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REDUCED p-MODULUS, p-HARMONIC RADIUS AND p-HARMONIC GREEN'S MAPPINGS

Abstract. We consider the definitions and properties of the metric characteristics of the spatial domains previously introduced by the author, and their connection with the class of mappings, the particular case of which are the harmonic Green's mappings introduced by A. I. Janushauskas. In determining these mappings, the role of the harmonic Green's function is played by the p-harmonic Green's function of the n-dimensional region ($1 < p < \infty$), the existence and properties of which are established by S. Kichenassamy and L. Veron. The properties of p-harmonic Green mappings established in the general case are analogous to the properties of harmonic Green's mappings ($p = 2$, $n = 3$). In particular, it is proved that the p-harmonic radius of the spatial domain has a geometric meaning analogous to the conformal radius of a plane domain.

Key words: *reduced p-modulus, p-harmonic inner radius, p-harmonic Green function, p-harmonic Green's mapping*

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1. Introduction. By definition, the conformal radius of a plane simply connected domain with respect to a fixed interior point is the radius of the disk onto which this domain can be conformally mapped, so that the indicated point passes to the origin, and its derivative at this point is equal to one. It is known that the conformal radius coincides with the inner radius of the region with respect to this point, determined by the (harmonic) Green's function [6]. The notion of p-harmonic inner radius of a spatial domain, introduced by the author (see [10]), is a natural generalization of the inner (conformal) radius of a plane domain. This concept has been applied in a number of works on the potential theory (see [4, 8, 18]). We show that the p-harmonic inner radius of a spatial domain homeomorphic to a ball has a geometric meaning analogous to the conformal radius. The role of the conformal mapping is played by the p-harmonic Green's mapping. This class of maps is defined by analogy with harmonic Green's mappings [19] and has a number of similar properties. The connection between the reduced p-module of the domain with respect to the interior point and the inner p-harmonic radius ([10]) makes it possible to extend geometric estimates and properties established by using the moduli method for other classes of maps to the case of p-harmonic Green's mappings (see [10, 12]).

2. The reduced p-module. Let $\mathbb{E}^n$ be the n-dimensional Euclidean space and $\overline{\mathbb{E}^n} = \mathbb{E}^n \cup \infty$ be its one-point compactification. Denote by $x = (x_1, x_2, \ldots, x_n)$ a vector in $\mathbb{E}^n$, $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$ is the length of x. $B^n(x_0, t)$ is an open ball centered at a point $x_0 \in \mathbb{E}^n$ with radius t; $B^n(\infty, t) = \{x \in \overline{\mathbb{E}^n} : |x| > t\}$; $S^{n-1}(x, t) = \partial B^n(x, t), x \in \overline{\mathbb{E}^n}$; ω_n is the volume of an n-dimensional ball of unit radius, $n\omega_n$ is the area of its surface.

The concept of p-capacity and its generalizations in different versions is encountered in the works of many authors (see, for example, [2,5,7,13]). We consider a condenser, which is a ring domain $D \in \mathbb{E}^n$, the complement of which consists of two connected components C_0 and C_1 (condenser plates).

For $1 < p < \infty$, we define the p-capacity of the condenser D by the formula

$$\operatorname{Cap}_p D = \inf \int\limits_D |\nabla u|^p \, d\omega, \tag{1}$$

where inf is taken over the class of continuous functions in $\overline{D}$ that are continuously differentiable in D and take values 0 on C_0 and 1 on C_1. It is known that if C_0 and C_1 are nondegenerate, then there is a unique potential function $u_0(x)$, which is the extremal for p-capacity of the condenser D and is p-harmonic, that is, it satisfies the p-Laplace equation (in the generalized sense).

Lemma 1. *For almost all $t \in (0, 1)$ we have:*

$$\operatorname{Cap}_p D = \int\limits_{S_t(u_0)} |\nabla u_0|^{p-1} \, dS, \tag{2}$$

where $S_t(u_0) = \{x \in D : u_0(x) = t\}$ is a level surface of the potential function and dS is the surface area element.

Proof. Note that if $\varphi(x)$ is a twice continuously differentiable function defined in a domain G with a piecewise smooth boundary, and $\psi(x)$ is a continuously differentiable function in G, then, integrating by parts, we obtain:

$$\int_G \psi \operatorname{div}\left(|\nabla\varphi|^{p-2}\nabla\varphi\right) d\omega =$$
$$= \int_{\partial G} \psi\, |\nabla\varphi|^{p-2} \frac{\partial\varphi}{\partial n} dS - \int_G |\nabla\varphi|^{p-2} \sum_{k=1}^{n} \frac{\partial\varphi}{\partial x_k}\frac{\partial\psi}{\partial x_k} d\omega, \quad (3)$$

where $\overrightarrow{n}$ is the vector of the external normal to ∂G. Let $D_{a,b}$ be a domain bounded by level surfaces $S_a(u_0)$ and $S_b(u_0)$ $(0 < a < b < 1)$. Due to the fact that $u_0(x)$ is a monotone function (see [15]), $D_{a,b}$ is a ring domain. Applying formula (3) in this domain in the case $\psi = 1$ and $\varphi = u_0$, we obtain

$$\int_{\partial D_{a,b}} |\nabla u_0|^{p-2} \frac{\partial u_0}{\partial n} dS = \int_{S_a(u_0)} |\nabla u_0|^{p-2} \frac{\partial u_0}{\partial n} dS + \int_{S_b(u_0)} |\nabla u_0|^{p-2} \frac{\partial u_0}{\partial n} dS = 0.$$

