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SHARP ESTIMATES OF PRODUCTS OF INNER RADII OF NON-OVERLAPPING DOMAINS IN THE COMPLEX PLANE

Abstract. In the paper we study a generalization of the extremal problem of geometric theory of functions of a complex variable on non-overlapping domains with free poles: Fix any $\gamma \in \mathbb{R}^+$ and find the maximum (and describe all extremals) of the functional

$$[r(B_0, 0)\, r(B_\infty, \infty)]^\gamma \prod_{k=1}^{n} r(B_k, a_k),$$

where $n \in \mathbb{N}$, $n \geqslant 2$, $a_0 = 0$, $|a_k| = 1$, B_0, B_∞, $\{B_k\}_{k=1}^n$ is a system of mutually non-overlapping domains, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{0, n}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, ($r(B, a)$ is an inner radius of the domain $B \subset \overline{\mathbb{C}}$ at $a \in B$). Instead of the classical condition that the poles are on the unit circle, we require that the system of free poles is an n-radial system of points normalized by some "control" functional. A partial solution of this problem is obtained.

Key words: *inner radius of a domain, non-overlapping domains, radial system of points, separating transformation, quadratic differential, Green's function*

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Let $\mathbb{N}$, $\mathbb{R}$ be the sets of natural and real numbers, respectively, $\mathbb{C}$ be the complex plane, $\overline{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}$ be a one-point compactification, and $\mathbb{R}^+ = (0, \infty)$. Let $\chi(t) = \frac{1}{2}(t + t^{-1})$, $t \in \mathbb{R}^+$, be the Zhukovskii function. Let $r(B, a)$ be an inner radius of the domain $B \subset \overline{\mathbb{C}}$ relative to the point $a \in B$.

The system of points $A_n := \{a_k \in \mathbb{C}, k = \overline{1, n}\}$, $n \in \mathbb{N}$, $n \geqslant 2$ is called n-radial, if $|a_k| \in \mathbb{R}^+$ for $k = \overline{1, n}$ and $0 = \arg a_1 < \ldots < \arg a_n < 2\pi$.

Denote

$$P_k = P_k(A_n) := \{w : \arg a_k < \arg w < \arg a_{k+1}\}, \quad a_{n+1} := a_1,$$

$$\alpha_k := \frac{1}{\pi} \arg \frac{a_{k+1}}{a_k}, \quad \alpha_{n+1} := \alpha_1, \quad k = \overline{1,n}, \quad \sum_{k=1}^{n} \alpha_k = 2.$$

For any n-radial system of points $A_n = \{a_k\}$, $k = \overline{1,n}$, we introduce the "control" functional

$$\mathcal{L}(A_n) := \prod_{k=1}^{n} \chi\left(\left|\frac{a_k}{a_{k+1}}\right|^{\frac{1}{2\alpha_k}}\right) \cdot |a_k|.$$

The class of n-radial systems of points for which $\mathcal{L}(A_n) = 1$ contains automatically all systems of n different points of the unit circle.

Consider the following extremal problem.

Problem 1. For any fixed value of $\gamma \in \mathbb{R}^+$, find the maximum of the functional

$$J_n(\gamma) = \left[r\left(B_0, 0\right) r\left(B_\infty, \infty\right)\right]^\gamma \prod_{k=1}^{n} r\left(B_k, a_k\right), \tag{1}$$

where $n \in \mathbb{N}$, $n \geqslant 2$, $a_0 = 0$, $A_n = \{a_k\}_{k=1}^n$ are n-radial systems of points, such that $\mathcal{L}\left(A_n\right) = 1$, B_0, B_∞, $\{B_k\}_{k=1}^n$ is a system of mutually non-overlapping domains, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{0,n}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$; also, describe all extremals.

