DOI:

UDC 517.54

A. K. Bakhtin, I. V. Denega

SHARP ESTIMATES OF PRODUCTS OF INNER RADII OF NON-OVERLAPPING DOMAINS IN THE COMPLEX PLANE

Abstract. In the paper we study a generalization of the extremal problem of geometric theory of functions of a complex variable on non-overlapping domains with free poles: Fix any $\gamma \in \mathbb{R}^+$ and find the maximum (and describe all extremals) of the functional

$$[r\,(B_0,0)\,r\,(B_\infty,\infty)]^\gamma \prod_{k=1}^{n} r\,(B_k,a_k)\,,$$

where $n \in \mathbb{N}$, $n \geqslant 2$, $a_0 = 0$, $|a_k| = 1$, B_0, B_∞, $\{B_k\}_{k=1}^n$ is a system of mutually non-overlapping domains, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{0,n}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, ($r(B,a)$ is an inner radius of the domain $B \subset \overline{\mathbb{C}}$ at $a \in B$). Instead of the classical condition that the poles are on the unit circle, we require that the system of free poles is an n-radial system of points normalized by some "control" functional. A partial solution of this problem is obtained.

Key words: *inner radius of a domain, non-overlapping domains, radial system of points, separating transformation, quadratic differential, Green's function*

2010 Mathematical Subject Classification: *30C75*

Let $\mathbb{N}$, $\mathbb{R}$ be the sets of natural and real numbers, respectively, $\mathbb{C}$ be the complex plane, $\overline{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ be a one-point compactification, and $\mathbb{R}^+ = (0, \infty)$. Let $\chi(t) = \frac{1}{2}(t + t^{-1})$, $t \in \mathbb{R}^+$, be the Zhukovskii function. Let $r(B, a)$ be an inner radius of the domain $B \subset \overline{\mathbb{C}}$ relative to the point $a \in B$.

The system of points $A_n := \{a_k \in \mathbb{C}, k = \overline{1,n}\}$, $n \in \mathbb{N}$, $n \geqslant 2$ is called n-radial, if $|a_k| \in \mathbb{R}^+$ for $k = \overline{1,n}$ and $0 = \arg a_1 < \ldots < \arg a_n < 2\pi$.

Denote

$$P_k = P_k(A_n) := \{w : \arg a_k < \arg w < \arg a_{k+1}\}, \quad a_{n+1} := a_1,$$

$$\alpha_k := \frac{1}{\pi}\arg\frac{a_{k+1}}{a_k}, \quad \alpha_{n+1} := \alpha_1, \quad k=\overline{1,n}, \quad \sum_{k=1}^{n}\alpha_k = 2.$$

For any n-radial system of points $A_n = \{a_k\}$, $k=\overline{1,n}$, we introduce the "control" functional

$$\mathcal{L}(A_n) := \prod_{k=1}^{n}\chi\left(\left|\frac{a_k}{a_{k+1}}\right|^{\frac{1}{2\alpha_k}}\right)\cdot|a_k|.$$

The class of n-radial systems of points for which $\mathcal{L}(A_n)=1$ contains automatically all systems of n different points of the unit circle.

Consider the following extremal problem.

Problem 1. For any fixed value of $\gamma\in\mathbb{R}^+$, find the maximum of the functional

$$J_n(\gamma) = [r(B_0,0)\,r(B_\infty,\infty)]^{\gamma}\prod_{k=1}^{n} r(B_k,a_k), \tag{1}$$

where $n\in\mathbb{N}$, $n\geqslant 2$, $a_0=0$, $A_n=\{a_k\}_{k=1}^{n}$ are n-radial systems of points, such that $\mathcal{L}(A_n)=1$, B_0, B_∞, $\{B_k\}_{k=1}^{n}$ is a system of mutually non-overlapping domains, $a_k\in B_k\subset\overline{\mathbb{C}}$, $k=\overline{0,n}$, $\infty\in B_\infty\subset\overline{\mathbb{C}}$; also, describe all extremals.

This problem belongs to the class of extremal problems with free poles. Problems of this type have been studied in many papers (see, for example, [1–16]). For $\gamma=\frac{1}{2}$ and $n\geqslant 2$, an estimate of the functional $J_n(\gamma)$ for the system of non-overlapping domains was found in the paper [6, p. 59]. Kuz'mina [15, p. 267] strengthened this result for simply connected domains and showed that the estimate is correct for $\gamma\in\left(0,\frac{n^2}{8}\right]$, $n\geqslant 2$. Note that for $n=2$ the Kuz'mina's estimate of the functional (1) coincides with the Dubinin's estimate. Some partial cases of the above-posed problem were considered in [2, 3, 5].

Let

$$S(x) = x^{2x^2+2}\cdot|1-x|^{-(1-x)^2}\cdot(1+x)^{-(1+x)^2} \quad \text{and} \quad \Psi(x)=\ln(S(x)).$$

$$\Psi'(x) = 4x\ln(x) - 2(x-1)\ln|x-1| - 2(x+1)\ln(x+1) + \frac{2}{x}\ \text{(see Fig. 1)}.$$

The function $S(x)$ is logarithmically convex on the interval $[0,x_0]$, $x_0\approx 0.88441$. Let $\Psi'(x)=t$, $y_0\leqslant t<0$, $y_0\approx -1.06$. The equation $\Psi'(x)=t_k$ has two solutions $x_1(t)\in(0,x_0]$ and $x_2(t)\in(x_0,\infty]$.

