know that h_i^{ℓ} and h_a^s are independent of the choice of $S(TM)$. The above three types local second fundamental forms are related to their shape operators by

$$h_i^{\ell}(X,Y) = g(A^*_{\xi_i}X,Y) - \sum_{k=1}^{r} h_k^{\ell}(X,\xi_i)\eta_k(Y) + \alpha_i g(X,Y), \tag{14}$$

$$\epsilon_a h_a^s(X,Y) = g(A_{E_a}X,Y) - \sum_{k=1}^{r} \lambda_{ak}(X)\eta_k(Y) + \gamma_a g(X,Y), \tag{15}$$

$$h_i^*(X,PY) = g(A_{N_i}X,PY) + \eta_i(X)\theta(PY) + \beta_i g(X,PY). \tag{16}$$

Applying the operator $\overline{\nabla}_X$ to $g(\xi_i,\xi_j)=0$, $\bar{g}(\xi_i,E_a)=0$, $\bar{g}(N_i,N_j)=0$, $\bar{g}(N_i,E_a)=0$, $\bar{g}(E_a,E_b)=\epsilon\delta_{ab}$ and $\bar{g}(N_i,\xi_j)=\delta_{ij}$ by turns, we have

$$\begin{cases} h_i^\ell(X,\xi_j) + h_j^\ell(X,\xi_i) = 0, \qquad h_a^s(X,\xi_i) = -\epsilon_a\lambda_{ai}(X), \\ \eta_j(A_{N_i}X) + \eta_i(A_{N_j}X) = -\beta_i\eta_j(X) - \beta_j\eta_i(X), \\ \bar{g}(A_{E_a}X,N_i) = \epsilon_a\rho_{ia}(X) - \gamma_a\eta_i(X), \\ \epsilon_b\mu_{ab} + \epsilon_a\mu_{ba} = 0, \quad \tau_{ij}(X) = \sigma_{ij}(X) + \alpha_j\eta_i(X). \end{cases} \tag{17}$$

Furthermore, using $(17)_1$, we see that

$$h_i^\ell(X,\xi_i) = 0, \qquad h_i^\ell(\xi_j,\xi_k) = 0, \qquad A^*_{\xi_i}\xi_i = 0. \tag{18}$$

Definition 1. *We say that a lightlike submanifold of a semi-Riemannian manifold is irrotational* [9] *if* $\overline{\nabla}_X\xi_i \in \Gamma(TM)$ *for all* $i\in\{1,\cdots,r\}$.

Remark. *From (7) and* $(17)_2$*, the above definition is equivalent to*

$$h_j^\ell(X,\xi_i) = 0, \qquad h_a^s(X,\xi_i) = \lambda_{ai}(X) = 0.$$

3. Structure equations. Let M be a generic lightlike submanifold of $\overline{M}$. From (1) we show that $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^\perp))$ are subbundles of $S(TM)$. Thus there exist two non-degenerate almost complex distributions H_o and H with respect to J, i. e., $J(H_o)=H_o$ and $J(H)=H$, such that

$$\begin{aligned} S(TM) =& \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{orth} J(S(TM^\perp)) \oplus_{orth} H_o, \\ H =& Rad(TM) \oplus_{orth} J(Rad(TM)) \oplus_{orth} H_o. \end{aligned}$$

In this case, the tangent bundle TM of M is decomposed as follow:

$$TM = H \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^\perp)). \tag{19}$$

Consider r-th local null vector fields U_i and V_i, $(n-r)$-th local non-null unit vector fields W_a, and their 1-forms u_i, v_i and w_a defined by

$$U_i = -JN_i, \qquad V_i = -J\xi_i, \qquad W_a = -JE_a, \tag{20}$$

$$u_i(X) = g(X, V_i), \quad v_i(X) = g(X, U_i), \quad w_a(X) = \epsilon_a g(X, W_a). \tag{21}$$

Denote by S the projection morphism of TM on H and by F the tensor field of type $(1,1)$ globally defined on M by $F = J \circ S$. Then X is expressed as $X = SX + \sum_{i=1}^{r} u_i(X)U_i + \sum_{a=r+1}^{n} w_a(X)W_a$. Therefore,

$$JX = FX + \sum_{i=1}^{r} u_i(X)N_i + \sum_{a=r+1}^{n} w_a(X)E_a. \tag{22}$$

Applying J to (22) and using $(5)_1$, (20) and (22), we have

$$F^2X = -X + \sum_{i=1}^{r} u_i(X)U_i + \sum_{a=r+1}^{n} w_a(X)W_a. \tag{23}$$

In the sequel, we say that F is the *structure tensor field* of M.

Applying the operator $\bar{\nabla}_X$ to $(20)_{1,2,3}$ and (22) by turns and using (6), (7)–(11), (14)–(16) and (20)–(22), we have

$$\begin{aligned} h_j^\ell(X, U_i) &= u_j(A_{N_i}X) + \beta_i u_j(X) = h_i^*(X, V_j) - \theta(V_j)\eta_i(X), \\ h_a^s(X, U_i) &= w_a(A_{N_i}X) + \beta_i w_a(X) = \epsilon_a\{h_i^*(X, W_a) - \theta(W_a)\eta_i(X)\}, \end{aligned} \tag{24}$$

$$\begin{gathered} h_j^\ell(X, V_i) = h_i^\ell(X, V_j), \qquad h_a^s(X, V_i) = \epsilon_a h_i^\ell(X, W_a), \\ \epsilon_b h_b^s(X, W_a) = \epsilon_a h_a^s(X, W_b), \end{gathered}$$

$$\begin{aligned} \nabla_X U_i &= F(A_{N_i}X) + \sum_{j=1}^{r} \tau_{ij}(X)U_j + \sum_{a=r+1}^{n} \rho_{ia}(X)W_a + \\ &\quad + \beta_i FX + \theta(U_i)X, \end{aligned} \tag{25}$$

$$\begin{aligned} \nabla_X V_i &= F(A^*_{\xi_i}X) - \sum_{j=1}^{r} \sigma_{ji}(X)V_j + \sum_{j=1}^{r} h_j^\ell(X, \xi_i)U_j - \\ &\quad - \sum_{a=r+1}^{n} \epsilon_a \lambda_{ai}(X)W_a + \alpha_i FX + \theta(V_i)X, \end{aligned} \tag{26}$$

$$\begin{aligned} \nabla_X W_a &= F(A_{E_a}X) + \sum_{i=1}^{r} \lambda_{ai}(X)U_i + \sum_{b=r+1}^{n} \mu_{ab}(X)W_b + \\ &\quad + \gamma_a FX + \theta(W_a)X, \end{aligned} \tag{27}$$

$$(\nabla_X F)(Y) = \sum_{i=1}^{r} u_i(Y)A_{N_i}X + \sum_{a=r+1}^{n} w_a(Y)A_{E_a}X - \\ - \sum_{i=1}^{r} h_i^\ell(X,Y)U_i - \sum_{a=r+1}^{n} h_a^s(X,Y)W_a + \\ + \theta(JY)X - \theta(Y)FX. \tag{28}$$