This problem belongs to the class of extremal problems with free poles. Problems of this type have been studied in many papers (see, for example, [1–16]). For $\gamma = \frac{1}{2}$ and $n \geqslant 2$, an estimate of the functional $J_n(\gamma)$ for the system of non-overlapping domains was found in the paper [6, p. 59]. Kuz'mina [15, p. 267] strengthened this result for simply connected domains and showed that the estimate is correct for $\gamma \in \left(0, \frac{n^2}{8}\right]$, $n \geqslant 2$. Note that for $n = 2$ the Kuz'mina's estimate of the functional (1) coincides with the Dubinin's estimate. Some partial cases of the above-posed problem were considered in [2, 3, 5].

Let

$$S(x) = x^{2x^2+2} \cdot |1-x|^{-(1-x)^2} \cdot (1+x)^{-(1+x)^2} \quad \text{and} \quad \Psi(x) = \ln(S(x)).$$

$$\Psi'(x) = 4x\ln(x) - 2(x-1)\ln|x-1| - 2(x+1)\ln(x+1) + \frac{2}{x} \text{ (see Fig. 1)}.$$

The function $S(x)$ is logarithmically convex on the interval $[0, x_0]$, $x_0 \approx 0.88441$. Let $\Psi'(x) = t$, $y_0 \leqslant t < 0$, $y_0 \approx -1.06$. The equation $\Psi'(x) = t_k$ has two solutions $x_1(t) \in (0, x_0]$ and $x_2(t) \in (x_0, \infty]$.

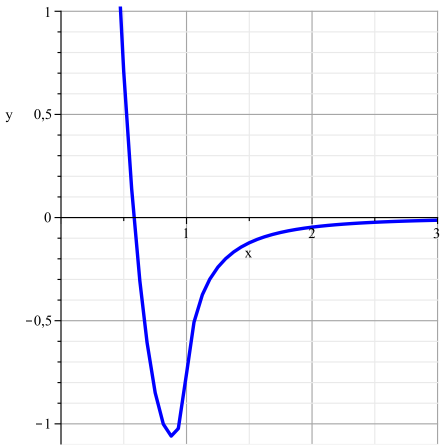

Figure 1: The function plot $y = \Psi'(x)$

Let $\delta_n^0 = \min\left((n-1)x_1(t) + x_2(t)\right) = 2\sqrt{\gamma_n^0}$, then $\gamma_n^0 = \left(\frac{\delta_n^0}{2}\right)^2$. Then the following proposition is true.

Theorem 1. *Let $n \in \mathbb{N}$, $n \geqslant 2$, $\gamma \in (0, \gamma_n^0]$, $\gamma_n^0 = \left(\frac{\delta_n^0}{2}\right)^2$. Then, for any n-radial system of points $A_n = \{a_k\}_{k=1}^n$ such that $\mathcal{L}(A_n) = 1$, and any system of mutually non-overlapping domains B_0, B_∞, B_k, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, the following inequality holds:*

$$[r(B_0, 0)\, r(B_\infty, \infty)]^\gamma \prod_{k=1}^{n} r(B_k, a_k) \leqslant [r(\Lambda_0, 0)\, r(\Lambda_\infty, \infty)]^\gamma \prod_{k=1}^{n} r(\Lambda_k, \lambda_k), \tag{2}$$

where the domains Λ_0, Λ_∞, Λ_k, and the points 0, ∞, λ_k, $k = \overline{1,n}$, are, respectively, circular domains and poles of the quadratic differential

$$Q(w)dw^2 = -\frac{\gamma w^{2n} + (n^2 - 2\gamma)w^n + \gamma}{w^2(w^n - 1)^2}\, dw^2. \tag{3}$$