It follows that

$$\int_{S_b(u_0)} |\nabla u_0|^{p-2} \frac{\partial u_0}{\partial n} dS = -\int_{S_a(u_0)} |\nabla u_0|^{p-2} \frac{\partial u_0}{\partial n} dS = \int_{S_a(u_0)} |\nabla u_0|^{p-1} dS = I.$$

The value of I, therefore, does not depend on the choice of the level surface of the potential function, except for the case when the gradient $u_0(x)$ vanishes on this surface. Applying formula (3) again in the domain $D_{a,b}$ when $\psi = \varphi = u_0$, we find

$$\int_{D_{a,b}} |\nabla u_0|^{p} d\omega = a \int_{S_a(u_0)} |\nabla u_0|^{p-2} \frac{\partial u_0}{\partial n} dS + b \int_{S_b(u_0)} |\nabla u_0|^{p-2} \frac{\partial u_0}{\partial n} dS = (b-a)I.$$

Passing to the limit as $a \to 0$ and $b \to 1$, we obtain the relation (2). □

A convenient metric characteristic of a condenser is the quantity

$$\operatorname{mod}_p D = \left(\frac{n\omega_n}{\operatorname{Cap}_p D}\right)^{\frac{1}{p-1}}, \quad (4)$$

which is called the p-module of a condenser D.

Let $\mu_p(x, x_0) = \mu_p(|x - x_0|) = \mu_p(t)$ be a fundamental solution of the p-Laplace equation:

$$\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = n\omega_n \delta(x - x_0), \tag{5}$$

where $\delta(x - x_0)$ is the Dirac measure or Dirac δ-function at $x_0 \in \overline{\mathbb{E}^n}$. With $x_0 \neq \infty$, we have

$$\mu_p(t) = \begin{cases} -\ln t, & p = n; \\ \frac{1}{\gamma} t^{-\gamma}, & p \neq n, \end{cases} \tag{6}$$

where $\gamma = \frac{n-p}{p-1}$. If $x_0 = \infty$, then the role of $\mu_p(x, \infty)$ is played by the function

$$\mu_p^\infty(|x|) = \mu_p^\infty(t) = -\mu_p\left(1/t\right). \tag{7}$$

Note that the p-module of the spherical ring K_r^R, bounded by the concentric spheres of radii r and $R > r$ is

$$\operatorname{mod}_p K_r^R = \mu_p(r) - \mu_p(R) = \begin{cases} \ln \frac{R}{r}, & p = n, \\ -\frac{1}{\gamma}(R^{-\gamma} - r^{-\gamma}), & p \neq n. \end{cases} \tag{8}$$

We will need the following well-known property from the potential theory, formulated here for the case of p-modules of ring domains.

Lemma 2. *If ring domains $D_1, D_2, \ldots, D_m$ are pairwise disjoint and each of them separates the boundary components of a ring domain D, then*

$$\operatorname{mod}_p D \geqslant \sum_{k=1}^{m} \operatorname{mod}_p D_k. \tag{9}$$

Proof. Let u_k be an admissible function for the ring domain D_k, $a_k \geqslant 0$ and $\sum_{k=1}^m a_k = 1$. Then $u = \sum_{k=1}^m u_k$ is an admissible function for the ring domain D and

$$\int_D |\nabla u|^p \, d\omega = \sum_{k=1}^{m} a_k \int_{D_k} |\nabla u_k|^p \, d\omega.$$

Hence,

$$\operatorname{Cap}_p D \leqslant \sum_{k=1}^{m} a_k^p \operatorname{Cap}_p D_k. \tag{10}$$