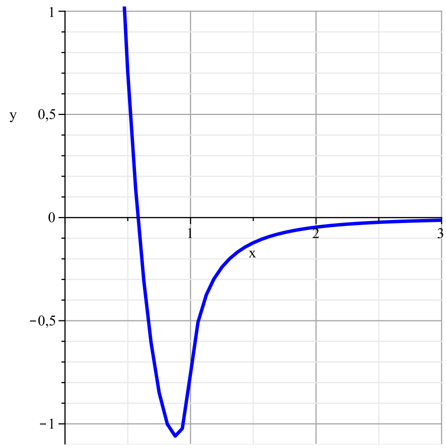

Figure 1: The function plot $y = \Psi'(x)$

Let $\delta_n^0 = \min\left((n-1)x_1(t) + x_2(t)\right) = 2\sqrt{\gamma_n^0}$, then $\gamma_n^0 = \left(\frac{\delta_n^0}{2}\right)^2$. Then the following proposition is true.

Theorem 1. *Let $n \in \mathbb{N}$, $n \geqslant 2$, $\gamma \in (0, \gamma_n^0]$, $\gamma_n^0 = \left(\frac{\delta_n^0}{2}\right)^2$. Then, for any n-radial system of points $A_n = \{a_k\}_{k=1}^n$ such that $\mathcal{L}(A_n) = 1$, and any system of mutually non-overlapping domains B_0, B_∞, B_k, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, the following inequality holds:*

$$[r(B_0, 0)\, r(B_\infty, \infty)]^\gamma \prod_{k=1}^n r(B_k, a_k) \leqslant [r(\Lambda_0,)\, r(\Lambda_\infty, \infty)]^\gamma \prod_{k=1}^n r(\Lambda_k, \lambda_k), \tag{2}$$

where the domains Λ_0, Λ_∞, Λ_k, and the points 0, ∞, λ_k, $k = \overline{1,n}$, are, respectively, circular domains and poles of the quadratic differential

$$Q(w)dw^2 = -\frac{\gamma w^{2n} + (n^2 - 2\gamma)w^n + \gamma}{w^2(w^n - 1)^2}\, dw^2. \tag{3}$$

Proof. Let $\zeta = \pi_k(w)$ denote a univalent branch of the multivalent analytic function $-i\,(e^{-i\arg a_k}w)^{\frac{1}{\alpha_k}}$, $k = \overline{1,n}$, that maps P_k onto the right half-plane $\operatorname{Re}\zeta > 0$ conformally in the one-sheet way. Consider the system of functions $\zeta = \pi_k(w) = -i\,(e^{-i\arg a_k}w)^{\frac{1}{\alpha_k}}$, $k = \overline{1,n}$. Let $\Omega_k^{(1)}$, $k = \overline{1,n}$, denote a domain of the plane $\mathbb{C}_\zeta$, obtained as a result of the union of the connected component of the set $\pi_k(B_k \bigcap \overline{P}_k)$, containing the point $\pi_k(a_k)$, with its symmetric reflection with respect to the imaginary axis. In turn, by $\Omega_k^{(2)}$, $k = \overline{1,n}$, we denote the domain of the plane $\mathbb{C}_\zeta$, obtained as a result of the union of the connected component of the set $\pi_k(B_{k+1} \bigcap \overline{P}_k)$, containing the point $\pi_k(a_{k+1})$, with its symmetric reflection with respect to the imaginary axis, $B_{n+1} := B_1$, $\pi_n(a_{n+1}) := \pi_n(a_1)$. In addition, $\Omega_k^{(0)}$ denotes a domain of the plane $\mathbb{C}_\zeta$ obtained as a result of the union of the connected component of the set $\pi_k(B_0 \bigcap \overline{P}_k)$, containing the point $\zeta = 0$, with its symmetric reflection with respect to the imaginary axis. Similarly, $\Omega_k^{(\infty)}$ denotes a domain of the plane $\mathbb{C}_\zeta$ obtained as a result of the union of the connected component of the set $\pi_k(B_\infty \bigcap \overline{P}_k)$, containing the point $\zeta = \infty$, with its symmetric reflection with respect to the imaginary axis. It is clear that $\pi_k(a_k) := \omega_k^{(1)}$, $\pi_k(a_{k+1}) := \omega_k^{(2)}$, $k = \overline{1,n}$, $\pi_n(a_{n+1}) := \omega_n^{(2)}$. The definition of the functions π_k yields

$$|\pi_k(w) - \omega_k^{(1)}| \sim \frac{1}{\alpha_k}|a_k|^{\frac{1}{\alpha_k}-1} \cdot |w - a_k|, \quad w \to a_k, \quad w \in \overline{P}_k,$$
$$|\pi_k(w) - \omega_k^{(2)}| \sim \frac{1}{\alpha_k}|a_{k+1}|^{\frac{1}{\alpha_k}-1} \cdot |w - a_{k+1}|, \quad w \to a_{k+1}, \quad w \in \overline{P}_k,$$

$$|\pi_k(w)| \sim |w|^{\frac{1}{\alpha_k}}, \quad w \to 0, \quad w \in \overline{P}_k,$$
$$|\pi_k(w)| \sim |w|^{\frac{1}{\alpha_k}}, \quad w \to \infty, \quad w \in \overline{P}_k.$$

Using the corresponding results for the separating transformation [6,7], we get the inequalities

$$r\left(B_k, a_k\right) \leqslant \left[\frac{r\left(\Omega_k^{(1)}, \omega_k^{(1)}\right) \cdot r\left(\Omega_{k-1}^{(2)}, \omega_{k-1}^{(2)}\right)}{\frac{1}{\alpha_k}|a_k|^{\frac{1}{\alpha_k}-1} \cdot \frac{1}{\alpha_{k-1}}|a_k|^{\frac{1}{\alpha_{k-1}}-1}}\right]^{\frac{1}{2}}, \tag{4}$$

$$r\left(B_0, 0\right) \leqslant \left[\prod_{k=1}^{n} r^{\alpha_k^2}\left(\Omega_k^{(0)}, 0\right)\right]^{\frac{1}{2}}, \tag{5}$$

$$r\left(B_\infty, \infty\right) \leqslant \left[\prod_{k=1}^{n} r^{\alpha_k^2}\left(\Omega_k^{(\infty)}, \infty\right)\right]^{\frac{1}{2}}. \tag{6}$$