Proof. Let $\zeta = \pi_k(w)$ denote a univalent branch of the multivalent analytic function $-i\left(e^{-i\arg a_k} w\right)^{\frac{1}{\alpha_k}}$, $k = \overline{1,n}$, that maps P_k onto the right half-plane $\operatorname{Re}\zeta > 0$ conformally in the one-sheet way. Consider the system of functions $\zeta = \pi_k(w) = -i\left(e^{-i\arg a_k} w\right)^{\frac{1}{\alpha_k}}$, $k = \overline{1,n}$. Let $\Omega_k^{(1)}$, $k = \overline{1,n}$, denote a domain of the plane $\mathbb{C}_\zeta$, obtained as a result of the union of the connected component of the set $\pi_k(B_k \bigcap \overline{P}_k)$, containing the point $\pi_k(a_k)$, with its symmetric reflection with respect to the imaginary axis. In turn, by $\Omega_k^{(2)}$, $k=\overline{1,n}$, we denote the domain of the plane $\mathbb{C}_\zeta$, obtained as a result of the union of the connected component of the set $\pi_k(B_{k+1} \bigcap \overline{P}_k)$, containing the point $\pi_k(a_{k+1})$, with its symmetric reflection with respect to the imaginary axis, $B_{n+1} := B_1$, $\pi_n(a_{n+1}) := \pi_n(a_1)$. In addition, $\Omega_k^{(0)}$ denotes a domain of the plane $\mathbb{C}_\zeta$ obtained as a result of the union of the connected component of the set $\pi_k(B_0 \bigcap \overline{P}_k)$, containing the point $\zeta=0$, with its symmetric reflection with respect to the imaginary axis. Similarly, $\Omega_k^{(\infty)}$ denotes a domain of the plane $\mathbb{C}_\zeta$ obtained as a result of the union of the connected component of the set $\pi_k(B_\infty \bigcap \overline{P}_k)$, containing the point $\zeta=\infty$, with its symmetric reflection with respect to the imaginary axis. It is clear that $\pi_k(a_k) := \omega_k^{(1)}$, $\pi_k(a_{k+1}) := \omega_k^{(2)}$, $k=\overline{1,n}$, $\pi_n(a_{n+1}) := \omega_n^{(2)}$. The definition of the functions π_k yields

$$|\pi_k(w)-\omega_k^{(1)}| \sim \frac{1}{\alpha_k}|a_k|^{\frac{1}{\alpha_k}-1}\cdot|w-a_k|, \quad w\to a_k, \quad w\in\overline{P}_k,$$
$$|\pi_k(w)-\omega_k^{(2)}| \sim \frac{1}{\alpha_k}|a_{k+1}|^{\frac{1}{\alpha_k}-1}\cdot|w-a_{k+1}|, \quad w\to a_{k+1}, \quad w\in\overline{P}_k,$$

$$|\pi_k(w)| \sim |w|^{\frac{1}{\alpha_k}}, \quad w\to 0, \quad w\in\overline{P}_k,$$
$$|\pi_k(w)| \sim |w|^{\frac{1}{\alpha_k}}, \quad w\to \infty, \quad w\in\overline{P}_k.$$

Using the corresponding results for the separating transformation [6,7], we get the inequalities

$$r\left(B_k,a_k\right) \leqslant \left[\frac{r\left(\Omega_k^{(1)},\omega_k^{(1)}\right)\cdot r\left(\Omega_{k-1}^{(2)},\omega_{k-1}^{(2)}\right)}{\frac{1}{\alpha_k}|a_k|^{\frac{1}{\alpha_k}-1}\cdot\frac{1}{\alpha_{k-1}}|a_k|^{\frac{1}{\alpha_{k-1}}-1}}\right]^{\frac{1}{2}}, \tag{4}$$

$$r\left(B_0,0\right) \leqslant \left[\prod_{k=1}^{n} r^{\alpha_k^2}\left(\Omega_k^{(0)},0\right)\right]^{\frac{1}{2}}, \tag{5}$$

$$r\left(B_\infty,\infty\right) \leqslant \left[\prod_{k=1}^{n} r^{\alpha_k^2}\left(\Omega_k^{(\infty)},\infty\right)\right]^{\frac{1}{2}}. \tag{6}$$