4. Recurrent and Lie recurrent structure tensors.

Definition 2. *The structure tensor field F of M is said to be recurrent [6] if there exists a 1-form ϖ on M such that*

$$(\nabla_X F)Y = \varpi(X)FY.$$

A generic lightlike submanifold M of an indefinite Kaehler manifold $\overline{M}$ is called recurrent if it admits a recurrent structure tensor field F.

Theorem 1. *Let M be a recurrent lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric non-metric connection. Then*

(1) F is parallel with respect to the induced connection ∇ on M,

(2) M is irrotational and the 1-forms ρ_{ia} satisfy $\rho_{ia} = 0$,

(3) the 1-form θ vanishes on TM,

(4) H, $J(ltr(TM))$ and $J(S(TM^\perp))$ are parallel distributions on M,

(5) M is locally a product manifold $M_r \times M_{n-r} \times M^\sharp$, where M_r, M_{n-r} and $M^\sharp$ are leaves of $J(ltr(TM))$, $J(S(TM^\perp))$ and H, respectively.

Proof.

(1) From the above definition and (27), we obtain

$$\varpi(X)FY = \sum_{i=1}^{r} u_i(Y)A_{N_i}X + \sum_{a=r+1}^{n} w_a(Y)A_{E_a}X - \\ - \sum_{i=1}^{r} h_i^\ell(X,Y)U_i - \sum_{a=r+1}^{n} h_a^s(X,Y)W_a + \theta(JY)X - \theta(Y)FX. \tag{29}$$

Replacing Y by ξ_j to this and using the fact that $F\xi_j = -V_j$, we get

$$\varpi(X)V_j = \sum_{k=1}^{r} h_k^\ell(X,\xi_j)U_k + \sum_{b=r+1}^{n} h_b^s(X,\xi_j)W_b + \theta(V_j)X + \alpha_j FX. \tag{30}$$

Taking the scalar product with N_i to (30), we obtain

$$\theta(V_j)\eta_i(X) + \alpha_j v_i(X) = 0.$$

Taking $X = V_i$ and $X = \xi_i$ to this equation by turns, we have

$$\alpha_i = 0, \qquad \theta(V_i) = 0, \tag{31}$$

for all i. Taking the scalar product with U_j to (30), we get $\varpi = 0$. Thus F is parallel with respect to the induced connection ∇ on M.

(2) Taking the scalar product with V_i and W_a to (30) such that $\varpi = \alpha_j = \theta(V_j) = 0$, we obtain $h_i^\ell(X, \xi_j) = 0$ and $h_a^s(X, \xi_j) = 0$. Thus M is irrotational by Remark in Section 2.

Replacing Y by W_a to (29) such that $\varpi = 0$, we have

$$A_{E_a}X = \sum_{i=1}^{r} h_i^\ell(X, W_a)U_i + \sum_{b=r+1}^{n} h_b^s(X, W_a)W_b - \gamma_a X + \theta(W_a)FX. \tag{32}$$

Taking the scalar product with N_i and U_i to this equation by turns and using (15), $(17)_4$, we obtain

$$\epsilon_a \rho_{ia}(X) = \theta(W_a)v_i(X), \qquad \epsilon_a h_a^s(X, U_i) = -\theta(W_a)\eta_i(X). \tag{33}$$

Replacing X by ξ_i to $(33)_2$ and using the fact that $h_a^s(\xi_i, U_i) = 0$, we get $\theta(W_a) = 0$. From this result and $(33)_1$, we see that $\rho_{ia} = 0$. Thus

$$\theta(W_a) = 0, \qquad \rho_{ia} = 0, \qquad h_a^s(X, U_i) = 0. \tag{34}$$

(3) Replacing Y by U_i to (29) such that $\varpi = 0$, we have

$$A_{N_i}X = \sum_{k=1}^{r} h_k^\ell(X, U_i)U_k + \sum_{a=r+1}^{n} h_a^s(X, U_i)W_a - \beta_i X + \theta(U_i)FX. \tag{35}$$

Taking the scalar product with N_j and U_j to this by turns, we get

$$\begin{aligned} \eta_j(A_{N_i}X) &= -\beta_i \eta_j(X) - \theta(U_i)v_j(X), \\ g(A_{N_i}X, U_j) &= -\beta_i v_j(X) - \theta(U_i)\eta_j(X). \end{aligned} \tag{36}$$

Taking $i = j$ to $(36)_1$ and using $(17)_3$, we get $\theta(U_i)v_i(X) = 0$. It follows that $\theta(U_i) = 0$. Using (16), $(36)_2$ reduces $h_i^*(X, U_j) = 0$. Thus

$$\theta(U_i) = 0, \qquad h_i^*(X, U_j) = 0. \tag{37}$$

Replacing X by ξ_j to (29) and using M is irrotational, we get

$$\sum_{i=1}^{r} u_i(Y)A_{N_i}\xi_j + \sum_{a=r+1}^{n} w_a(Y)A_{E_a}\xi_j + \theta(JY)\xi_j + \theta(Y)V_j = 0.$$

Taking the scalar product with U_j to this equation, we have

$$\sum_{i=1}^{r} u_i(Y)\bar{g}(A_{N_i}\xi_j, U_j) + \sum_{a=r+1}^{n} w_a(Y)\bar{g}(A_{E_a}\xi_j, U_j) + \theta(Y) = 0. \tag{38}$$

Taking $Y = U_i$ and $Y = W_a$ by turns and using $(34)_1$ and $(37)_1$, we get

$$\bar{g}(A_{N_i}\xi_j, U_j) = 0, \qquad \bar{g}(A_{E_a}\xi_j, U_j) = 0.$$

Consequently, (38) is reduced to $\theta(X) = 0$. Thus θ vanishes on TM.