The conditions of realization of the sign of equality in inequalities (4)–(6) are described in [7, p. 29]. On the basis of those relations, we get the inequality

$$J_n(\gamma)\leqslant\Big(\prod_{k=1}^{n}\alpha_k\Big)\prod_{k=1}^{n}\frac{|a_k|^{\frac{1}{\alpha_k}}+|a_{k+1}|^{\frac{1}{\alpha_k}}}{(|a_k||a_{k+1}|)^{\frac{1}{2\alpha_k}}}\cdot|a_k|\times$$

$$\times\left\{\prod_{k=1}^{n}\left(r\left(\Omega_k^{(0)},0\right)r\left(\Omega_k^{(\infty)},\infty\right)\right)^{\gamma\alpha_k^2}\cdot\frac{r\left(\Omega_k^{(1)},\omega_k^{(1)}\right)\cdot r\left(\Omega_k^{(2)},\omega_k^{(2)}\right)}{\left(|a_k|^{\frac{1}{\alpha_k}}+|a_{k+1}|^{\frac{1}{\alpha_k}}\right)^2}\right\}^{\frac{1}{2}}.$$

Further, from the last relation we have

$$J_n(\gamma)\leqslant\left(\prod_{k=1}^{n}\alpha_k\right)\prod_{k=1}^{n}\left(\left|\frac{a_k}{a_{k+1}}\right|^{\frac{1}{2\alpha_k}}+\left|\frac{a_{k+1}}{a_k}\right|^{\frac{1}{2\alpha_k}}\right)|a_k|\times$$

$$\times\left\{\prod_{k=1}^{n}\left(r\left(\Omega_k^{(0)},0\right)r\left(\Omega_k^{(\infty)},\infty\right)\right)^{\gamma\alpha_k^2}\cdot\frac{r\left(\Omega_k^{(1)},\omega_k^{(1)}\right)\cdot r\left(\Omega_k^{(2)},\omega_k^{(2)}\right)}{\left(|a_k|^{\frac{1}{\alpha_k}}+|a_{k+1}|^{\frac{1}{\alpha_k}}\right)^2}\right\}^{\frac{1}{2}},$$

where $|\omega_k^{(1)}|=|a_k|^{\frac{1}{\alpha_k}}$, $|\omega_k^{(2)}|=|a_{k+1}|^{\frac{1}{\alpha_k}}$, $|\omega_k^{(1)}-\omega_k^{(2)}|=|a_k|^{\frac{1}{\alpha_k}}+|a_{k+1}|^{\frac{1}{\alpha_k}}$. Taking into account the fact that

$$\prod_{k=1}^{n}\frac{1}{2}\left(\left|\frac{a_k}{a_{k+1}}\right|^{\frac{1}{2\alpha_k}}+\left|\frac{a_{k+1}}{a_k}\right|^{\frac{1}{2\alpha_k}}\right)|a_k|=\prod_{k=1}^{n}\chi\left(\left|\frac{a_k}{a_{k+1}}\right|^{\frac{1}{2\alpha_k}}\right)|a_k|=\mathcal{L}\left(A_n\right),$$

we obtain the following inequality

$$J_n(\gamma)\leqslant 2^n\cdot\left(\prod_{k=1}^{n}\alpha_k\right)\cdot\mathcal{L}\left(A_n\right)\times$$

$$\times\prod_{k=1}^{n}\left\{\left(r\left(\Omega_k^{(0)},0\right)r\left(\Omega_k^{(\infty)},\infty\right)\right)^{\gamma\alpha_k^2}\cdot\frac{r\left(\Omega_k^{(1)},\omega_k^{(1)}\right)\cdot r\left(\Omega_k^{(2)},\omega_k^{(2)}\right)}{\left(|a_k|^{\frac{1}{\alpha_k}}+|a_{k+1}|^{\frac{1}{\alpha_k}}\right)^2}\right\}^{\frac{1}{2}}.$$

Equality in the last inequality is achieved when equality is realized in the inequalities (4) – (6) for all $k=\overline{1,n}$. Based on the last relation, Theorem 4.1.1 in [1], Corollary 4.1.3 in [1], and the invariance of the functional

$$\left(\frac{r\left(B_1,a_1\right)r\left(B_3,a_3\right)}{|a_1-a_3|^2}\right)^{\gamma}\left(\frac{r\left(B_2,a_2\right)r\left(B_4,a_4\right)}{|a_2-a_4|^2}\right),$$

we have

$$J_n(\gamma) \leqslant \left(\frac{2}{\sqrt{\gamma}}\right)^n \cdot \left(\prod_{k=1}^{n} \alpha_k\sqrt{\gamma}\right) \cdot \mathcal{L}\left(A_n\right) \times$$

$$\times \prod_{k=1}^{n} \left\{ \left(r\left(\widetilde{\Omega}_k^{(0)}, 0\right) r\left(\widetilde{\Omega}_k^{(\infty)}, \infty\right)\right)^{\gamma\alpha_k^2} \cdot \frac{r\left(\widetilde{\Omega}_k^{(1)}, \widetilde{\omega}_k^{(1)}\right) \cdot r\left(\widetilde{\Omega}_k^{(2)}, \widetilde{\omega}_k^{(2)}\right)}{\left(|a_k|^{\frac{1}{\alpha_k}} + |a_{k+1}|^{\frac{1}{\alpha_k}}\right)^2} \right\}^{\frac{1}{2}},$$

where the domains $\widetilde{\Omega}_k^{(0)}$, $\widetilde{\Omega}_k^{(\infty)}$, $\widetilde{\Omega}_k^{(1)}$, $\widetilde{\Omega}_k^{(2)}$ and points 0, ∞, $\widetilde{\omega}_k^{(1)}$, $\widetilde{\omega}_k^{(2)}$, are, respectively, the circular domains and the poles of the quadratic differential

$$Q(z)dz^2 = -\frac{z^4 + 2(1 - \dfrac{2}{\gamma\alpha_k^2})z^2 + 1}{z^2(z^2+1)^2}\, dz^2.$$

