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THE INTERPOLATION PROBLEM IN THE SPACES OF ANALYTICAL FUNCTIONS OF FINITE ORDER IN THE HALF-PLANE

Abstract. The aim of this paper is to study the interpolation problem in the spaces of analytical functions of finite order $\rho > 1$ in the half-plane. The necessary and sufficient conditions for its solvability in terms of the canonical Nevanlinna product of nodes of interpolation are obtained. The solution of the interpolation problem is constructed in the form of the Jones interpolation series, which is a generalization of the Lagrange interpolation series.

Key words: *half-plane, function of finite order, free interpolation, Nevanlinna product, interpolation series*

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1. Introduction and main result. Definitions and Notation. In 1948, A. F. Leont'ev [4] first considered the interpolation problem in the space of entire functions $[\rho,\infty]$ of finite order $\rho > 0$, which obtained subsequently the title of the *free interpolation problem.* In this paper we consider the problem of simple free interpolation in the spaces of analytical functions of finite order $\rho > 1$ in the half-plane.

Let $\mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}$. Denote by $[\rho,\infty]^+$ the space of analytical functions of finite order $\rho > 1$ in $\mathbb{C}_+$ [2, Chapter I, §1], i.e., $f \in [\rho,\infty]^+$ if

$$\limsup_{r\to\infty} \sup_{0<\theta<\pi} \frac{\ln^+ \ln^+ |f(re^{i\theta})|}{\ln r} \leqslant \rho\,,$$

where $\ln^+ a = \begin{cases} \ln^+ a = \ln a\,, & a > 1, \\ \ln^+ a = 0\,, & a \leqslant 1\,. \end{cases}$

Let $A = \{a_n\}_{n=1}^{\infty} \subset \mathbb{C}_+$ be a sequence of distinct complex numbers such that all limit points of A are on the real axis and infinity.

Definition 1. *The sequence A is called an interpolation sequence in the space $[\rho,\infty]^+$ if for any sequence of complex numbers $\{b_n\}_{n=1}^{\infty}$ satisfying the conditions:*

$$\sup_{n\in\mathbb{N}} \frac{\ln^+ \ln^+ |b_n|}{\ln |a_n| + 2} < \infty \,, \tag{1}$$

$$\limsup_{|a_n|\to\infty} \frac{\ln^+ \ln^+ |b_n|}{\ln |a_n|} \leqslant \rho \,, \tag{2}$$

there exists a function $F \in [\rho,\infty]^+$ solving the interpolation problem

$$F(a_n) = b_n, \quad n = 1,2,\dots \,. \tag{3}$$

The condition (1) and (2) are necessary restrictions on the sequence $\{b_n\}_{n=1}^{\infty}$. These restrictions are related to the fact that a function $F(z)$, solving the interpolation problem (3), must belong to the space $[\rho,\infty]^+$.

In 1975, B. Ya. Levin and N. Uen [6] considered the problem of simple interpolation in the space $[\rho,\infty]^+$ for $\rho > 1$. Necessary conditions and sufficient conditions in terms of the canonical Nevanlinna product of interpolation nodes were obtained for the solvability of the corresponding interpolation problems. But between the two types of these conditions there was a gap that did not allow the interpolation nodes to "accumulate" at points of the real axis. The aim of this paper is to study the interpolation problem in the space $[\rho,\infty]^+$ for $\rho > 1$. We find necessary and sufficient conditions for the interpolation problem to be solvable. This conditions are formulated in terms of canonical product determined by the interpolation nodes. According to its content, the problem is a free interpolation problem, because only natural restrictions are imposed on the values of the function at the interpolation nodes, due to the requirement for the solution to be in the given space.