(4) Using (2), (11), (14), (15), (22), (26) and (27), we get

$$\begin{cases} g(\nabla_X\xi_i, V_j) = -h_i^\ell(X, V_j) + \alpha_i u_j(X), \\ g(\nabla_X\xi_i, W_a) = -h_i^\ell(X, W_a) + \epsilon_a\alpha_i w_a w(X), \\ g(\nabla_X V_i, V_j) = h_j^\ell(X, \xi_i) + \theta(V_i)u_j(X), \\ g(\nabla_X V_i, W_a) = -\lambda_{ai}(X) + \epsilon_a\theta(V_i)w_a(X), \\ g(\nabla_X Z, V_j) = h_j^\ell(X, FZ) + \theta(Z)u_j(X), \\ g(\nabla_X Z, W_a) = \epsilon_a\{h_a^s(X, FZ) + \theta(Z)w_a(X)\}, \end{cases} \tag{39}$$

for any $X \in \Gamma(TM)$ and $Z \in \Gamma(H_o)$. Taking $Y = V_j$ and $Y = FZ$, $Z \in \Gamma(H_o)$ to (29) by turns and using (31) and the facts that $\theta = 0$ on TM, $u_i(FZ) = w_a(FZ) = 0$ and $JFZ = F^2Z = -Z$, we have

$$h_i^\ell(X, V_j) = 0, \qquad h_a^s(X, V_j) = h_j^\ell(X, W_a) = 0, \tag{40}$$

$$h_i^\ell(X, FZ) = 0, \qquad h_a^s(X, FZ) = 0. \tag{41}$$

Using (31), (40), (41) and $\lambda_{ai} = 0$, (39) are equivalent to

$$\nabla_X Y \in \Gamma(H), \quad \forall\, X \in \Gamma(TM), \quad \forall\, Y \in \Gamma(H).$$

It follows that H is a parallel distribution on M.

Applying F to (32) and (35) and using $(34)_1$ and $(37)_1$, we get

$$F(A_{N_i}X) = -\beta_i FX, \qquad F(A_{E_a}X) = -\gamma_a FX.$$

Using these results together with (34), (37) and $\lambda_{ai} = 0$, (25) and (27) reduce to

$$\nabla_X U_i = \sum_{j=1}^{r} \tau_{ij}(X) U_j, \qquad \nabla_X U_i \in \Gamma(J(ltr(TM))), \tag{42}$$

$$\nabla_X W_a = \sum_{b=r+1}^{n} \mu_{ab} W_b, \qquad \nabla_X W_a \in \Gamma(J(S(TM^\perp))). \tag{43}$$

Thus $J(ltr(TM))$ and $J(S(TM^\perp))$ are also parallel distributions on M.

(5) As H, $J(ltr(TM))$ and $J(S(TM^\perp))$ are parallel distributions and satisfy (19), by the decomposition theorem of de Rham [10], M is locally a product manifold $M_r \times M_{n-r} \times M^\sharp$, where M_r, M_{n-r} and $M^\sharp$ are leaves of the distributions $J(ltr(TM))$, $J(S(TM^\perp))$ and H respectively. □

Definition 3. *The structure tensor field F of M is said to be Lie recurrent [6] if there exists a 1-form ϑ on M such that*

$$(\mathcal{L}_X F)Y = \vartheta(X) FY,$$

where $\mathcal{L}_X$ denotes the Lie derivative on M with respect to X, that is,

$$(\mathcal{L}_X F)Y = [X, FY] - F[X, Y]. \tag{44}$$

In the case $\vartheta = 0$, i.e., $\mathcal{L}_X F = 0$, we say that F is Lie parallel. A generic lightlike submanifold M of an indefinite Kaehler manifold $\overline{M}$ is called Lie recurrent if it admits a Lie recurrent structure tensor field F.

Theorem 2. *Let M be a Lie recurrent generic lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric non-metric connection. Then*

(1) *F is Lie parallel,*

(2) *τ_{ij} and ρ_{ia} satisfy $\tau_{ij}(FX) = 0$ and $\rho_{ia}(FX) = 0$. Moreover,*

$$\tau_{ij}(X) = \sum_{k=1}^{r} u_k(X) g(A_{N_k} V_j, N_i).$$

Proof.

(1) Using (13), (22), (28) and (44), we obtain

$$\begin{aligned}\vartheta(X)FY &= -\nabla_{FY}X + F\nabla_Y X + \\ &\quad + \sum_{i=1}^{r} u_i(Y)A_{N_i}X + \sum_{a=r+1}^{n} w_a(Y)A_{E_a}X - \\ &\quad - \sum_{i=1}^{r} h_i^\ell(X,Y)U_i - \sum_{a=r+1}^{n} h_a^s(X,Y)W_a + \\ &\quad + \Big\{\sum_{i=1}^{r}\beta_i u_i(Y) + \sum_{a=r+1}^{n}\gamma_a w_a(Y)\Big\}X. \end{aligned} \tag{45}$$

Replacing Y by ξ_j and Y by V_j to (45) respectively, we have

$$\begin{aligned}-\vartheta(X)V_j &= \nabla_{V_j}X + F\nabla_{\xi_j}X - \\ &\quad - \sum_{i=1}^{r} h_i^\ell(X,\xi_j)U_i - \sum_{a=r+1}^{n} h_a^s(X,\xi_j)W_a, \end{aligned} \tag{46}$$

$$\begin{aligned}\vartheta(X)\xi_j &= -\nabla_{\xi_j}X + F\nabla_{V_j}X - \\ &\quad - \sum_{i=1}^{r} h_i^\ell(X,V_j)U_i - \sum_{a=r+1}^{n} h_a^s(X,V_j)W_a. \end{aligned} \tag{47}$$

Taking the scalar product with U_i to (46) and N_i to (47), we get

$$\begin{aligned}-\delta_{ij}\vartheta(X) &= g(\nabla_{V_j}X, U_i) - \bar{g}(\nabla_{\xi_j}X, N_i), \\ \delta_{ij}\vartheta(X) &= g(\nabla_{V_j}X, U_i) - \bar{g}(\nabla_{\xi_j}X, N_i),\end{aligned} \tag{48}$$

respectively. It follows that $\vartheta = 0$. Thus F is Lie parallel.

(2) Taking the scalar product with N_i to (46) such that $X = W_a$ and using (15), $(17)_4$ and (27), we get $h_a^s(U_i, V_j) = \rho_{ia}(\xi_j)$. Also, taking the scalar product with W_a to (47) such that $X = U_i$ and using (25), we have $h_a^s(U_i, V_j) = -\rho_{ia}(\xi_j)$. Thus $\rho_{ia}(\xi_j) = 0$ and $h_a^s(U_i, V_j) = 0$.