Each term in the braces of the last inequality is a value of the functional

$$K_\tau = \left[r\left(B_0, 0\right) r\left(B_\infty, \infty\right)\right]^{\tau^2} \cdot \frac{r\left(B_1, a_1\right) r\left(B_2, a_2\right)}{|a_1 - a_2|^2} \tag{7}$$

on the system of nonoverlapping domains $\{\widetilde{\Omega}_k^{(0)}, \widetilde{\Omega}_k^{(1)}, \widetilde{\Omega}_k^{(2)}, \widetilde{\Omega}_k^{(\infty)}\}$, and the corresponding system of points $\{0, \widetilde{\omega}_k^{(1)}, \widetilde{\omega}_k^{(2)}, \infty\}$ $(k = \overline{1, n})$.

An estimate of the functional (7) in the case of fixed poles was first obtained in [6], and then in the papers [9,15]. On the basis of Lemma 4.1.2 [1], we get the estimate

$$K_\tau \leqslant \Phi(\tau), \quad \tau \geqslant 0,$$

where $\Phi(\tau) = \tau^{2\tau^2} \cdot |1-\tau|^{-(1-\tau)^2} \cdot (1+\tau)^{-(1+\tau)^2}$. Then

$$J_n(\gamma) \leqslant \left(\frac{2}{\sqrt{\gamma}}\right)^n \cdot \left(\prod_{k=1}^{n} \alpha_k\sqrt{\gamma}\right) \left[\prod_{k=1}^{n} \Phi(\tau_k)\right]^{1/2} = \tag{8}$$

$$= \left(\frac{2}{\sqrt{\gamma}}\right)^n \cdot \left[\prod_{k=1}^{n} \left(\tau_k^{2\tau_k^2+2} \cdot |1-\tau_k|^{-(1-\tau_k)^2} \cdot (1+\tau_k)^{-(1+\tau_k)^2}\right)\right]^{\frac{1}{2}},$$

where $\tau_k = \sqrt{\gamma} \cdot \alpha_k$, $k = \overline{1,n}$.

Consider the function $S(x) = x^{2x^2+2} \cdot |1-x|^{-(1-x)^2} \cdot (1+x)^{-(1+x)^2}$. The function $S(x)$ is logarithmically convex on the interval $[0,x_0]$, $x_0 \approx 0.88441$. Now we consider an extremal problem

$$\prod_{k=1}^{n} S(x_k) \longrightarrow \max, \quad \sum_{k=1}^{n} x_k = 2\sqrt{\gamma}, \quad x_k = \alpha_k\sqrt{\gamma}.$$

Let $X^{(0)} = \left\{x_k^{(0)}\right\}_{k=1}^{n}$ be an arbitrary extremal point of the problem. The following result holds (obtained similarly [12]):

$$\Psi'(x_1^{(0)}) = \Psi'(x_2^{(0)}) = \ldots = \Psi'(x_n^{(0)}), \tag{9}$$

where $\Psi'(x) = 4x\ln(x) - 2(x-1)\ln|x-1| - 2(x+1)\ln(x+1) + \frac{2}{x}$ (see Fig. 1).

Further it will be necessary for us to show that the following condition holds:

$$x_1^{(0)} = x_2^{(0)} = \cdots = x_n^{(0)} \quad \text{for all} \quad \gamma \in (0,\gamma_n].$$

Let $\Psi'(x) = t$, $y_0 \leqslant t < 0$, $y_0 \approx -1.06$. We find a solution of equation $\Psi'(x) = t_k$, $k = \overline{1,53}$. Since $\forall t_k \in [y_0, 0)$, it follows that the equation has two solutions $x_1(t) \in (0, x_0]$, $x_2(t) \in (x_0, \infty]$.

Consider the following values of t: $t_1 = -0.02$, $t_2 = -0.04$, $t_3 = -0.06$, $t_4 = -0.08$, $\cdots$, $t_{52} = -1.04$, $t_{53} = y_0$. Direct calculations are presented in Table 1.

Consider the case $n = 2$. From the analysis of the tabular data for $n = 2$, we get that the minimum of the sum $x_1(t_k) + x_2(t_{k+1})$ is achieved for the interval $[-0.62; -0.64]$ and is equal to 1.709336 (see Table 2). The relation $x_1(t) + x_2(t) = 2\sqrt{\gamma}$ holds for each $\gamma \in (0; 0.73]$. Let $\gamma = 0.73$; then the value $2\sqrt{\gamma}$ is less than the minimum 1.709336. Thus, for $n = 2$ and $\gamma \in (0; 0.73]$, we obtain that x_2 does not belong to (x_0, ∞), that is x_1 and x_2 belong to the interval $(0, x_0]$ and $x_1 = x_2$. From inequalities (8) and (9) for $n = 2$, we have

$$J_2(\gamma) \leqslant \frac{4}{\gamma} \cdot S\left(\frac{2\sqrt{\gamma}}{2}\right).$$

For $n = 3$, the minimum of the value $2x_1(t_k) + x_2(t_{k+1})$ on the whole graph is achieved on the interval $[-0.48; -0.50]$ and is equal to 2.381211 (see Table 3). Similarly, $2x_1(t) + x_2(t) = 2\sqrt{\gamma}$. Let $\gamma = 1.41$; then $2\sqrt{\gamma} = 2.3748$. Thus, for $\gamma \in (0; 1.41]$ the situation $x_2 \in (x_0, \infty)$ is not possible. In this way, we obtain $x_1, x_2, x_3 \in (0, x_0]$ and $x_1 = x_2 = x_3$.