The conditions of realization of the sign of equality in inequalities (4)–(6) are described in [7, p. 29]. On the basis of those relations, we get the inequality

$$J_n(\gamma) \leqslant \left(\prod_{k=1}^{n} \alpha_k\right) \prod_{k=1}^{n} \frac{|a_k|^{\frac{1}{\alpha_k}} + |a_{k+1}|^{\frac{1}{\alpha_k}}}{(|a_k||a_{k+1}|)^{\frac{1}{2\alpha_k}}} \cdot |a_k| \times$$

$$\times \left\{ \prod_{k=1}^{n} \left(r\left(\Omega_k^{(0)}, 0\right) r\left(\Omega_k^{(\infty)}, \infty\right)\right)^{\gamma\alpha_k^2} \cdot \frac{r\left(\Omega_k^{(1)}, \omega_k^{(1)}\right) \cdot r\left(\Omega_k^{(2)}, \omega_k^{(2)}\right)}{\left(|a_k|^{\frac{1}{\alpha_k}} + |a_{k+1}|^{\frac{1}{\alpha_k}}\right)^2} \right\}^{\frac{1}{2}}.$$

Further, from the last relation we have

$$J_n(\gamma) \leqslant \left(\prod_{k=1}^{n} \alpha_k\right) \prod_{k=1}^{n} \left(\left|\frac{a_k}{a_{k+1}}\right|^{\frac{1}{2\alpha_k}} + \left|\frac{a_{k+1}}{a_k}\right|^{\frac{1}{2\alpha_k}} \right) |a_k| \times$$

$$\times \left\{ \prod_{k=1}^{n} \left(r\left(\Omega_k^{(0)}, 0\right) r\left(\Omega_k^{(\infty)}, \infty\right)\right)^{\gamma\alpha_k^2} \cdot \frac{r\left(\Omega_k^{(1)}, \omega_k^{(1)}\right) \cdot r\left(\Omega_k^{(2)}, \omega_k^{(2)}\right)}{\left(|a_k|^{\frac{1}{\alpha_k}} + |a_{k+1}|^{\frac{1}{\alpha_k}}\right)^2} \right\}^{\frac{1}{2}},$$

where $|\omega_k^{(1)}| = |a_k|^{\frac{1}{\alpha_k}}$, $|\omega_k^{(2)}| = |a_{k+1}|^{\frac{1}{\alpha_k}}$, $|\omega_k^{(1)} - \omega_k^{(2)}| = |a_k|^{\frac{1}{\alpha_k}} + |a_{k+1}|^{\frac{1}{\alpha_k}}$. Taking into account the fact that

$$\prod_{k=1}^{n} \frac{1}{2} \left(\left|\frac{a_k}{a_{k+1}}\right|^{\frac{1}{2\alpha_k}} + \left|\frac{a_{k+1}}{a_k}\right|^{\frac{1}{2\alpha_k}} \right) |a_k| = \prod_{k=1}^{n} \chi\left(\left|\frac{a_k}{a_{k+1}}\right|^{\frac{1}{2\alpha_k}} \right) |a_k| = \mathcal{L}\left(A_n\right),$$

we obtain the following inequality

$$J_n(\gamma) \leqslant 2^n \cdot \left(\prod_{k=1}^{n} \alpha_k\right) \cdot \mathcal{L}\left(A_n\right) \times$$

$$\times \prod_{k=1}^{n} \left\{ \left(r\left(\Omega_k^{(0)}, 0\right) r\left(\Omega_k^{(\infty)}, \infty\right)\right)^{\gamma\alpha_k^2} \cdot \frac{r\left(\Omega_k^{(1)}, \omega_k^{(1)}\right) \cdot r\left(\Omega_k^{(2)}, \omega_k^{(2)}\right)}{\left(|a_k|^{\frac{1}{\alpha_k}} + |a_{k+1}|^{\frac{1}{\alpha_k}}\right)^2} \right\}^{\frac{1}{2}}.$$