Taking the scalar product with U_i to (46) with $X = W_a$ and using (15), $(17)_{2,4}$ and (27), we get $\epsilon_a\rho_{ia}(V_j) = \lambda_{aj}(U_i)$. Also, taking the scalar product with W_a to (46) such that $X = U_i$ and using $(17)_2$ and (25), we get $\epsilon_a\rho_{ia}(V_j) = -\lambda_{aj}(U_i)$. Thus $\rho_{ia}(V_j) = 0$ and $\lambda_{aj}(U_i) = 0$.

Taking the scalar product with V_i to (46) such that $X = W_a$ and using $(17)_2$, $(24)_4$ and (27), we obtain $\lambda_{ai}(V_j) = -\lambda_{aj}(V_i)$. Also, taking the scalar product with W_a to (46) such that $X = V_i$ and using $(17)_2$ and (26), we have $\lambda_{ai}(V_j) = \lambda_{aj}(V_i)$. Thus we obtain $\lambda_{ai}(V_j) = 0$.

Taking the scalar product with W_a to (46) such that $X = \xi_i$ and using (11), (14) and $(17)_2$, we get $h_i^\ell(V_j, W_a) = \lambda_{ai}(\xi_j)$. Also, taking the scalar product with V_i to (47) such that $X = W_a$ and using (27), we have $h_i^\ell(V_j, W_a) = -\lambda_{ai}(\xi_j)$. Thus $\lambda_{ai}(\xi_j) = 0$ and $h_i^\ell(V_j, W_a) = 0$.

Summarizing the above results, we obtain

$$\rho_{ia}(\xi_j) = 0, \quad \rho_{ia}(V_j) = 0, \quad \lambda_{ai}(U_j) = 0, \quad \lambda_{ai}(V_j) = 0, \quad \lambda_{ai}(\xi_j) = 0,$$
$$h_a^s(U_i, V_j) = h_j^\ell(U_i, W_a) = 0, \quad h_i^\ell(V_j, W_a) = h_a^s(V_j, V_i) = 0. \quad (49)$$

Taking the scalar product with N_i to (45) and using $(17)_4$, we have

$$-\bar{g}(\nabla_{FY}X, N_i) + g(\nabla_Y X, U_i) + \sum_{a=r+1}^{n} \epsilon_a w_a(Y)\rho_{ia}(X) +$$
$$+ \sum_{k=1}^{r} u_k(Y)\{\bar{g}(A_{N_k}X, N_i) + \beta_k \eta_i(X)\} = 0. \quad (50)$$

Taking $X = \xi_j$ and $Y = U_k$ to (50) and using (11) and (14), we have

$$h_j^\ell(U_k, U_i) = g(A_{N_k}\xi_j, N_i) + \beta_k \delta_{ij}. \quad (51)$$

As h_j^ℓ is symmetric, applying $(24)_1$ {take $X = U_i$} to (51), we obtain

$$h_k^*(U_i, V_j) = h_j^\ell(U_i, U_k) = g(A_{N_k}\xi_j, N_i) + \beta_k \delta_{ij}. \quad (52)$$

On the other hand, applying $(24)_1$ {take $X = U_k$} to (51), we obtain

$$h_i^*(U_k, V_j) = g(A_{N_k}\xi_j, N_i) + \beta_k \delta_{ij}.$$

Exchanging i by k and k by i to this equation and using $(17)_3$, we have

$$h_k^*(U_i, V_j) = \bar{g}(A_{N_i}\xi_j, N_k) + \beta_i \delta_{kj} = -\bar{g}(A_{N_k}\xi_j, N_i) - \beta_k \delta_{ij}. \quad (53)$$

Comparing (52) with (53), we obtain

$$g(A_{N_k}\xi_j, N_i) + \beta_k \delta_{ij} = 0. \quad (54)$$

Replacing X by ξ_j to (50) and using (11), (14), (17))$_6$, $(49)_1$ and (54), we get

$$h_j^\ell(X, U_i) = \tau_{ij}(FX). \quad (55)$$

Taking $X = V_j$ to (50) and using $(17)_6$, (26) and $(49)_2$, we have

$$h_j^\ell(FX, U_i) + \tau_{ij}(X) = \sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k}V_j, N_i). \quad (56)$$

Taking $X = U_i$ to (45) and using (16), (23), $(24)_{1,2}$ and (25), we get

$$\begin{aligned}\sum_{k=1}^{r} u_k(Y)A_{N_k}U_i + \sum_{a=r+1}^{n} w_a(Y)A_{E_a}U_i - A_{N_i}Y -\\ - F(A_{N_i}FY) - \sum_{j=1}^{r}\tau_{ij}(FY)U_j - \sum_{a=r+1}^{n}\rho_{ia}(FY)W_a +\\ + \{\sum_{j=1}^{r}\beta_j u_j(Y) + \sum_{a=r+1}^{n}\gamma_a w_a(Y)\}U_i - \beta_i\{F^2Y + Y\} = 0.\end{aligned} \quad (57)$$

Taking scalar product with V_j to (57) and using (54), we get

$$h_j^{\ell}(X, U_i) = -\tau_{ij}(FX).$$

Comparing this equation with (55), we obtain

$$\tau_{ij}(FX) = 0, \qquad h_j^{\ell}(X, U_i) = 0. \quad (58)$$

Using $(58)_2$, the equation (56) reduced to

$$\tau_{ij}(X) = \sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k}V_j, N_i). \quad (59)$$

Taking the scalar product with U_j to (57) and then, taking $Y = W_a$ and using (15), (16) and $(24)_2$, we have

$$h_i^*(W_a, U_j) = \epsilon_a h_a^s(U_i, U_j) = \epsilon_a h_a^s(U_j, U_i) = h_i^*(U_j, W_a). \quad (60)$$

Taking the scalar product with W_a to (57) and using (23), we have

$$\epsilon_a\rho_{ia}(FY) = -h_i^*(Y, W_a) + \sum_{k=1}^{r} u_k(Y)h_k^*(U_i, W_a) + \sum_{b=r+1}^{n}\epsilon_b w_b(Y)h_b^s(U_i, W_a)$$

by (15) and (16). Taking the scalar product with U_i to (45) such that $X = W_a$ and using $(17)_4$, (23), $(24)_2$ and (60), we get

$$\epsilon_a\rho_{ia}(FY) = h_i^*(Y, W_a) - \sum_{k=1}^{r} u_k(Y)h_k^*(U_i, W_a) - \sum_{b=r+1}^{n}\epsilon_b w_b(Y)h_b^s(U_i, W_a).$$