Denote by $B_q(u,v)$ the Nevanlinna primary factor

$$B_q(u,v) = \begin{cases} \dfrac{\bar{v}(u-v)}{v(u-\bar{v})}, & q = 0, \\ B_0(u,v)\exp\left(\displaystyle\sum_{i=1}^{q} \frac{u^i}{i}\left(\frac{1}{v^i} - \frac{1}{\bar{v}^i}\right)\right), & q \in \mathbb{N}\,. \end{cases}$$

Let $A = \{a_n = r_n e^{i\theta_n}\}_{n=1}^{\infty} \subset \mathbb{C}_+$, $r_n > \delta_0 > 0$, be the sequence of distinct complex numbers such that all limit points of A are on the real axis and infinity, and for any $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \frac{\sin\theta_n}{r_n^{\rho+\varepsilon}} < \infty, \quad \rho > 1\,, \tag{4}$$

then the function

$$E(z) = E_A(z) =: \prod_{|a_n|<1} \left(\frac{z-a_n}{z-\bar{a}_n}\right) \prod_{|a_n|\geqslant 1} B_q(z,a_n), \quad q = [\rho]$$

belongs to the space $[\rho,\infty]^+$. We denote by $[\cdot]$ the integer part of a number. The function $E(z)$ is called *the canonical function of the sequence A*.

Our main result is the theorem stated below.

Theorem 1. *The following two statements are equivalent.*

1) *The sequence A is an interpolation sequence in the space $[\rho,\infty]^+$.*

2) *Condition (4) is true and the canonical function $E(z)$ of the sequence A satisfies the conditions*:

$$\sup_{n\in\mathbb{N}} \frac{1}{\ln|a_n|+2} \ln^+\ln^+ \frac{1}{|E'(a_n)|\operatorname{Im} a_n} < \infty, \tag{5}$$

$$\limsup_{|a_n|\to\infty} \frac{1}{\ln|a_n|} \ln^+\ln^+ \frac{1}{|E'(a_n)|\operatorname{Im} a_n} \leqslant \rho\,. \tag{6}$$

We now state the result of B. Ya. Levin and N. Uen from the paper [6]: In order for the sequence A to be an interpolation sequence in space $[\rho,\infty]^+$, it is necessary that condition (4) is true and the canonical function $E(z)$ of the sequence A satisfies condition (6); and sufficient that condition (4) is true and the canonical function $E(z)$ of the sequence A satisfies the condition

$$\limsup_{|a_n|\to\infty} \frac{1}{\ln|a_n|} \ln^+\ln^+ \frac{1}{|E'(a_n)|\operatorname{Im}^2 a_n} \leqslant \rho\,. \tag{7}$$

Note that condition (7) implies the following restriction on the interpolation nodes: for all $\varepsilon > 0$ the inequality $\operatorname{Im} a_n \geqslant \exp(-|a_n|^{\rho+\varepsilon})$ is true. This restriction allowed the authors to construct an interpolating function in the form of the Lagrange interpolation series. We also note that the restriction (7) does not allow the interpolation nodes to have limit points on the real axis, and therefore condition (1) on the values of the interpolating function is unnecessary. In this formulation, the problem can not be regarded as a problem of free interpolation in space $[\rho,\infty]^+$. These shortcomings are eliminated in this paper.

2. Preliminaries. Let

$$A_n(z) = \prod_{0<|a_n-a_k|\leqslant r_n/2} \frac{\bar{a}_k(z-a_k)}{a_k(z-\bar{a}_k)},\ E_n(z) = E(z)\frac{a_n(z-\bar{a}_n)}{\bar{a}_n(z-a_n)}A_n^{-1}(z)\,.$$

We need the following statements.

Lemma 1. *If the sequence A satisfies (4) then*

$$\begin{gathered}\sup_{z\in\mathbb{C}_+}\frac{1}{\ln|z|+2}\sup_{n\in\mathbb{N}}\ln^+\ln^+|E_n(z)|<\infty,\\ \limsup_{|z|\to\infty}\frac{1}{\ln|z|}\sup_{n\in\mathbb{N}}\ln^+\ln^+|E_n(z)|\leqslant\rho\,.\end{gathered}\tag{8}$$

The lemma is proved by standard methods for estimating canonical products (see e.g. [2], [5]), and we omit the proof.