Equality in the last inequality is achieved when equality is realized in the inequalities (4)–(6) for all $k = \overline{1, n}$. Based on the last relation, Theorem 4.1.1 in [1], Corollary 4.1.3 in [1], and the invariance of the functional

$$\left(\frac{r\left(B_1, a_1\right) r\left(B_3, a_3\right)}{|a_1 - a_3|^2}\right)^{\gamma} \left(\frac{r\left(B_2, a_2\right) r\left(B_4, a_4\right)}{|a_2 - a_4|^2}\right),$$

we have

$$J_n(\gamma) \leqslant \left(\frac{2}{\sqrt{\gamma}}\right)^n \cdot \Big(\prod_{k=1}^{n} \alpha_k \sqrt{\gamma}\Big) \cdot \mathcal{L}\left(A_n\right) \times$$

$$\times \prod_{k=1}^{n} \left\{ \left(r\left(\widetilde{\Omega}_k^{(0)}, 0\right) r\left(\widetilde{\Omega}_k^{(\infty)}, \infty\right)\right)^{\gamma\alpha_k^2} \cdot \frac{r\left(\widetilde{\Omega}_k^{(1)}, \widetilde{\omega}_k^{(1)}\right) \cdot r\left(\widetilde{\Omega}_k^{(2)}, \widetilde{\omega}_k^{(2)}\right)}{\left(|a_k|^{\frac{1}{\alpha_k}} + |a_{k+1}|^{\frac{1}{\alpha_k}}\right)^2} \right\}^{\frac{1}{2}},$$

where the domains $\widetilde{\Omega}_k^{(0)}$, $\widetilde{\Omega}_k^{(\infty)}$, $\widetilde{\Omega}_k^{(1)}$, $\widetilde{\Omega}_k^{(2)}$ and points 0, ∞, $\widetilde{\omega}_k^{(1)}$, $\widetilde{\omega}_k^{(2)}$, are, respectively, the circular domains and the poles of the quadratic differential

$$Q(z)dz^2 = -\frac{z^4 + 2(1 - \frac{2}{\gamma\alpha_k^2})z^2 + 1}{z^2(z^2+1)^2}\, dz^2.$$

Each term in the braces of the last inequality is a value of the functional

$$K_\tau = \left[r\left(B_0, 0\right) r\left(B_\infty, \infty\right)\right]^{\tau^2} \cdot \frac{r\left(B_1, a_1\right) r\left(B_2, a_2\right)}{|a_1 - a_2|^2} \tag{7}$$

on the system of nonoverlapping domains $\{\widetilde{\Omega}_k^{(0)}, \widetilde{\Omega}_k^{(1)}, \widetilde{\Omega}_k^{(2)}, \widetilde{\Omega}_k^{(\infty)}\}$, and the corresponding system of points $\{0, \widetilde{\omega}_k^{(1)}, \widetilde{\omega}_k^{(2)}, \infty\}$ $(k = \overline{1, n})$.

An estimate of the functional (7) in the case of fixed poles was first obtained in [6], and then in the papers [9,15]. On the basis of Lemma 4.1.2 [1], we get the estimate

$$K_\tau \leqslant \Phi(\tau), \quad \tau \geqslant 0,$$

where $\Phi(\tau) = \tau^{2\tau^2} \cdot |1-\tau|^{-(1-\tau)^2} \cdot (1+\tau)^{-(1+\tau)^2}$. Then

$$J_n(\gamma) \leqslant \left(\frac{2}{\sqrt{\gamma}}\right)^n \cdot \left(\prod_{k=1}^{n} \alpha_k \sqrt{\gamma}\right) \left[\prod_{k=1}^{n} \Phi(\tau_k)\right]^{1/2} = \tag{8}$$

$$= \left(\frac{2}{\sqrt{\gamma}}\right)^n \cdot \left[\prod_{k=1}^{n} \left(\tau_k^{2\tau_k^2+2} \cdot |1-\tau_k|^{-(1-\tau_k)^2} \cdot (1+\tau_k)^{-(1+\tau_k)^2}\right)\right]^{\frac{1}{2}},$$

where $\tau_k = \sqrt{\gamma} \cdot \alpha_k$, $k = \overline{1,n}$.