Comparing the last two equations, we obtain $\rho_{ia}(FY) = 0$. □

5. Indefinite complex space forms.

Definition 4. *An indefinite complex space form $\bar{M}(c)$ is an indefinite Kaehler manifold of constant holomorphic sectional curvature c such that*

$$\widetilde{R}(\overline{X},\overline{Y})\overline{Z} = \frac{c}{4}\{\bar{g}(\overline{Y},\overline{Z})\overline{X} - \bar{g}(\overline{X},\overline{Z})\overline{Y} + \bar{g}(J\overline{Y},\overline{Z})J\overline{X} - \bar{g}(J\overline{X},\overline{Z})J\overline{Y} + 2\bar{g}(\overline{X},J\overline{Y})J\overline{Z}\}, \quad (61)$$

where $\widetilde{R}$ is the curvature tensor of the Levi-Civita connection $\widetilde{\nabla}$ on $\overline{M}$.

Let $\overline{R}$ be the curvature tensor of the semi-symmetric non-metric connection $\overline{\nabla}$ on $\overline{M}$. By directed calculations from (3) and (4), we get

$$\overline{R}(\overline{X},\overline{Y})\overline{Z} = \widetilde{R}(\overline{X},\overline{Y})\overline{Z} + (\overline{\nabla}_{\overline{X}}\theta)(\overline{Z})\overline{Y} - (\overline{\nabla}_{\overline{Y}}\theta)(\overline{Z})\overline{X}. \quad (62)$$

Denote by R and R^* the curvature tensors of the induced linear connections ∇ and ∇^* on M and $S(TM)$ respectively. Using the Gauss-Weingarten formulae, we obtain Gauss equations for M and $S(TM)$ respectively:

$$\begin{aligned}\overline{R}(X,Y)Z &= R(X,Y)Z + \sum_{i=1}^{r}\{h_i^\ell(X,Z)A_{N_i}Y - h_i^\ell(Y,Z)A_{N_i}X\} + \\ &+ \sum_{a=r+1}^{n}\{h_a^s(X,Z)A_{E_a}Y - h_a^s(Y,Z)A_{E_a}X\} + \sum_{i=1}^{r}\Big\{(\nabla_X h_i^\ell)(Y,Z) - \\ &- (\nabla_Y h_i^\ell)(X,Z) + \sum_{j=1}^{r}[\tau_{ji}(X)h_j^\ell(Y,Z) - \tau_{ji}(Y)h_j^\ell(X,Z)] + \\ &+ \sum_{a=r+1}^{n}[\lambda_{ai}(X)h_a^s(Y,Z) - \lambda_{ai}(Y)h_a^s(X,Z)] - \theta(X)h_i^\ell(Y,Z) + \\ &+ \theta(Y)h_i^\ell(X,Z)\Big\}N_i + \sum_{a=r+1}^{n}\Big\{(\nabla_X h_a^s)(Y,Z) - (\nabla_Y h_a^s)(X,Z) + \\ &+ \sum_{i=1}^{r}[\rho_{ia}(X)h_i^\ell(Y,Z) - \rho_{ia}(Y)h_i^\ell(X,Z)] + \sum_{b=r+1}^{n}[\mu_{ba}(X)h_b^s(Y,Z) - \\ &- \mu_{ba}(Y)h_b^s(X,Z)] - \theta(X)h_a^s(Y,Z) + \theta(Y)h_a^s(X,Z)\Big\}E_a, \quad (63)\end{aligned}$$

$$\begin{aligned}R(X,Y)PZ &= R^*(X,Y)PZ + \\ &+ \sum_{i=1}^{r}\{h_i^*(X,PZ)A^*_{\xi_i}Y - h_i^*(Y,PZ)A_{\xi_i}X\} +\end{aligned}$$

$$+\sum_{i=1}^{r}\Big\{(\nabla_X h_i^*)(Y,PZ)-(\nabla_Y h_i^*)(X,PZ)+$$
$$+\sum_{k=1}^{r}[\sigma_{ik}(Y)h_k^*(X,PZ)-\sigma_{ik}(X)h_k^*(Y,PZ)]-$$
$$-\theta(X)h_i^*(Y,PZ)+\theta(Y)h_i^*(X,PZ)\Big\}\xi_i. \quad (64)$$

Comparing the tangential, lightlike transversal and radical components of the two equations (62) and (63) and using (22), we get

$$\begin{aligned}R(X,Y)Z &= \sum_{i=1}^{r}\{h_i^\ell(Y,Z)A_{N_i}X-h_i^\ell(X,Z)A_{N_i}Y\}+\\&+\sum_{a=r+1}^{n}\{h_a^s(Y,Z)A_{E_a}X-h_a^s(X,Z)A_{E_a}Y\}+\\&+(\overline{\nabla}_X\theta)(Z)Y-(\overline{\nabla}_Y\theta)(Z)X+\\&+\frac{c}{4}\{g(Y,Z)X-g(X,Z)Y+\bar{g}(JY,Z)FX-\\&-\bar{g}(JX,Z)FY+2\bar{g}(X,JY)FZ\}, \quad (65)\end{aligned}$$

$$\begin{aligned}&(\nabla_X h_i^\ell)(Y,Z)-(\nabla_Y h_i^\ell)(X,Z)+\sum_{k=1}^{r}\{h_k^\ell(Y,Z)\tau_{ki}(X)-\\&-h_k^\ell(X,Z)\tau_{ki}(Y)\}+\sum_{a=r+1}^{n}\{h_a^s(Y,Z)\lambda_{ai}(X)-h_a^s(X,Z)\lambda_{ai}(Y)\}-\\&-h_i^\ell(Y,Z)\theta(X)+\theta(Y)h_i^\ell(X,Z)\theta(Y)=\\&=\frac{c}{4}\{u_i(X)\bar{g}(JY,Z)-u_i(Y)\bar{g}(JX,Z)+2u_i(Z)\bar{g}(X,JY)\}. \quad (66)\end{aligned}$$