Then, taking into account the inequalities (8) and (9) for $n = 3$, we have

$$J_3(\gamma) \leqslant \left(\frac{2}{\sqrt{\gamma}}\right)^3 \left[S\left(\frac{2\sqrt{\gamma}}{3}\right)\right]^{3/2}.$$

Similarly, the situation holds for all $\gamma \in (0,\gamma_n]$, $n = 4, 5, 6$.

k	t_k	$x_1(t_k)$	$x_2(t_k)$	k	t_k	$x_1(t_k)$	$x_2(t_k)$
0	0	0.581421	∞	27	-0.54	0.671495	1.047944
1	-0.02	0.584192	2.607677	28	-0.56	0.675680	1.041549
2	-0.04	0.586996	2.095431	29	-0.58	0.679954	1.035639
3	-0.06	0.589833	1.849825	30	-0.6	0.684325	1.030184
4	-0.08	0.592706	1.696659	31	-0.62	0.688797	1.025157
5	-0.1	0.595614	1.588941	32	-0.64	0.693377	1.020539
6	-0.12	0.598559	1.507710	33	-0.66	0.698072	1.016313
7	-0.14	0.601542	1.443586	34	-0.68	0.702890	1.012468
8	-0.16	0.604564	1.391304	35	-0.7	0.707842	1.008999
9	-0.18	0.607626	1.347643	36	-0.72	0.712936	1.005911
10	-0.2	0.610729	1.310499	37	-0.74	0.718185	1.003228
11	-0.22	0.613876	1.278433	38	-0.76	0.723604	1.001015
12	-0.24	0.617066	1.250421	39	-0.78	0.729208	0.999457
13	-0.26	0.620302	1.225709	40	-0.8	0.735017	0.997390
14	-0.28	0.623585	1.203729	41	-0.82	0.741053	0.994797
15	-0.3	0.626917	1.184045	42	-0.84	0.747345	0.991762
16	-0.32	0.630299	1.166313	43	-0.86	0.753926	0.988295
17	-0.34	0.633734	1.150260	44	-0.88	0.760838	0.984381
18	-0.36	0.637223	1.135664	45	-0.9	0.768138	0.979982
19	-0.38	0.640770	1.122345	46	-0.92	0.775896	0.975038
20	-0.4	0.644375	1.110153	47	-0.94	0.784212	0.969461
21	-0.42	0.648041	1.098962	48	-0.96	0.793228	0.963114
22	-0.44	0.651772	1.088668	49	-0.98	0.803162	0.955787
23	-0.46	0.655569	1.079182	50	-1	0.814378	0.947120
24	-0.48	0.659437	1.070427	51	-1.02	0.827585	0.936407
25	-0.5	0.663378	1.062338	52	-1.04	0.844608	0.921828
26	-0.52	0.667396	1.054860	53	-1.06	0.884406	0.884406

Table 1: Two solutions of the equation $\Psi'(x) = t_k$, $k = \overline{1{,}53}$

From Table 1, for an arbitrary $n \geqslant 7$, the following inequality holds:

$$(n-1)x_1(t_k) + x_2(t_{k+1}) > nx_1(t_k) + (x_2(t_{k+1}) - x_1(t_k)) > 0.58n,$$

since $x_1(t_k) \geqslant 0.5830$ and $x_2(t_{k+1}) - x_1(t_k) \geqslant 0$. Using the condition

$$(n-1)x_1(t) + x_2(t) = 2\sqrt{\gamma_n},$$

we assume that $2\sqrt{\gamma_n} = 0.58n$. Thus, $\gamma_n = 0.084n^2$, that is, when $\gamma \in (0; 0.084n^2]$ then the sum $(n-1)x_1(t) + x_2(t)$ is less than $0.58n$. Thus, for $n \geqslant 7$ and $\gamma \in (0,\gamma_n]$, we obtain

$$J_n(\gamma) \leqslant \left(\frac{2}{\sqrt{\gamma}}\right)^n \left[S\left(\frac{2\sqrt{\gamma}}{n}\right)\right]^{n/2}.$$

k	t_k	$x_1(t_k)+x_2(t_{k+1})$	k	t_k	$x_1(t_k)+x_2(t_{k+1})$
0	0		27	-0.54	1.715340
1	-0.02	3.189098	28	-0.56	1.713044
2	-0.04	2.679623	29	-0.58	1.711318
3	-0.06	2.436820	30	-0.6	1.710138
4	-0.08	2.286492	31	-0.62	1.709482
5	-0.1	2.181647	32	-0.64	**1.709336**
6	-0.12	2.103324	33	-0.66	1.709690
7	-0.14	2.042145	34	-0.68	1.710540
8	-0.16	1.992846	35	-0.7	1.711889
9	-0.18	1.952207	36	-0.72	1.713753
10	-0.2	1.918125	37	-0.74	1.716163
11	-0.22	1.889163	38	-0.76	1.719200
12	-0.24	1.864297	39	-0.78	1.723061
13	-0.26	1.842775	40	-0.8	1.726598
14	-0.28	1.824031	41	-0.82	1.729814
15	-0.3	1.807630	42	-0.84	1.732815
16	-0.32	1.793230	43	-0.86	1.735640
17	-0.34	1.780559	44	-0.88	1.738307
18	-0.36	1.769398	45	-0.9	1.740820
19	-0.38	1.759569	46	-0.92	1.743176
20	-0.4	1.750923	47	-0.94	1.745356
21	-0.42	1.743337	48	-0.96	1.747326
22	-0.44	1.736709	49	-0.98	1.749015
23	-0.46	1.730953	50	-1	1.750281
24	-0.48	1.725996	51	-1.02	1.750785
25	-0.5	1.721775	52	-1.04	1.749413
26	-0.52	1.718238	53	-1.06	1.729015

Table 2: Minimum of the sum $x_1(t_k)+x_2(t_{k+1})$, $k=\overline{1{,}53}$

The equality case is straightforward to verify. Theorem 1 is proved. □

From Theorem 1, we obtain the following results.