Lemma 2. *If the sequence A satisfie (4), (5) and (6), then*

$$\sup_{n\in\mathbb{N}}\sum_{k=1}^{\infty}\frac{\operatorname{Im}a_k\operatorname{Im}a_n}{|a_n-\bar{a}_k|^2(1+r_k^2)^{\frac{\rho+1}{2}}}<\infty\,.$$

Proof. We get from (5), (6) and (8) that

$$\begin{gathered}\sup_{n\in\mathbb{N}}\frac{1}{\ln|r_n|+2}\ln^+\ln^+\frac{1}{|A_n(a_n)|}<\infty,\\ \limsup_{n\in\mathbb{N}}\frac{1}{\ln|r_n|}\ln^+\ln^+\frac{1}{|A_n(a_n)|}\leqslant\rho\,.\end{gathered}$$

From these inequalities, the identity

$$\left|\frac{a-b}{\bar{a}-b}\right|^2 = 1-\frac{4\operatorname{Im}a\operatorname{Im}b}{|\bar{a}-b|^2}\,,$$

and the elementary inequality $x \leqslant -\ln(1-x)$ $(0 \leqslant x < 1)$ we get the following relations:

$$
\begin{gathered}
\sup_{n\in\mathbb{N}} \frac{1}{\ln|r_n|+2} \ln \sum_{0<|a_n-a_k|\leqslant r_n/2} \frac{\operatorname{Im} a_k \operatorname{Im} a_n}{|a_n-\bar{a}_k|^2} < \infty, \\
\limsup_{r_n\to\infty} \frac{1}{\ln|r_n|} \ln \sum_{0<|a_n-a_k|\leqslant r_n/2} \frac{\operatorname{Im} a_k \operatorname{Im} a_n}{|a_n-\bar{a}_k|^2} \leqslant \rho\,.
\end{gathered} \tag{9}
$$

The condition (4) implies that for any $\varepsilon > 0$ the series

$$
\sum_{n=1}^{\infty} \frac{\operatorname{Im} a_n}{r_n^{\rho+1+\varepsilon}} \tag{10}
$$

converges. From (9) and (10) we obtain the statement of the lemma. □

Definition 2. *An absolutely continuous function $\rho(r)$ on the real axis $[0,+\infty)$ that satisfies the conditions*

$$
\lim_{r\to+\infty} \rho(r) = \rho, \qquad \lim_{r\to+\infty} \rho'(r) r \ln r = 0\,,
$$

is called a proximate order.

We need the following lemma [5, Chapter I, Theorem 16].

Lemma 3. *If $f(r)$ is any continuous function that is positive for $r > 0$ and satisfies the condition*

$$
\limsup_{r\to+\infty} \frac{\ln f(r)}{\ln r} = \rho < \infty\,,
$$

then the proximate order $\rho(r)$ can be chosen so that for all positive values of r

$$
f(r) \leqslant r^{\rho(r)}
$$

and for some sequence of values r_n $(n = 1, 2, \ldots)$ tending to infinity

$$
f(r_n) = r_n^{\rho(r_n)}\,.
$$

3. The proof of implication 1) $\Rightarrow$ 2) **of the Theorem 1.** We prove the implication 1) $\Rightarrow$ 2) by contradiction. Let the sequence A be an interpolation sequence in the space $[\rho,\infty]^+$. There exists a function

$F \in [\rho,\infty]^+$ such that $F(a_1) = 1$ and $F(a_n) = 0$, $n \geqslant 2$. Therefore the sequence $A \backslash \{a_1\}$ belongs to the set of zeros of F and satisfies condition (4) [2, Chapter I, §3].

We now prove (5). Assume the contrary, that there exists a sequence $\{c_n\} \subset A$ such that

$$\lim_{n\to\infty} \frac{1}{\ln|c_n|+2} \ln^+ \ln^+ \frac{1}{|E'(c_n)|\operatorname{Im} c_n} = \infty\,. \tag{11}$$

Using the Carleson theorem [1] and passing if necessary to a subsequence, we can assume that Blaschke product $H(z)$ of the sequence $\{c_n\}$ satisfies the condition

$$\inf_n |H'(c_n)| \operatorname{Im} c_n \geqslant \delta > 0\,. \tag{12}$$

Suppose further that $F \in [\rho,\infty]^+$ is such that $F(c_n) = 1$ and $F(a_n) = 0$, $a_n \neq c_n$. Then the function $f(z) = \dfrac{F(z)H(z)}{E(z)}$ belongs to $[\rho,\infty]^+$. We have $\dfrac{1}{E'(c_n)} = \dfrac{f(c_n)}{H'(c_n)}$. The last equality with (12) contradicts (11) and $f(z) \in [\rho,\infty]^+$.

The inequality (5) is proved. The inequality (6) is proved similarly.