$$\begin{aligned}&(\nabla_X h_i^*)(Y,PZ)-(\nabla_Y h_i^*)(X,PZ)-\\&-\sum_{k=1}^{r}\{h_k^*(Y,PZ)\sigma_{ik}(X)-h_k^*(X,PZ)\sigma_{ik}(Y)\}-\\&-\sum_{k=1}^{r}\{h_k^\ell(Y,PZ)\eta_i(A_{N_k}X)-h_k^\ell(X,PZ)\eta_i(A_{N_k}Y)\}-\\&-\sum_{a=r+1}^{n}\{h_a^s(Y,PZ)\eta_i(A_{E_a}X)-h_a^s(X,PZ)\eta_i(A_{E_a}Y)\}-h_i^*(Y,PZ)\theta(X)+\end{aligned}$$

$$+ h_i^*(X, PZ)\theta(Y) - (\overline{\nabla}_X\theta)(PZ)\eta_i(Y) + (\overline{\nabla}_Y\theta)(PZ)\eta_i(X) =$$
$$= \frac{c}{4}\{g(Y, PZ)\eta_i(X) - g(X, PZ)\eta_i(Y) + v_i(X)\bar{g}(JY, PZ) -$$
$$- v_i(Y)\bar{g}(JX, PZ) + 2v_i(PZ)\bar{g}(X, JY)\}. \quad (67)$$

Theorem 3. *Let M be a generic lightlike submanifold of an indefinite complex space form $\bar{M}(c)$ with a semi-symmetric non-metric connection. If one of the following four statements*

(1) *M is recurrent,*
(2) *M is Lie recurrent,*
(3) *U_i is parallel with respect to the connection ∇, or*
(4) *V_i is parallel with respect to the connection ∇*

is satisfied, then $\bar{M}(c)$ is flat, i. e., $c = 0$.

Proof. (1) By Theorem 1, we get $\rho_{ia} = 0$ and $\theta = 0$ on TM, and we have $(34)_3$, (36) and $(37)_{1,2}$. From $(36)_1$ and $(37)_1$: $\theta(U_i) = 0$, we obtain

$$\eta_i(A_{N_j}X) = -\beta_j\eta_i(X). \quad (68)$$

Applying $\overline{\nabla}_X$ to $\theta(U_i) = 0$ and using (7), $(34)_3$ and $\theta_{|TM} = 0$, we have

$$(\overline{\nabla}_X\theta)(U_i) = -\sum_{k=1}^{r}\beta_k h_k^\ell(X, U_i). \quad (69)$$

Applying ∇_X to $(37)_2$: $h_i^*(Y, U_j) = 0$ and using $(42)_1$, we obtain

$$(\nabla_X h_i^*)(Y, U_j) = 0.$$

Taking $PZ = U_j$ to (67) and using $(34)_3$, $(37)_2$ (68) and (69), we have

$$\frac{c}{4}\{v_j(Y)\eta_i(X) - v_j(X)\eta_i(Y) + v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)\} = 0.$$

Taking $X = \xi_i$ and $Y = V_j$, we have $c = 0$ and $\overline{M}(c)$ is flat.

(2) Taking $X = \xi_j$ to (14) and using $(18)_2$ and h_i^ℓ is symmetric, we get $h_i^\ell(X, \xi_j) = g(A_{\xi_i}^*\xi_j, X)$. From this result and $(17)_1$, we obtain $g(A_{\xi_i}^*\xi_j + A_{\xi_j}^*\xi_i, X) = 0$. As $S(TM)$ is non-degenerate, we get $A_{\xi_i}^*\xi_j = -A_{\xi_j}^*\xi_i$. Thus $A_{\xi_i}^*\xi_j$ is skew-symmetric with respect to i and j.

In the case M is Lie recurrent, taking $Y = U_j$ to (57), we have

$$A_{N_j}U_i + \beta_j U_i = A_{N_i}U_j + \beta_i U_j.$$

Applying F to this equation, we have $F(A_{N_j}U_i) = F(A_{N_i}U_j)$. Thus $F(A_{N_i}U_j)$ is symmetric with respect to i and j. Therefore, we obtain

$$h_i^\ell(\xi_j, F(A_{N_j}U_i)) = g(A^*_{\xi_i}\xi_j, F(A_{N_j}U_i)) = 0. \tag{70}$$

From (17)$_2$, (24)$_4$, (49)$_4$ and the fact that h_a^s is symmetric, we get

$$h_i^\ell(\xi_j, W_a) = \epsilon_a h_a^s(\xi_j, V_i) = \epsilon_a h_a^s(V_i, \xi_j) = -\lambda_{aj}(V_i) = 0. \tag{71}$$

Applying ∇_X to (58)$_2$: $h_i^\ell(Y, U_j) = 0$ and using (25), we have

$$\begin{aligned}(\nabla_X h_i^\ell)(Y, U_j) &= -h_i^\ell(Y, F(A_{N_j}X)) - \\ &- \sum_{a=r+1}^{a} \rho_{ja}(X) h_i^\ell(Y, W_a) - \beta_j h_i^\ell(Y, FX) - \theta(U_j) h_i^\ell(X, Y).\end{aligned}$$

Substituting this into (66) with $Z = U_j$ and using (58)$_2$, we get

$$\begin{aligned}&h_i^\ell(X, F(A_{N_j}Y)) - h_i^\ell(Y, F(A_{N_j}X)) + \\ &+ \sum_{a=r+1}^{n} \{\rho_{ja}(Y) h_i^\ell(X, W_a) - \rho_{ja}(X) h_i^\ell(Y, W_a)\} + \\ &+ \sum_{a=r+1}^{n} \{\lambda_{ai}(X) h_a^s(Y, U_j) - \lambda_{ai}(Y) h_a^s(X, U_j)\} + \\ &+ \beta_j\{h_i^\ell(X, FY) - h_i^\ell(Y, FX)\} = \\ &= \frac{c}{4}\{u_i(Y)\eta_j(X) - u_i(X)\eta_j(Y) + 2\delta_{ij}\bar{g}(X, JY)\}.\end{aligned}$$

Taking $Y = U_i$ and $X = \xi_j$ to this equation and using (49)$_{3,5}$, (58)$_2$, (70) and (71), we have $c = 0$. Consequently, $\bar{M}(c)$ is flat.