Assuming

$$a_k = \frac{\left(\operatorname{Cap}_p D_k\right)^{\frac{-1}{p-1}}}{\sum\limits_{k=1}^{m} \left(\operatorname{Cap}_p D_k\right)^{\frac{-1}{p-1}}},$$

from (10), we obtain (9). □

Let $G \subset \overline{\mathbb{E}^n}$ be a domain homeomorphic to a ball, $x_0 \in G$, $G_t = G \backslash \overline{B^n(x_0, t)}$. If $x_0 \neq \infty$, by Lemma 2, for sufficiently small $0 < t_1 < t_2$ we have

$$\operatorname{mod}_p G_{t_1} \geqslant \operatorname{mod}_p G_{t_2} + \operatorname{mod}_p K_{t_1}^{t_2},$$

where $K_{t_1}^{t_2}$ is the ring bounded by concentric spheres of radii t_1 and t_2 with center at x_0. Therefore,

$$\operatorname{mod}_p G_{t_1} - \mu_p(t_1) \geqslant \operatorname{mod}_p G_{t_2} - \mu_p(t_2).$$

Consequently, the following limit exists:

$$\lim_{t \to 0} \left[\operatorname{mod} G_t - \mu_p(t)\right] = m_p(x_0, G). \tag{11}$$

Similarly, when $x_0 = \infty$,

$$\operatorname{mod}_p G_{t_1^{-1}} \geqslant \operatorname{mod}_p G_{t_2^{-1}} + \operatorname{mod}_p K_{t_2^{-1}}^{t_1^{-1}},$$

where $K_{t_2^{-1}}^{t_1^{-1}}$ is the ring bounded by concentric spheres of radii t_2^{-1} and t_1^{-1} $(t_2^{-1} < t_1^{-1})$ with the center at the origin. Hence, taking (8) into account, we find

$$\operatorname{mod}_p G_{\frac{1}{t_1}} - \mu_p^{\infty}(t_1) \geqslant \operatorname{mod}_p G_{\frac{1}{t_2}} - \mu_p^{\infty}(t_2).$$

Consequently, the following limit exists:

$$\lim_{t \to 0} \left[\operatorname{mod} G_{\frac{1}{t}} - \mu_p^{\infty}(t)\right] = m_p(\infty, G). \tag{12}$$

In the general case, the quantity $m_p(x_0, G) = h_p(x_0, G)$ will be called the reduced p-module of the domain G with respect to the point x_0. If $p > n$ and $x_0 \neq \infty$, we have

$$m_p(x_0, G) = \lim_{t \to 0} \operatorname{mod}_p G_t.$$

If $1 < p < n$ and $x_0 = \infty$, then

$$m_p(\infty, G) = \lim_{t\to 0} \operatorname{mod}_p G_{\frac{1}{t}} = \left(\frac{n\omega_n}{C_p(\overline{\mathbb{E}^n}\backslash G)}\right)^{\frac{1}{p-1}},$$

where $C_p(A)$ is the p-capacity of the compact $A \subset \mathbb{E}^n$, see [13], defined by

$$C_p(A) = \inf \int\limits_{\mathbb{E}^n} |\nabla u|^p \, d\omega.$$

Here the infimum is taken over the class of continuously differentiable functions, greater than or equal to 1 on A, with compact support in $\mathbb{E}^n$.

The notion of reduced modulus of a plane domain ($p = n = 2$) appeared for the first time in Teichmiiller's article [17]. Various generalizations of the concept of the reduced module and their applications were considered in [1, 10–12, 14]. The definition of the reduced p-module of a domain with respect to a point, given in this article above, can be extended to the case of domains of arbitrary connectivity. To do this, we use the definition of the p-module of the domain, connected either to the p-capacity of the corresponding condenser, or to the corresponding modules of families of curves or surfaces (see [7, 16]).

3. The inner *p*-harmonic radius. Let G be a domain with the regular boundary in $\overline{\mathbb{E}^n}$, $x_0 \in G$. From the results of S. Kichenassamy and L. Veron [9] it follows that in the domain G there exists a unique (generalized) solution $u = u_G(x, x_0) \in C^{1,\alpha}(G\backslash x_0)$, $\alpha > 0$ of the Dirichlet problem for equation (5), which equals zero on the boundary of the domain G, and such that the function

$$h_p(x, x_0) = u_G(x, x_0) - \mu_p(x, x_0) \in L^\infty(G).$$

In addition, there is the limit

$$\lim_{x\to x_0} h_p(x, x_0) = h_p(x_0, G) \tag{13}$$

and

$$\lim_{x\to x_0} |x - x_0|^{\frac{n-1}{p-1}} \left(\nabla u_G(x, x_0) - \nabla \mu_p(x, x_0)\right) = 0. \tag{14}$$

The function $u_G(x, x_0)$ will be called the p-harmonic Green's function of the domain G with a pole at the point x_0, and the function $h_p(x_0, G)$ will be called the Roben p-function of the domain G. Note that when $p > n$ we have $h_p(x_0, G) = u_G(x_0, x_0)$. By definition [10], the inner p-harmonic radius of the domain G at the point x_0 is the value of $R_p(x_0, G)$ for which

$$h_p(x_0, G) = \begin{cases} -\mu_p(R_p(x_0, G))\,, & x_0 \neq \infty, \\ -\mu_p^\infty(R_p(\infty, G))\,, & x_0 = \infty. \end{cases} \tag{15}$$