4. The proof of implication 2) $\Rightarrow$ 1) of the Theorem 1. Since the series (4) converges (after a renumbering of the points a_n if it is necessary), we can assume that

$$\frac{\operatorname{Im} a_{n+1}}{1+r_{n+1}^2} \leqslant \frac{\operatorname{Im} a_n}{1+r_n^2}\,, \quad n \in \mathbb{N}\,. \tag{13}$$

We define

$$\alpha_n(z) = \sum_{k=n}^{\infty} \frac{1+\bar{a}_k z}{i(\bar{a}_k - z)} \frac{\operatorname{Im} a_k}{(1+r_k^2)^{\frac{[\rho]+3}{2}}}\,, \quad n = 1,2,\ldots\,.$$

The series defining the functions $\alpha_n(z)$ converges uniformly in each domain $D_{r,\delta} = \{z : |z| \leqslant r, \operatorname{Im} z \geqslant \delta > 0\}$, because

$$\left|\frac{1+\bar{a}_k z}{i(\bar{a}_k - z)}\right| \frac{\operatorname{Im} a_k}{(1+r_k^2)^{\frac{[\rho]+3}{2}}} \leqslant \frac{(1+|z|)(1+r_k)}{\delta} \frac{\operatorname{Im} a_k}{(1+r_k^2)^{\frac{[\rho]+3}{2}}}$$

if $z \in D_{r,\delta}$, and it follows from (4) that the series

$$\sum_{k=1}^{\infty} \frac{\operatorname{Im} a_k}{(1+r_k^2)^{\frac{\rho+1}{2}+\varepsilon}}$$

converges for any $\varepsilon > 0$.

Let us estimate $\operatorname{Re} \alpha_n(z)$. We have

$$\operatorname{Re} \alpha_n(z) = \sum_{k=n}^{\infty} \frac{\operatorname{Im} a_k + \operatorname{Im} z + r_k^2 \operatorname{Im} z + |z|^2 \operatorname{Im} a_k}{|\bar{a}_k - z|^2} \frac{\operatorname{Im} a_k}{(1+r_k^2)^{\frac{[\rho]+3}{2}}} .$$

Using Lemma 2 and inequality (13), we obtain

$$\operatorname{Re} \alpha_n(a_n) = \sum_{k=n}^{\infty} \frac{\operatorname{Im} a_k(\operatorname{Im} a_k(1+r_n^2) + \operatorname{Im} a_n(1+r_k^2))}{|\bar{a}_k - a_n|^2 (1+r_k^2)^{\frac{[\rho]+3}{2}}} =$$

$$= \sum_{k=n}^{\infty} \left(\frac{\operatorname{Im} a_k}{1+r_k^2} + \frac{\operatorname{Im} a_n}{1+r_n^2} \right) \frac{\operatorname{Im} a_k (1+r_k^2)(1+r_n^2)}{|\bar{a}_k - a_n|^2 (1+r_k^2)^{\frac{[\rho]+3}{2}}} \leqslant$$

$$\leqslant 2 \frac{1+r_n^2}{1+r_n^2} \sum_{k=n}^{\infty} \frac{\operatorname{Im} a_n}{|\bar{a}_k - a_n|^2} \frac{\operatorname{Im} a_k}{(1+r_k^2)^{\frac{[\rho]+1}{2}}} \leqslant M < \infty , \tag{14}$$

for some $M > 0$, and

$$\operatorname{Re} \alpha_n(z) \geqslant \sum_{k=n}^{\infty} \frac{(\operatorname{Im} a_k)^2}{(1+r_k^2)^{\frac{[\rho]+3}{2}} |\bar{a}_k - z|^2} . \tag{15}$$

We put further

$$\varphi_n(z) = \left(\frac{1+z\bar{a}_n}{1+r_n^2} \right)^{[\rho]+3} \left(\frac{z}{a_n} \right)^{S_n} \left(\frac{2 \operatorname{Im} a_n}{z - \bar{a}_n} \right)^2 \exp(\alpha_n(a_n) - \alpha_n(z)) ,$$

where the integers S_n will be chosen below. We note that

$$\varphi_n(a_n) = 1, \quad n = 1,2,\ldots .$$

In addition, using the elementary inequality $1 + x \leqslant \sqrt{2(1+x^2)}$, we obtain for $|z| \geqslant 1$:

$$\left| \frac{1+z\bar{a}_n}{1+r_n^2} \right| \leqslant \frac{|z|(1+r_n)}{1+r_n^2} \leqslant \frac{\sqrt{2}|z|}{\sqrt{1+r_n^2}} .$$