(3) As $\nabla_X U_i = 0$, taking the scalar product with U_j to (25), we get

$$\eta_j(A_{N_i}X) = -\beta_i \eta_j(X) + \theta(U_i) v_j(X).$$

Substituting this equation into the left term of (17)$_3$, we have

$$\theta(U_i) v_j(X) + \theta(U_j) v_i(X) = 0.$$

Taking $X = V_j$ to this equation, we obtain

$$\theta(U_i) = 0, \qquad \eta_j(A_{N_i}X) = -\beta_i \eta_j(X). \tag{72}$$

Applying $\bar{\nabla}_X$ to $\theta(U_i) = 0$ and using (7) and $\nabla_X U_i = 0$, we get

$$(\bar{\nabla}_X \theta)(U_i) = -\sum_{k=1}^{r} \beta_k h_k^\ell(X, U_i) - \sum_{a=r+1}^{n} \gamma_a h_a^s(X, U_i). \tag{73}$$

Taking the scalar product with W_a and N_j to (25) by turns and using (16) and $(72)_1$, we have

$$\rho_{ia} = 0, \qquad h_i^*(X, U_j) = 0. \tag{74}$$

From $(17)_4$ and $(74)_1$, we see that

$$\eta_i(A_{E_a} X) = -\gamma_a \eta_i(X). \tag{75}$$

Applying ∇_Y to $(74)_2$ and using the fact that $\nabla_X U_j = 0$, we obtain

$$(\nabla_X h_i^*)(Y, U_j) = 0.$$

Replacing PZ by U_j to (66) and using $(72)_2$, (73), $(74)_2$, (75) and the last equation, we have

$$\frac{c}{4}\{v_j(Y)\eta_i(X) - v_j(X)\eta_i(Y) + v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)\} = 0.$$

Taking $X = \xi_i$ and $Y = V_j$ to this equation, we have $c = 0$.

(4) As $\nabla_X V_i = 0$, taking the scalar product with V_j, W_a and N_j to (26) by turns and using (14) and $(17)_2$, we obtain

$$h_j^\ell(X, \xi_i) = -\theta(V_i) u_j(X), \qquad h_a^s(X, \xi_i) = -\theta(V_i) w_a(X), \tag{76}$$
$$h_i^\ell(X, U_j) = -\theta(V_i)\eta_j(X).$$

By using $(24)_4$, $(76)_3$ and the fact that h_i^ℓ is symmetric, we see that

$$h_a^s(U_j, V_k) = \epsilon_a h_k^\ell(U_j, W_a) = 0. \tag{77}$$

From $(24)_1$ and $(76)_3$, we obtain $h_i^*(Y, V_j) = 0$. Applying ∇_X to this equation and using the fact that $\nabla_X V_j = 0$, we get

$$(\nabla_X h_i^*)(Y, V_j) = 0.$$

Taking $PZ = V_j$ to (66) and using the last two equations, we obtain

$$\sum_{j=1}^{r}\{h_k^\ell(X,V_j)\eta_i(A_{N_k}Y)-h_k^\ell(Y,V_j)\eta_i(A_{N_k}X)\}+ \\ +\sum_{a=r+1}^{n}\{h_a^s(X,V_j)\eta_i(A_{E_a}Y)-h_a^s(Y,V_j)\eta_i(A_{E_a}X)\}- \\ -(\bar{\nabla}_X\theta)(V_j)v_i(Y)+(\bar{\nabla}_Y\theta)(V_j)v_i(X)= \\ =\frac{c}{4}\{u_j(Y)\eta_i(X)-u_j(X)\eta_i(Y)+2\delta_{ij}\bar{g}(X,JY)\}.$$

Taking $X=\xi_i$ and $Y=U_j$ and using (76) and (77), we get $c=0$. □

Theorem 4. *Let M be a generic lightlike submanifold of an indefinite complex space form $\bar{M}(c)$ with a semi-symmetric non-metric connection. If W_a is parallel with respect to ∇ and $\sum_{k=1}^{r}\beta_k h_a^s(W_a,V_k)\neq 0$, then $r=1$ and $c=0$.*

Proof. As $\nabla_X W_a=0$, taking the scalar product with W_b to (27), we get

$$\mu_{ab}(X)=-\theta(W_a)w_b(X).$$

Substituting this equation into the left term of $(17)_5$, we have

$$\epsilon_b\theta(W_a)w_b(X)+\epsilon_a\theta(W_b)w_a(X)=0.$$

Replacing X by W_b to the last equation, we obtain

$$\theta(W_a)=0,\qquad\qquad \mu_{ab}=0. \tag{78}$$

Applying $\bar{\nabla}_X$ to $\theta(W_a)=0$ and using (7) and $\nabla_X W_a=0$, we get

$$(\bar{\nabla}_X\theta)(W_a)=-\sum_{i=1}^{r}\beta_i h_i^\ell(X,W_a)-\sum_{a=r+1}^{n}\gamma_b h_b^s(X,W_a). \tag{79}$$

Taking the scalar product with U_i, V_i and N_i to (27) by turns and using (15), $(17)_4$ and $(78)_1$, we have

$$\eta_i(A_{E_a}X)=-\gamma_a\eta_i(X),\ \text{i.e.,}\ \rho_{ia}=0,\quad \lambda_{ai}=0,\quad h_a^s(X,U_i)=0. \tag{80}$$

As $\lambda_{ai}=0$, from $(17)_2$, we obtain

$$h_a^s(X,\xi_i)=0. \tag{81}$$

From (24)$_2$, (78)$_1$ and (80)$_3$, we obtain $h_i^*(X, W_a) = 0$. Applying ∇_Y to this equation and using the fact that $\nabla_X W_a = 0$, we get

$$(\nabla_X h_i^*)(Y, W_a) = 0.$$

Replacing PZ by W_a to (67) and using (24)$_4$, (79), (80)$_1$ and the last two equations, we have

$$\sum_{k=1}^{r} h_a^s(X, V_k)\{\eta_i(A_{N_k}Y) + \beta_k \eta_i(Y)\} - \sum_{k=1}^{r} h_a^s(Y, V_k)\{\eta_i(A_{N_k}X) + \beta_k \eta_i(X)\} = \\ = \frac{c}{4}\{w_a(Y)\eta_i(X) - w_a(X)\eta_i(Y)\}.$$

Taking $X = \xi_i$ and $Y = W_a$ to this and using (81), we have

$$\sum_{k=1}^{r} h_a^s(W_a, V_k)\{\eta_i(A_{N_k}\xi_i) + \beta_k\} = -\frac{c}{4}. \tag{82}$$