Thus, $R_n(x_0, G) = \exp\{h_n(x_0, G)\}$ and for $p \neq n$

$$R_p(x_0, G) = \begin{cases} \big(-\gamma h_p(x_0, G)\big)^{-1/\gamma}, & x_0 \neq \infty, \\ \big(\gamma h_p(\infty, G)\big)^{1/\gamma}, & x_0 = \infty. \end{cases} \tag{16}$$

The inner p-harmonic radius of an arbitrary domain $G \subset \overline{\mathbb{E}^n}$ at the point x_0 is the number $R_p(x_0, G) = \sup R_p(x_0, G')$, where the supremum is taken over all domains $G' \subset G$ with the smooth boundary.

Theorem 1. [10] *For any domain $G \subset \overline{\mathbb{E}^n}$ with regular boundary and any $x_0 \in G$ we have $m_p(x_0, G) = h_p(x_0, G)$.*

Proof. Let $G \subset \mathbb{E}^n$ be a domain with regular boundary, $x_0 \neq \infty$, and $\Omega_a(u_G) = \{x \in G : u_G(x, x_0) \geqslant a\}$. Let $\Omega_a(u_G)$ be a closed bounded set. We show that $\Omega_a(u_G)$ is star-shaped with respect to the point x_0 for sufficiently large a. It follows from (14), that for any direction $\overrightarrow{l}$

$$\lim_{x \to x_0} |x - x_0|^{\frac{n-1}{p-1}} \left(\frac{\partial u_G(x, x_0)}{\partial l} - \frac{\partial \mu_p(x, x_0)}{\partial l} \right) = 0. \tag{17}$$

In particular, passing to spherical coordinates and calculating the derivative along the radius $\overrightarrow{x - x_0}$ for $\rho = |x - x_0|$ we obtain:

$$\lim_{\rho \to 0} \rho^{\frac{n-1}{p-1}} \frac{\partial u_G}{\partial \rho} = -1. \tag{18}$$

It follows that for small ρ the function $u_G(x,x_0)$ decreases monotonically with respect to ρ and the level surface $S_a(u_G) = \partial\Omega_a(u_G)$ is star-shaped with respect to the point x_0. We consider the condenser $G(a) = G \backslash \Omega_a(u_G)$. The extremal function for the p-capacity of the condenser $G(a)$ has the form $v_a(x) = \frac{1}{a} u_G(x, x_0)$. By Lemma 1,

$$\operatorname{Cap}_p G(a) = \frac{1}{a^{p-1}} \int\limits_{S_a(u_G)} |\nabla u_G|^{p-1}\, dS. \tag{19}$$

Applying formula (3) to the domain bounded by the surface $S_a(u_G)$ and sphere $S(x_0,t)$, where $t>0$ is sufficiently small, and setting $\psi=1$, and $\varphi=u_G(x,x_0)$, we find

$$\operatorname{Cap}_p G(a)=\frac{1}{a^{p-1}}\int\limits_{S_a(u_G)}|\nabla u_G|^{p-1}\,dS=-\frac{1}{a^{p-1}}\int\limits_{S(x_0,t)}|\nabla u_G|^{p-2}\,\frac{\partial u_G}{\partial t}dS. \tag{20}$$

It follows from (14) that $\dfrac{\partial u_G}{\partial t}=-t^{\frac{1-n}{p-1}}(1+o(t))$, $t\to 0$. Thus,

$$\operatorname{Cap}_p G(a)-\frac{1}{a^{p-1}}\int\limits_{S(x_0,t)}t^{1-n}\big(1+\alpha(t)\big)^{p-2}\big(1+o(t)\big)dS=$$
$$=\frac{n\omega_n}{a^{p-1}}\big(1+\alpha(t)\big)^{p-2}\big(1+o(t)\big)$$

and passing to the limit as $t\to 0$, we find that $\operatorname{Cap}_p G(a)=n\omega_n/a^{p-1}$ or $\operatorname{mod}_p G(a)=a$. We consider now the condenser $G_t=G\backslash\overline{B^n(x_0,t)}$, where $t>0$ is sufficiently small, and the values $a_1<a_2$ are such that the level surface $S_{a_1}(u_G)$ contains the sphere $S^{n-1}(x_0,t)$ and touches it, and the level surface $S_{a_2}(u_G)$ lies inside this sphere and touches it from within. Such values a_1 and a_2 exist except for the trivial case when the domain G is a ball with center at the point x_0.
Since the p-module of the condenser does not decrease as it expands, then $\operatorname{mod}_p G(a_2)\geqslant\operatorname{mod}_p G_t\geqslant\operatorname{mod}_p G(a_1)$ or

$$a_1\leqslant\operatorname{mod}_p G_t\leqslant a_2. \tag{21}$$

It follows from (13) that for any $\varepsilon>0$ there exists $\delta>0$, such that