Corollary 1. *Let $n\in\mathbb{N}$, $n\geqslant 2$, $\gamma\in(0,\gamma_n]$, $\gamma_2=0.7304$, $\gamma_3=1.4175$, $\gamma_4=2.2983$, $\gamma_5=3.3683$, $\gamma_6=4.6244$, and $\gamma_n=0.084\,n^2$, $n\geqslant 7$. Then for any n-radial system of points $A_n=\{a_k\}_{k=1}^n$ such that $\mathcal{L}(A_n)=$ $=1$, and any system of mutually non-overlapping domains B_0, B_∞, B_k, $a_0=0\in B_0\subset\overline{\mathbb{C}}$, $\infty\in B_\infty\subset\overline{\mathbb{C}}$, $a_k\in B_k\subset\overline{\mathbb{C}}$, $k=\overline{1,n}$, the inequality (2) holds. Equality is attained in the same case as in Theorem 1.*

Corollary 2. *Under the conditions of Theorem 1, the following inequal-*

ity holds:

$$[r(B_0,0)r(B_\infty,\infty)]^\gamma \prod_{k=1}^{n} r(B_k,a_k) \leqslant \left(\frac{4}{n}\right)^n \frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{2\gamma}{n}}}{\left|1-\frac{4\gamma}{n^2}\right|^{\frac{2\gamma}{n}+\frac{n}{2}}} \left|\frac{n-2\sqrt{\gamma}}{n+2\sqrt{\gamma}}\right|^{2\sqrt{\gamma}}. \quad (10)$$

Equality in this inequality is achieved when 0, ∞, a_k *and* B_0, B_∞, B_k, $k=\overline{1,n}$, *are, respectively, poles and circular domains of the quadratic differential* (3).

k	t_k	$2x_1(t_k)+x_2(t_{k+1})$	k	t_k	$2x_1(t_k)+x_2(t_{k+1})$
0	0		27	-0.54	2.382735
1	-0.02	3.770519	28	-0.56	2.384539
2	-0.04	3.263814	29	-0.58	2.386998
3	-0.06	3.023816	30	-0.6	2.390093
4	-0.08	2.876325	31	-0.62	2.393807
5	-0.1	2.774353	32	-0.64	2.398133
6	-0.12	2.698938	33	-0.66	2.403067
7	-0.14	2.640704	34	-0.68	2.408612
8	-0.16	2.594388	35	-0.7	2.414780
9	-0.18	2.556771	36	-0.72	2.421594
10	-0.2	2.525751	37	-0.74	2.429099
11	-0.22	2.499892	38	-0.76	2.437386
12	-0.24	2.478172	39	-0.78	2.446665
13	-0.26	2.459841	40	-0.8	2.455806
14	-0.28	2.444333	41	-0.82	2.464831
15	-0.3	2.431215	42	-0.84	2.473869
16	-0.32	2.420146	43	-0.86	2.482985
17	-0.34	2.410858	44	-0.88	2.492232
18	-0.36	2.403133	45	-0.9	2.501659
19	-0.38	2.396792	46	-0.92	2.511314
20	-0.4	2.391692	47	-0.94	2.521252
21	-0.42	2.387712	48	-0.96	2.531538
22	-0.44	2.384750	49	-0.98	2.542243
23	-0.46	2.382725	50	-1	2.553443
24	-0.48	2.381565	51	-1.02	2.565162
25	-0.5	**2.381211**	52	-1.04	2.576998
26	-0.52	2.381615	53	-1.06	2.573623

Table 3: Minimum of the sum $2x_1(t_k)+x_2(t_{k+1})$, $k=\overline{1,53}$

Corollary 3. *Let* $n\in\mathbb{N}$, $n\geqslant 2$, $\gamma\in(0,\gamma_n]$, $\gamma_2=0.7304$, $\gamma_3=1.4175$, $\gamma_4=2.2983$, $\gamma_5=3.3683$, $\gamma_6=4.6244$, *and* $\gamma_n=0.084\,n^2$, $n\geqslant 7$. *Then,* *for any other points of the unit circle $|w| = 1$ and any set of mutually non-overlapping domains B_0, B_∞, B_k, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1, n}$, the inequality (2) holds. Equality is attained in the same case as in Theorem 1.*

If we consider a sufficiently strict restriction on the distribution of the angles α_k, $k = \overline{1, n}$, then we can get a stronger result.

Let $y_0 \approx 0.884414$ be a root of the equation

$$\ln \frac{y^2}{1 - y^2} = \frac{1}{y^2}. \tag{11}$$

Then the following proposition is true.