Comparing the co-screen components of (62) and (63), we obtain

$$\begin{aligned} &(\nabla_X h_a^s)(Y, Z) - (\nabla_Y h_a^s)(X, Z) + \\ &\quad + \sum_{i=1}^{r}\{\rho_{ia}(X)h_i^\ell(Y, Z) - \rho_{ia}(Y)h_i^\ell(X, Z)\} + \\ &\quad + \sum_{b=r+1}^{n}\{\mu_{ba}(X)h_b^s(Y, Z) - \mu_{ba}(Y)h_b^s(X, Z)\} - \\ &\quad - \theta(X)h_a^s(Y, Z) + \theta(Y)h_a^s(X, Z) \\ &\quad = \frac{c}{4}\{w_a(X)\bar{g}(JY, Z) - w_a(Y)\bar{g}(JX, Z) + 2w_a(Z)\bar{g}(X, JY)\}. \end{aligned} \tag{83}$$

As $\lambda_{ai} = \mu_{ab} = \theta(W_a) = 0$ and $FW_b = 0$, from (27), we have

$$F(A_{E_a}X) = -\gamma_a FX, \qquad F(A_{E_a}W_b) = 0. \tag{84}$$

Applying ∇_X to $h_a^s(Y, U_i) = 0$ and using (25) and (80)$_3$, we get

$$(\nabla_X h_a^s)(Y, U_i) = -h_a^s(Y, F(A_{N_i}X)) - \beta_i h_a^s(FX, Y) - \theta(U_i)h_a^s(X, Y)$$

due to $\rho_{ia} = 0$. Substituting this into (83) with $Z = U_i$ and using the fact that $\rho_{ia} = \mu_{ab} = 0$, we have

$$h_a^s(X, F(A_{N_i}Y)) \; - \; h_a^s(Y, F(A_{N_i}X)) + \beta_i\{h_a^s(X, FY) - h_a^s(FX, Y)\} =$$

$$= \frac{c}{4}\{w_a(Y)\eta_i(X) - w_a(X)\eta_i(Y)\}.$$

Taking $X = \xi_i$ and $Y = W_a$ to this equation and using (81), we get

$$h_a^s(W_a, F(A_{N_i}\xi_i)) - \beta_i h_a^s(V_i, W_a) = -\frac{c}{4}.$$

From $(5)_2$, (15), $(17)_{3,4}$, (22), $(84)_2$ and the fact: $\rho_{ia} = 0$, we have

$$h_a^s(W_a, F(A_{N_i}\xi_i)) = -\epsilon_a g(A_{N_i}\xi_i, F(A_{E_a}W_a)) -$$
$$- \sum_{k=1}^{r} h_a^s(W_a, V_k)\eta_k(A_{N_i}\xi_i) = - \sum_{k=1}^{r} h_a^s(W_a, V_k)\eta_k(A_{N_i}\xi_i) =$$
$$= \sum_{k=1}^{r} h_a^s(W_a, V_k)\{\eta_i(A_{N_k}\xi_i) + 2\beta_k\}.$$

From the last two equations, we see that

$$\sum_{k=1}^{r} h_a^s(W_a, V_k)\{\eta_i(A_{N_k}\xi_i) + 2\beta_k\} - \beta_i h_a^s(W_a, V_i) = -\frac{c}{4}.$$

Comparing this equation with (82), we obtain

$$\sum_{k=1}^{r} \beta_k h_a^s(W_a, V_k) = \beta_i h_a^s(W_a, V_i), \quad \forall\, i.$$

It follow that

$$(r-1)\sum_{k=1}^{r} \beta_k h_a^s(W_a, V_k) = 0.$$

Assume that $\sum_{k=1}^{r} \beta_k h_a^s(W_a, V_k) \neq 0$. Then $r = 1$ and $i = j = k = 1$. Thus, from $(17)_3$, we see that

$$\eta_i(A_{N_1}X) = -\beta_1\eta_1(X).$$

From this result and (82), we obtain $c = 0$. □

Acknowledgment. In this paper, we studied the geometry of generic lightlike submanifolds of an indefinite Kaehler manifold with a semi-symmetric non-metric connection. But the geometry of generic lightlike submanifolds and several CR-type lightlike submanifolds of an indefinite Kaehler manifold with a quarter-symmetric non-metric connection are still open problems. We hope that the publication of this paper will help in solving the above more general cases.

References

[1] Ageshe N. S., Chafle M. R. *A semi-symmetric non-metric connection on a Riemannian manifold.* Indian J. Pure Appl. Math., 1992, vol. 23, no. 6, pp. 399 – 409.

[2] Duggal K. L., Bejancu A. *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications.* Kluwer Acad. Publishers, Dordrecht, 1996.

[3] Duggal K. L., Jin D. H. *Generic lightlike submanifolds of an indefinite Sasakian manifold.* Int. Elec. J. Geo., 2012, vol. 5, no. 1, pp. 108 –119.

[4] Jin D. H. *Indefinite generalized Sasakian space form admitting a generic lightlike submanifold.* Bull. Korean Math. Soc., 2014, vol. 51, no. 6, pp. 1711–1726. DOI: https://doi.org/10.4134/BKMS.2014.51.6.1711.

[5] Jin D. H. *Generic lightlike submanifolds of an indefinite trans-Sasakian manifold of a quasi-constant curvature.* Appl. Math. Sci., 2015, vol. 9, no. 60, pp. 2985 – 2997. DOI: http://dx.doi.org/10.12988/ams.2015.52165.

[6] Jin D. H. *Special lightlike hypersurfaces of indefinite Kaehler manifolds.* Filomat, 2016, vol. 30, no. 7, pp. 1919 –1930.

[7] Jin D. H. *Lightlike hypersurface of an indefinite Kaehler manifold with a semi-symmetric non-metric connection.* J. Korean Math. Soc., 2017, vol. 54, no. 1, pp. 101–115.

[8] Jin D. H., Lee J. W. *Generic lightlike submanifolds of an indefinite cosymplectic manifold.* Math. Probl. in Engin., 2011, Art ID 610986, pp. 1–16.

[9] Kupeli D. N. *Singular Semi-Riemannian Geometry*, Kluwer Academic, 1996, 366 p.

[10] De Rham G. *Sur la réductibilité d'un espace de Riemannian.* Comm. Math. Helv., 1952, vol. 26, pp. 328 – 344.

Received April 12, 2018.
In revised form, July 27, 2018.
Accepted September 27, 2018.
Published online October 11, 2018.

Department of Mathematics
Dongguk University
Gyeongju 780-714, Republic of Korea
E-mail: jindh@dongguk.ac.kr