Further

$$|\varphi_n(z)| \leqslant \left(\frac{\sqrt{2}|z|}{\sqrt{1+r_n^2}}\right)^{[\rho]+3} \left|\frac{z}{r_n}\right|^{S_n} \frac{4\operatorname{Im}^2 a_n}{|z-\bar{a}_n|^2}\times$$

$$\times \exp\{\operatorname{Re}[\alpha_n(a_n)-\alpha_n(z)]\}, \quad n=1,2,\dots. \tag{16}$$

Let

$$P_n(z) = \frac{b_n\varphi_n(z)}{(z-a_n)E'(a_n)}.$$

The formal series

$$F(z) = E(z)\sum_{n=1}^{\infty} P_n(z) \tag{17}$$

solves the interpolation problem (3).

We now show that the function $F(z)$ belongs to $[\rho,\infty]^+$ for the suitable choice of the sequence $\{S_n\}$ of positive integers. Let $C(z,r)$ be the disk of radius r around a point z. It follows from (14), (15) and (16) for all $z\in\mathbb{C}_+$, $z\notin C(a_n,\operatorname{Im} a_n)$, that

$$|P_n(z)| \leqslant M_1\left|\frac{b_n}{\operatorname{Im} a_n E'(a_n)}\right| \left(\frac{|z|}{\sqrt{1+r_n^2}}\right)^{[\rho]+3} \left|\frac{z}{r_n}\right|^{S_n} \frac{4\operatorname{Im}^2 a_n}{|z-\bar{a}_n|^2}\times$$

$$\times \exp\left(-\sum_{k=n}^{\infty}\frac{(\operatorname{Im} a_k)^2}{(1+r_k^2)^{\frac{[\rho]+3}{2}}|\bar{a}_k-z|^2}\right), \quad n=1,2,\dots, \tag{18}$$

where $M_1>0$ is some constant.

Let

$$\lambda_n(z) = \sum_{k=n}^{\infty}\frac{(\operatorname{Im} a_k)^2}{(1+r_k^2)^{\frac{[\rho]+3}{2}}|\bar{a}_k-z|^2},$$

so that

$$\lambda_n(z)-\lambda_{n+1}(z) = \frac{(\operatorname{Im} a_n)^2}{(1+r_n^2)^{\frac{[\rho]+3}{2}}|\bar{a}_n-z|^2}, \quad n\in\mathbb{N}.$$

It is clear that $\lambda_n(z)\downarrow 0$ as $n\to\infty$, $z\in\mathbb{C}_+$.

Returning to (18), we get for $z \in \mathbb{C}_+$, $z \notin C(a_n, \operatorname{Im} a_n)$

$$|P_n(z)| \leqslant M_1 |z|^{[\rho]+3} \left| \frac{b_n}{\operatorname{Im} a_n E'(a_n)} \right| \left| \frac{z}{r_n} \right|^{S_n} \times$$
$$\times (\lambda_n(z) - \lambda_{n+1}(z)) \exp(-\lambda_n(z)), \quad n = 1,2,\dots .$$

Using the elementary inequality $t \leqslant e^t - 1$, $t \geqslant 0$, for $t = \lambda_n(z) - \lambda_{n+1}(z)$ gives us

$$|P_n(z)| \leqslant M_1 |z|^{[\rho]+3} \left| \frac{b_n}{\operatorname{Im} a_n E'(a_n)} \right| \left| \frac{z}{r_n} \right|^{S_n} \times$$
$$\times (\exp(-\lambda_{n+1}(z)) - \exp(-\lambda_n(z))), \quad n = 1,2,\dots . \tag{19}$$

We now choose a sequence of numbers S_n such that the function $F(z)$ defined by series (17) belongs to the space $[\rho,\infty]^+$.