$$\mu_p(|x-x_0|)+h_p(x_0,G)-\varepsilon<u_G(x,x_0)<\mu_p(|x-x_0|)+h_p(x_0,G)+\varepsilon$$

for any x, such that $|x-x_0|<\delta$. Choosing t sufficiently small and using (21), we have

$$h_p(x_0,G)-\varepsilon<\operatorname{mod}_p G_t-\mu_p(t)<h_p(x_0,G)+\varepsilon,$$

so $m_p(x_0,G)=h_p(x_0,G)$. The proof of the theorem in the case of an arbitrary domain with a regular boundary, as well as the consideration of the case $x_0=\infty$ is obtained by modifying the above arguments. $\square$

Note that from the definition of the inner p-harmonic radius of an arbitrary domain $G \subset \overline{\mathbb{E}^n}$ at the point x_0 and the relation (15), and also the well-known property of continuity of the p-capacity (p-module) with respect to the monotonic convergence of sets (see, for example, [7]), it follows that $R_n(x_0, G) = \exp\{m_n(x_0, G)\}$ and for $p \neq n$

$$R_p(x_0, G) = \begin{cases} \big(\gamma m_p(x_0, G)\big)^{-1/\gamma}, & x_0 \neq \infty, \\ \big(-\gamma m_p(\infty, G)\big)^{1/\gamma}, & x_0 = \infty. \end{cases} \tag{22}$$

4. p-Harmonic Green's mappings. Let G and $\widetilde{G}$ be homeomorphic to a ball domains regular boundaries in $\overline{\mathbb{E}^n}$. Let $u_G(x, x_0)$ and $u_{\widetilde{G}}(y, y_0)$ be p-harmonic Green's functions for these domains with poles at points $x_0 \in G$ ($x_0 \neq \infty$) and $y_0 \in \widetilde{G}$ ($y_0 \neq \infty$), respectively, $1 < p \leqslant n$. Consider the mapping $f : G \to \widetilde{G}$ such that:

- $f(x_0) = y_0$;
- the level set $S_t(u_G)$ is mapped onto the level set $S_t(u_{\widetilde{G}})$;
- the trajectory of the gradient field $\nabla u_G(x, x_0)$ that enters the pole x_0 corresponds to the trajectory of the gradient field $\nabla u_{\widetilde{G}}(y, y_0)$ that enters the pole y_0.

Such mappings are constructed by analogy with the Green's mappings ($p = n = 3$) considered in the monograph by A. I. Januszauskas [19], as a special case of harmonic mappings with respect to M. A. Lavrentyev. It follows from relation (17) that p-harmonic Green's functions of G and $\widetilde{G}$ have the property that for any ray l from the point $x_0 \in G$ (respectively, $y_0 \in \widetilde{G}$) there is the unique trajectory of the field $\nabla u_G(x, x_0)$ (respectively, the unique trajectory of the field $\nabla u_{\widetilde{G}}(y, y_0)$), entering x_0 (respectively, y_0) with the tangent l. Let $\sigma : S \to S$ be the rotation (linear mapping) of the unit sphere $S = S(0, 1)$ under which a point $X \in S$ mapped to the point $\sigma(X)$. If l is the ray from the center of S passing through the point X, then $\sigma(l)$ denotes the ray from the center of S passing through the point $\sigma(X)$.