Theorem 2. *Let $n \in \mathbb{N}$, $n \geqslant 2$, $\gamma \in (0, \gamma_n)$, $\gamma_n = \frac{1}{4} y_0^2 n^2$. Then for any n-radial system of points $A_n = \{a_k\}_{k=1}^n$ such that $\mathcal{L}(A_n) = 1$, $0 < \alpha_k \leqslant y_0/\sqrt{\gamma}$, where y_0 is a root of equation (11), $k = \overline{1, n}$, and for any collection of pairwise nonoverlapping domains B_0, B_∞, B_k, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1, n}$, the inequality (10) holds. Equality is attained in the same case as in Corollary 2.*

Proof. The proof of Theorem 2 practically repeats the proof of Theorem 1, only the logarithmic convexity of the function $S(x)$ on the segment $(0, y_0]$ and relation below are used in the final stage of the proof. The reation is

$$\frac{1}{n} \sum_{k=1}^{n} \ln S(x_k) \leqslant \ln S\left(\frac{\sum\limits_{k=1}^{n} x_k}{n}\right).$$

It is equivalent to

$$\ln \left(\prod_{k=1}^{n} S(x_k)\right)^{\frac{1}{n}} \leqslant \ln \left(S\left(\frac{2}{n}\sqrt{\gamma}\right)\right).$$

Equality in this inequality is attained if

$$\tau_1 = \tau_2 = \ldots = \tau_n = \frac{2\sqrt{\gamma}}{n},$$

i. e., if $\alpha_k = \frac{2}{n}$, $k = \overline{1,n}$. In this case, relation (7) yields

$$J_n(\gamma) \leqslant J_n^0(\gamma) = \left(\frac{4}{n}\right)^n \left[\left(r(D_0, 0)\, r(D_\infty, \infty)\right)^{\frac{4\gamma}{n^2}} \cdot \frac{r(D_1, -i)\, r(D_2, i)}{|(-i) - i|^2} \right]^{\frac{n}{2}},$$

where D_0, D_∞, D_1 and D_2 are the circular domains of the quadratic differential

$$Q(z)dz^2 = -\frac{\frac{4\gamma}{n^2}z^4 + 2\left(\frac{4\gamma}{n^2} - 2\right)z^2 + \frac{4\gamma}{n^2}}{z^2(z^2+1)^2}\,dz^2. \tag{12}$$

From whence, we have, eventually,

$$J_n(\gamma) \leqslant \left(\frac{2}{\sqrt{\gamma}}\right)^n \left[S\left(\frac{2}{n}\sqrt{\gamma}\right)\right]^{\frac{n}{2}}.$$

Using a specific formula for $S(x)$, we get the basic inequality of Theorem 2. Changing the variable in (12) by the formula $z = -iw^{\frac{n}{2}}$, we get the quadratic differential (3). The sign of equality in inequality (10) is verified directly. Theorem 2 is proved. $\square$

Corollary 4. *Let $n \in \mathbb{N}$, $n \geqslant 2$, $\gamma \in (0, \gamma_n]$, $\gamma_n = 0.19n^2$. Then for any n-radial system of points $A_n = \{a_k\}_{k=1}^n$ such that $\mathcal{L}(A_n) = 1$, $0 < \alpha_k \leqslant y_0/\sqrt{\gamma}$, $y_0 \approx 0.88441$, $k = \overline{1,n}$, and any set of mutually non-overlapping domains B_0, B_∞, B_k, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, the inequality (2) holds. Equality is attained in the same case as in Theorem 1.*

Consider the following problem, which was formulated as an open problem in the case $\gamma = 1$ in the paper by Dubinin [7].

Problem 2. Find, for any fixed value of $\gamma \in (0, n]$, the maximum of the functional

$$r^\gamma(B_0, 0)\prod_{k=1}^{n} r(B_k, a_k),$$

where B_0, B_1, B_2,..., B_n, $n \geqslant 2$, is any system of pairwise non-overlapping domains in $\overline{\mathbb{C}}$, where the domains $B_1, \dots, B_n$ have symmetry with respect to the unit circle, $a_0 = 0$, $|a_k| = 1$, $k = \overline{1,n}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{0,n}$; describe all extremals of the functional.

This problem was solved for $\gamma = 1$ and $n \geqslant 2$ by Kovalev [13, 14]. The following theorem substantially complements the results of the papers [4, 13, 14]. We obtain the following results assuming that $B_0 \subset U$ (here U denotes the unit circle).

Theorem 3. *Let $n \in \mathbb{N}$, $n \geqslant 2$, $\gamma \in (0, \gamma_n)$, $\gamma_n = \frac{1}{2}y_0^2 n^2$. Then, for any n-radial system of points $A_n = \{a_k\}_{k=1}^n$, such that $|a_k| = 1$,*

$0 < \alpha_k \leqslant y_0/\sqrt{\gamma}$, where y_0 is a root of equation (11), $k = \overline{1,n}$, and any set of mutually non-overlapping domains B_0, B_k, $a_0 = 0 \in B_0 \subset U$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, where the domains B_k have symmetry with respect to the unit circle $|w| = 1$ for all $k = \overline{1,n}$, the following inequality holds:

$$r^{\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right) \leqslant r^{\gamma}\left(\Lambda_0, 0\right) \prod_{k=1}^{n} r\left(\Lambda_k, \lambda_k\right). \tag{13}$$

Equality in (13) is attained when 0, λ_k and Λ_0, Λ_k, $k = \overline{1,n}$, are, respectively, the poles and the circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{\gamma w^{2n} + 2(n^2-\gamma)w^n + \gamma}{w^2(w^n-1)^2}\, dw^2. \tag{14}$$

Proof. Note (see [6, p.59]) that if the domains B_k have symmetry with respect to the unit circle $|w| = 1$ for all $k = \overline{1,n}$, and the domain $B_0 \subset U$, then the extremal problem for the functional $r^{\gamma}\left(B_0, 0\right) \prod\limits_{k=1}^{n} r\left(B_k, a_k\right)$ can be reduced, by easy transformations, to the study of the functional $r^{\gamma/2}\left(B_0, 0\right) r^{\gamma/2}\left(B_{\infty}, \infty\right) \prod\limits_{k=1}^{n} r\left(B_k, a_k\right)$. Thus, using this property and proofs of Theorem 1 and Theorem 2, we obtain the result of Theorem 3. □

Using Corollary 3 and proofs of Theorem 3 and Theorem 1, it is not difficult to obtain the following result.