From the conditions (1), (2), the inequalities (5), (6) and Lemma 3, it follows that there exist $M_2 > 0$ and a proximate order $\rho(r)$, $\lim\limits_{r\to\infty} \rho(r) = \rho$, such that

$$\left| \frac{b_n}{\operatorname{Im} a_n E'(a_n)} \right| \leqslant M_2 \exp(r_n^{\rho(r_n)}), \quad n = 1, 2, \dots .$$

From this and (19), we obtain

$$|P_n(z)| \leqslant M_2 \exp(r_n^{\rho(r_n)}) |z|^{[\rho]+3} \left| \frac{z}{r_n} \right|^{S_n} \times$$
$$\times (\exp(-\lambda_{n+1}(z)) - \exp(-\lambda_n(z))), \quad n = 1,2,\dots . \tag{20}$$

Set $S_n = [r_n^{\rho(r_n)}] + 1$, $n = 1, 2, \dots$. Then, by (20),

$$|P_n(z)| \leqslant M_2 |z|^{[\rho]+4} \left| \frac{z}{r_n} \right|^{[r_n^{\rho(r_n)}]} \times$$
$$\times (\exp(-\lambda_{n+1}(z)) - \exp(-\lambda_n(z))), \quad n = 1, 2, \dots . \tag{21}$$

We represent the sequence A as a sum of two sequences $A = A_1 \cup A_2$: if $a_n \in A_1$ then $|z| \geqslant r_n$, and if $a_n \in A_2$ then $|z| < r_n$.

By (21), we have for the finite subset $\tilde{A}_2 \subset A_2$ that

$$\sum_{a_n \in \tilde{A}_2} |P_n(z)| \leqslant M_2 |z|^{[\rho]+4} \exp(-\lambda_{\tilde{n}_2}(z)) , \tag{22}$$

where $\tilde{n}_2$ is the largest member number from $\tilde{A}_2$.

For the finite subset $\tilde{A}_1 \subset A_1$, we have in the same way

$$\sum_{a_n \in \tilde{A}_1} |P_n(z)| \leqslant M_2 |z|^{[\rho]+4} \sum_{a_n \in \tilde{A}_1} |z|^{[r_n^{\rho(r_n)}]} (\exp(-\lambda_{n+1}(z)) - \exp(-\lambda_n(z))) \leqslant$$

$$\leqslant M_2 |z|^{[\rho]+4} |z|^{|z|^{\rho(|z|)}} \exp(-\lambda_{\tilde{n}_1}(z)), \qquad (23)$$

where $\tilde{n}_1$ is the largest member number from $\tilde{A}_1$.

From (22) and (23), we have for the finite subset $\tilde{A} = \tilde{A}_1 \cup \tilde{A}_2 \subset A$

$$\sum_{a_n \in \tilde{A}} |E(z) P_n(z)| \leqslant$$

$$\leqslant M_2 |z|^{[\rho]+4} |E(z)| (|z|^{|z|^{\rho(|z|)}} \exp(-\lambda_{\tilde{n}_1}(z)) + \exp(-\lambda_{\tilde{n}_2}(z))), \qquad (24)$$

if $z \in \mathbb{C}_+$, $z \notin \bigcup\limits_{a_n \in A} C(a_n, \operatorname{Im} a_n)$.

Applying the maximum modulus principle to the analytic function

$$\sum_{a_n \in \tilde{A}} |E(z) P_n(z)|$$

inside the disk $C(a_n, \operatorname{Im} a_n)$, we obtain that the estimate (24) is true for all $z \in \mathbb{C}_+$. This implies the convergence of the series (17) on compact sets in $\mathbb{C}_+$ and its belonging to the space $[\rho, \infty]^+$.

The Theorem 1 is proved.

Remark 1. *In this paper, we consider the interpolation problem in the space* $[\rho, \infty]^+$, $\rho > 1$. *There are various definitions of the order of functions analytic in the half-plane [2, 3, 8, 9]. These definitions coincide for* $\rho > 1$ *and differ for* $0 \leqslant \rho \leqslant 1$. *In our opinion, each case requires an independent study.*

Remark 2. *In* 1994, *K. G. Malytin [7] considered the problem of multiple interpolation in the space* $[\rho(r), \infty)^+$ *of functions of at most normal type for the proximate order* $\rho(r)$, $\rho > 1$, *in the upper half-plane* $\mathbb{C}^+$.

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