p-Harmonic Green's mapping $f : G \to \widetilde{G}$ is defined in a sufficiently small neighborhood $U(x_0)$ of the pole x_0 , as follows. If l is the tangent at the point x_0 of the trajectory of gradient field $\nabla u_G(x, x_0)$ that enters the pole x_0 and passes through the point $x \in S_t(u_G)$, then $y = f(x) \in S_t(u_{\widetilde{G}})$ belongs to the trajectory of the gradient field $\nabla u_{\widetilde{G}}(y, y_0)$, that enters the pole y_0 with the tangent $\sigma(l)$. The constructed mapping is a homeomorphism of a sufficiently small neighborhood $U(x_0)$ onto a sufficiently small neighborhood of $U(y_0)$. This homeomorphism can be extended outside these neighborhoods by means of the following construction, similar to that described in [19]. If the function $u_G(x, x_0)$ has no critical values $\alpha : a \leqslant \alpha < \infty$ in the domain $G(a) = G\backslash\Omega_a(u_G)$, or $\nabla u_G(x, x_0) = 0$ in some points on the level surface $S_\alpha(u_G)$, then the whole domain $G(a)$ is homeomorphic to the ball. The same is true for the domain $\widetilde{G}(a) = \widetilde{G}\backslash\Omega_a(u_{\widetilde{G}})$. Let $\alpha_0 > \alpha_1 > \ldots > \alpha_k > 0$ be the critical values of the function u_G in the domain G. There are a finite number of such values, provided that $\nabla u_G(x, x_0) \neq 0$ on ∂G. Analogously, let $\beta_0 > \beta_1 > \ldots > \beta_m > 0$ be the critical values of the function $u_{\widetilde{G}}$ in the domain $\widetilde{G}$. Let $\gamma = \max(\alpha_0, \beta_0)$. Consider the field $\nabla u_G(x, x_0)$ that enters the pole x_0 with the tangent l and has the level surface $S_a(u_G)$, and the field $\nabla u_{\widetilde{G}}(y, y_0)$ that enters the pole y_0 with the tangent $\sigma(l)$ and has the level surface $S_a(u_{\widetilde{G}})$. Consider a point $x \in G(\gamma)$ at the intersection of the trajectory of the field $\nabla u_G(x, x_0)$ and associate it with the point $y \in \widetilde{G}(\gamma)$ at the intersection of the trajectory of the field $\nabla u_{\widetilde{G}}(y, y_0)$. Thus, the extension of the mapping f from the neighborhood $U(x_0)$ to the homeomorphism of the domain $G(\gamma)$ to the domain $\widetilde{G}(\gamma)$ is defined. Further extended beyond $G(\gamma)$ along such trajectories, this mapping may have singularities, because different trajectories of the gradient field intersect at critical points. Such construction is possible only if both functions $u_G(x, x_0)$ and $u_{\widetilde{G}}(y, y_0)$ have singularities at the points x_0 and y_0, or $u_{\widetilde{G}}(y_0, y_0) = u_G(x_0, x_0)$.

Let f_t be the trace of the mapping f on the level surface $S_t(u_G)$, $J_{f_t}(x)$ be the Jacobian of the trace f_t and $J_f(x)$ be the Jacobian of f. The following theorem extends the properties established in [19] for $p = 2$ and $n = 3$ to the case of p-harmonic Green's mappings.

Theorem 2. *The following relations hold:*

$$1)\ |f'(x_0)| = \lim_{x \to x_0} \frac{|f(x) - y_0|}{|x - x_0|} = \begin{cases} \Delta_n(\widetilde{G}, G), & p = n; \\ 1, & p < n. \end{cases} \tag{23}$$

$$2)\ \lim_{t \to \infty} \frac{|\nabla u_{\widetilde{G}}|}{|\nabla u_G|}\bigg|_{u_G = u_{\widetilde{G}} = t} = 1. \tag{24}$$

$$3)\ \lim_{x \to x_0} J_f(x) = \begin{cases} \Delta_n^n(\widetilde{G}, G), & p = n; \\ 1, & p < n. \end{cases} \tag{25}$$

$$4)\ J_f(x) = J_{f_t}(x) \times \frac{|\nabla u_G(x, x_0)|}{\left|\nabla u_{\widetilde{G}}(f(x), y_0)\right|}, \quad x \in S_t(u_G). \tag{26}$$

$$5)\ J_{f_t}(x) = \begin{cases} \left(\dfrac{|\nabla u_G(x, x_0)|}{|\nabla u_{\widetilde{G}}(f(x), y_0)|}\right)^{n-1} \times \Delta_n^n(\widetilde{G}, G),\ p = n, \\ \left(\dfrac{|\nabla u_G(x, x_0)|}{|\nabla u_{\widetilde{G}}(f(x), y_0)|}\right)^{p-1},\ p < n, \end{cases} \quad x \in S_t(u_G). \tag{27}$$

Here $\Delta_n(\widetilde{G}, G) = \exp\left[h_n(y_0, \widetilde{G}) - h_n(x_0, G)\right]$.

Proof. The following representations hold for the p-harmonic Green's functions of the domains G and $\widetilde{G}$ in the neighborhood of the poles x_0 and $f(x_0) = y_0$ due to (13):

$$u_G(x,x_0) = \mu_p(x, x_0) + h_p(x_0, G) + O(|x - x_0|)$$

and

$$u_{\widetilde{G}}(f(x), y_0) = \mu_p(f(x), y_0) + h_p(y_0,\widetilde{G}) + O(|f(x) - y_0|).$$

On the corresponding level surfaces $u_{\widetilde{G}}(f(x),y_0) = u_G(x,x_0)$; thus

$$|f(x) - y_0| = \\ = \begin{cases} |x - x_0| \exp\left[h_p(y_0,\widetilde{G}) - h_p(x_0,G) + O(|x - x_0|)\right],\ p = n, \\ |x-x_0|\left\{1-|x-x_0|^{\gamma}\left[h_p(y_0,\widetilde{G})-h_p(x_0,G)\right]+o(|x-x_0|^{\gamma})\right\},\ p < n. \end{cases} \tag{28}$$

This implies the first relation.
Due to (14), we have for all $1 < p \leqslant n$:

$$|\nabla u_G(x, x_0)| = |x - x_0|^{\frac{1-n}{p-1}} (1 + O(|x - x_0|)),$$

and, respectively,

$$\left|\nabla u_{\widetilde{G}}(f(x), y_0)\right| = |f(x) - y_0|^{\frac{1-n}{p-1}} \left(1 + O\big(|f(x) - y_0|\big)\right).$$

Hence, taking (28) into account, we obtain (24).