Theorem 4. *Let $n \in \mathbb{N}$, $\gamma \in (0, \gamma_n]$, $\gamma_2 = 1.4608$, $\gamma_3 = 2.8350$, $\gamma_4 = 4.5966$, $\gamma_5 = 6.7366$, $\gamma_6 = 9.2488$, $\gamma_n = 0.168\, n^2$, $n \geqslant 7$. Then, for any other points of the unit circle $|w| = 1$ and any system of mutually non-overlapping domains B_0, B_k, $a_0 = 0 \in B_0 \subset U$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, where the domains B_k have symmetry with respect to the unit circle $|w| = 1$ for all $k = \overline{1,n}$, the following inequality holds:*

$$r^{\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right) \leqslant \left(\frac{4}{n}\right)^{n} \frac{\left(\frac{2\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left|1-\frac{2\gamma}{n^2}\right|^{\frac{n}{2}+\frac{\gamma}{n}}} \left|\frac{n-\sqrt{2\gamma}}{n+\sqrt{2\gamma}}\right|^{\sqrt{2\gamma}}.$$

Equality in the inequality is achieved when a_k and B_k, $k = \overline{0,n}$, are, respectively, poles and circular domains of the quadratic differential (14).

References

[1] Bakhtin A. K. , Bakhtina G. P. , Zelinskii Yu. B. *Topological-algebraic structures and geometric methods in complex analysis.* Zb. prats of the Inst. of Math. of NASU, 2008. (in Russian) DOI: https://doi.org/10.13140/RG.2.1.1660.6242.

[2] Bakhtin A. K., Denega I. V. *Some estimates of the functionals for n-radial systems of points.* Zb. prats of the Inst. of Math. of NASU, 2011, vol. 8, no. 1, pp. 12 – 21. (in Russian)

[3] Bakhtina G. P., Dvorak I. Y., Denega I. V. *About the product of inner radii of pairwise non-overlapping domains.* Dopov. Nac. akad. nauk Ukr., 2016, no. 1, pp. 7 – 11. DOI: https://doi.org/10.15407/dopovidi2016.01.007.

[4] Bakhtina G. P. *On the conformal radii of symmetric nonoverlapping regions.* Modern issues of material and complex analysis, K.: Inst. Math. of NAS of Ukraine, 1984, pp. 21 – 27. (in Russian)

[5] Denega I. V. *Some inequalities for inner radii of partially non-overlapping domains.* Dopov. Nac. akad. nauk Ukr., 2012, no. 5, pp. 19 – 22. (in Russian)

[6] Dubinin V. N. *Separating transformation of domains and problems on extremal decomposition.* Notes scientific. sem. Leningr. Dep. of Math. Inst. AN USSR., 1988, vol. 168, pp. 48 – 66. (in Russian); translation in J. Soviet Math., 1991, vol. 53, no. 3, pp. 252 – 263. DOI: https://doi.org/10.1007/BF01303649.

[7] Dubinin V. N. *Symmetrization method in geometric function theory of complex variables.* Successes Mat. Science, 1994, vol. 49, no. 1(295), pp. 3 – 76. (in Russian); translation in Russian Math. Surveys. 1994, vol. 1, pp. 1 – 79. DOI: https://doi.org/10.1070/RM1994v049n01ABEH002002.

[8] Dubinin V. N. *Condenser capacities and symmetrization in geometric function theory.* Birkhäuser/Springer, Basel, 2014. DOI: : https://doi.org/10.1007/978-3-0348-0843-9.

[9] Emelyanov E. G. *On the Problem of Maximizing the Product of Powers of Conformal Radii Nonoverlapping Domains.* J. Math. Sci. (N.Y.), 2004, vol. 122, no. 6, pp. 3641 – 3647. DOI: https://doi.org/10.1023/B:JOTH.0000035239.55516.08.

[10] Goluzin G. M. *Geometric theory of functions of a complex variable.* Amer. Math. Soc. Providence, R.I., 1969.

[11] Jenkins J. *Univalent functions and conformal mapping.* Moscow:Publishing House of Foreign Literature, **256**, 1962. (in Russian) DOI: : https://doi.org/10.1007/978-3-642-88563-1.

[12] Kovalev L. V. *On the problem of extremal decomposition with free poles on a circle.* Dal'nevost. Mat. Zh., 1996, no. 2, pp. 96 – 98. (in Russian)

[13] Kovalev L. V. *On the inner radii of symmetric nonoverlapping domains.* Izv. Vyssh. Uchebn. Zaved. Mat., 2000, no. 6, pp. 77 – 78. (in Russian)

[14] Kovalev L. V. *On three nonoverlapping domains.* Dal'nevost. Mat. Zh., 2000, no. 1, pp. 3 – 7. (in Russian)

[15] Kuzmina G. V. *Problems on extremal decomposition of the riemann sphere. Notes scientific.* Sem. Leningr. Dep. of Math. Inst. AN USSR., 2001, vol. 276, pp. 253–275. (in Russian); translation in J. Math. Sci. (N.Y.), 2003, vol. 118, no. 1, pp. 4880 – 4894. DOI: https://doi.org/10.1023/A:1025580802209.

[16] Lavrent'ev M. A. *On the theory of conformal mappings.* Tr. Sci. Inst An USSR, 1934, vol. 5, pp. 159 – 245. (in Russian)

Received September 19, 2018.
In revised form, September 21, 2018.
Accepted December 28, 2018.
Published online January 11, 2019.

Institute of Mathematics of the National Academy of Sciences of Ukraine
Department of complex analysis and potential theory
01004 Ukraine, Kiev-4, 3, Tereschenkivska st.
A. K. Bakhtin
E-mail: abahtin@imath.kiev.ua
I. V. Denega
E-mail: iradenega@gmail.com