Let us prove the equality (25). Let the ball $B^n(x_0, r) \subset G$ and $\widetilde{B}$ be its image under the mapping f. Let

$$R_{t_1} = \max_{y \in \widetilde{B}} |y - y_0| = |y_{t_1} - y_0|,$$

where $y_{t_1} \in S_{t_1}(u_{\widetilde{G}})$, and

$$R_{t_2} = \min_{y \in \widetilde{B}} |y - y_0| = |y_{t_1} - y_0| \, ,$$

where $y_{t_2} \in S_{t_2}(u_{\widetilde{G}})$, $0 \leqslant t_1 < t_2 < \infty$. We set $x_{t_\nu} = f^{-1}(y_{t_\nu}), \nu = 1, 2$. For the n-dimensional Lebesgue measure $m_n(\widetilde{B})$ of the domain $\widetilde{B}$ we have the inequality

$$\frac{R_{t_2}^n}{r^n} \leqslant \frac{m_n(D)}{\omega_n r^n} \leqslant \frac{R_{t_1}^n}{r^n}. \tag{29}$$

From relation (27) we easily find

$$\lim_{r \to 0} \frac{R_{t_2}^n}{r^n} = \lim_{r \to 0} \frac{R_{t_1}^n}{r^n} = \begin{cases} \exp n \big[h_n(y_0, \widetilde{G}) - h_n(x_0, G)\big], & p = n, \\ 1, \quad p < n. \end{cases}$$

Note that $J_f(x_0) = \lim\limits_{r \to 0} m_n(D)/\omega_n r^n$; then (29) implies (25). By construction of the map f, $J_f(x) = J_{f_t}(x) K_t$, where K_t is the coefficient of extension along the orthogonal trajectories of the mapping f on the level surface $S_t(u_G)$. The increase rate of a function along orthogonal trajectories to level surfaces is proportional to the length of its gradient, then $K_t = |\nabla u_G(x, x_0)| \times \big[|\nabla u_{\widetilde{G}}(f(x), y_0)|\big]^{-1}$ for $x \in S_t(u_G)$, that is, equality (26) is satisfied. Moreover, by virtue of (24), $\lim\limits_{t \to \infty} K_t = 1$. We consider two level surfaces $S_t(u_G)$ and $S_{t_1}(u_G)$, where $0 \leqslant t < t_1 < \infty$. Let point $X \in S_t(u_G)$ and $\theta(X) \in S_{t_1}(u_G)$ be its image lying on the trajectory of the field $\nabla u_G(x, x_0)$, passing through X. If there are no critical points in the layer bounded by these surfaces, then the mapping $\theta : S_t(u_G) \to S_{t_1}(u_G)$ is a homeomorphism. Let $U(X) \subset S_t(u_G)$ be an open connected neighborhood of X, and $V(X) \subset S_{t_1}(u_G)$ be its image under the mapping θ. Denote by $\Omega(X)$ the domain that represents the part of the flow tube of the vector field $\nabla u_G(x, x_0)$, enclosed between $U(X)$ and $V(X)$. Applying formula (3) in the domain $\Omega(X)$ in the case when $\psi = 1$ and $\varphi = u_G(x, x_0)$, we obtain

$$\int\limits_{V(X)} |\nabla u_G|^{p-2} \frac{\partial u_G}{\partial n} dS_V = \int\limits_{U(X)} |\nabla u_G|^{p-2} \frac{\partial u_G}{\partial n} dS_U =$$

$$= \int\limits_{V(X)} |\nabla u_G|^{p-2} \frac{\partial u_G}{\partial n} \left[J_\theta(x)\right]^{-1} dS_V,$$

where $J_\theta(x)$ is the Jacobian of the mapping θ. Applying the mean-value theorem and contracting the neighborhood $U(X)$ to the point X, we find

$$|\nabla u_G|^{p-2}\frac{\partial u_G}{\partial n}\bigg|_{u_G=t_1} = |\nabla u_G|^{p-2}\frac{\partial u_G}{\partial n}\bigg|_{u_G=t}[J_\theta(X)]^{-1}.$$

Hence,

$$J_\theta(X) = |\nabla u_G|^{p-1}|_{u_G=t}\times\left[|\nabla u_G|^{p-1}|_{u_G=t_1}\right]^{-1}.$$

Analogously, for the mapping $\widetilde{\theta}: S_{t_1}(u_{\widetilde{G}}) \to S_{t_2}(u_{\widetilde{G}})$ we obtain

$$J_{\widetilde{\theta}}(X) = |\nabla u_{\widetilde{G}}|^{p-1}|_{u_{\widetilde{G}}=t}\times\left[|\nabla u_{\widetilde{G}}|^{p-1}|_{u_{\widetilde{G}}=t_1}\right]^{-1}.$$

As $f_t = \widetilde{\theta}^{-1}\circ f_{t_1}\circ\theta$,

$$J_{f_t}(x) = J_{f_{t_1}}(x)\times\frac{|\nabla u_G(x,x_0)|^{p-1}}{|\nabla u_{\widetilde{G}}(f(x),y_0)|^{p-1}}\Bigg|_{u_G=u_{\widetilde{G}}=t}\times\frac{|\nabla u_{\widetilde{G}}(f(x),y_0)|^{p-1}}{|\nabla u_G(x,x_0)|^{p-1}}\Bigg|_{u_G=u_{\widetilde{G}}=t_1}. \tag{30}$$

Passing to the limit in (30) for $t_1\to\infty$ and taking (24) and (25) into account, we obtain (27). □

From relation (23) and our reasoning, we obtain

Corollary 1.

$$R_p(y_0,\widetilde{G}) = \begin{cases}|f'(x_0)|\,R_n(x_0,G), & p=n,\\ \left[R_p{}^{-\gamma}(x_0,G)+\lambda_f^p(x_0)\right]^{-\frac{1}{\gamma}}, & p<n,\end{cases} \tag{31}$$

where $\lambda_f^p(x_0) = \lim\limits_{x\to x_0}\left[|f(x)-y_0|^{-\gamma}-|x-x_0|^{-\gamma}\right]$.

Theorem 3. *Assume that $\widetilde{G}$ is a ball of radius R centered at y_0; for $p=n$ we have $|f'(x_0)|=1$ if and only if $R_n(x_0,G)=R$, and for $p<n$ we have $J_f(x)=1+O(|x-x_0|^{\frac{n-1}{p-1}})$ if and only if $R_p(x_0,G)=R$.*

Proof. The first part of the statement follows immediately from (31). Further, since

$$u_{\widetilde{G}}(y_0,\widetilde{G}) = \begin{cases}\ln R-\ln|y-y_0|\,, & p=n,\\ -\frac{1}{\gamma}R^{-\gamma}+\frac{1}{\gamma}\,|y-y_0|\,, & p<n,\end{cases} \tag{32}$$

for $y = f(x)$, we have

$$|f(x) - y_0| = \begin{cases} \frac{R}{R_n(x_0,G)} |x - x_0| \left(1 + O(|x - x_0|)\right), & p = n, \\ |x - x_0| \left[1 + C\,|x - x_0|^{\gamma} + O(|x - x_0|^{\gamma+1})\right]^{-\frac{1}{\gamma}}, & p < n. \end{cases} \tag{33}$$

Here $C = R^{-\gamma} - R_p{}^{-\gamma}(x_0, G)$. It follows from (32) and (14) that

$$\left|\nabla u_{\widetilde{G}}(y_0, \widetilde{G})\right| = |y - y_0|^{-\frac{n-1}{p-1}}$$

and, respectively,

$$|\nabla u_G(x, x_0)| = |x - x_0|^{-\frac{n-1}{p-1}} \left(1 + O(|x - x_0|)\right).$$

From this, using (33), we find:

$$\frac{|\nabla u_G(x, x_0)|}{\left|\nabla u_{\widetilde{G}}(f(x), y_0)\right|} = \begin{cases} \dfrac{R_n(x_0, G)}{R}\left(1 + O(|x - x_0|)\right), & p = n, \\ 1 - \dfrac{n-1}{n-p} C\,|x - x_0|^{\gamma} + O(|x - x_0|^{\gamma+1}), & p < n. \end{cases} \tag{34}$$

Since, by virtue of (26) and (27),

$$J_f(x) = \begin{cases} \left(\dfrac{|\nabla u_G(x, x_0)|}{|\nabla u_{\widetilde{G}}(f(x), y_0)|}\right)^n \times \left(\dfrac{R}{R_n(x_0, G)}\right)^n, & p = n; \\ \left(\dfrac{|\nabla u_G(x, x_0)|}{\left|\nabla u_{\widetilde{G}}(f(x), y_0)\right|}\right)^p, & p < n, \end{cases}$$

from (34) we deduce

$$J_f(x) = \begin{cases} 1 + O(|x - x_0|\,, & p = n, \\ 1 - p\frac{n-1}{n-p} C\,|x - x_0|^{\gamma} + O(|x - x_0|^{\gamma+1}), & p < n, \end{cases} \tag{35}$$

from which the assertion to be proved follows. □

Remark. *For $p = n$ the construction of p-harmonic Green's mappings described above and the assertion of Theorem 2 can be extended to the case where one or both poles x_0 or y_0 are equal to ∞.*

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