Consider the function $S(x) = x^{2x^2+2} \cdot |1-x|^{-(1-x)^2} \cdot (1+x)^{-(1+x)^2}$. The function $S(x)$ is logarithmically convex on the interval $[0,x_0]$, $x_0 \approx 0.88441$. Now we consider an extremal problem

$$\prod_{k=1}^{n} S(x_k) \longrightarrow \max, \quad \sum_{k=1}^{n} x_k = 2\sqrt{\gamma}, \quad x_k = \alpha_k \sqrt{\gamma}.$$

Let $X^{(0)} = \left\{x_k^{(0)}\right\}_{k=1}^{n}$ be an arbitrary extremal point of the problem. The following result holds (obtained similarly [12]):

$$\Psi'(x_1^{(0)}) = \Psi'(x_2^{(0)}) = \ldots = \Psi'(x_n^{(0)}), \tag{9}$$

where $\Psi'(x) = 4x\ln(x) - 2(x-1)\ln|x-1| - 2(x+1)\ln(x+1) + \frac{2}{x}$ (see Fig. 1).

Further it will be necessary for us to show that the following condition holds:

$$x_1^{(0)} = x_2^{(0)} = \cdots = x_n^{(0)} \quad \text{for all} \quad \gamma \in (0, \gamma_n].$$

Let $\Psi'(x) = t$, $y_0 \leqslant t < 0$, $y_0 \approx -1.06$. We find a solution of equation $\Psi'(x) = t_k$, $k = \overline{1, 53}$. Since $\forall t_k \in [y_0, 0)$, it follows that the equation has two solutions $x_1(t) \in (0, x_0]$, $x_2(t) \in (x_0, \infty]$.

Consider the following values of t: $t_1 = -0.02$, $t_2 = -0.04$, $t_3 = -0.06$, $t_4 = -0.08$, $\cdots$, $t_{52} = -1.04$, $t_{53} = y_0$. Direct calculations are presented in Table 1.

Consider the case $n = 2$. From the analysis of the tabular data for $n = 2$, we get that the minimum of the sum $x_1(t_k) + x_2(t_{k+1})$ is achieved for the interval $[-0.62; -0.64]$ and is equal to 1.709336 (see Table 2). The relation $x_1(t) + x_2(t) = 2\sqrt{\gamma}$ holds for each $\gamma \in (0; 0.73]$. Let $\gamma = 0.73$; then the value $2\sqrt{\gamma}$ is less than the minimum 1.709336. Thus, for $n = 2$ and $\gamma \in (0; 0.73]$, we obtain that x_2 does not belong to (x_0, ∞), that is x_1 and x_2 belong to the interval $(0, x_0]$ and $x_1 = x_2$. From inequalities (8) and (9) for $n = 2$, we have

$$J_2(\gamma) \leqslant \frac{4}{\gamma} \cdot S\left(\frac{2\sqrt{\gamma}}{2}\right).$$

For $n = 3$, the minimum of the value $2x_1(t_k) + x_2(t_{k+1})$ on the whole graph is achieved on the interval $[-0.48; -0.50]$ and is equal to 2.381211 (see Table 3). Similarly, $2x_1(t) + x_2(t) = 2\sqrt{\gamma}$. Let $\gamma = 1.41$; then $2\sqrt{\gamma} = 2.3748$. Thus, for $\gamma \in (0; 1.41]$ the situation $x_2 \in (x_0, \infty)$ is not possible. In this way, we obtain $x_1, x_2, x_3 \in (0, x_0]$ and $x_1 = x_2 = x_3$.

Then, taking into account the inequalities (8) and (9) for $n = 3$, we have

$$J_3(\gamma) \leqslant \left(\frac{2}{\sqrt{\gamma}}\right)^3 \left[S\left(\frac{2\sqrt{\gamma}}{3}\right)\right]^{3/2}.$$

Similarly, the situation holds for all $\gamma \in (0, \gamma_n]$, $n = 4, 5, 6$.

k	t_k	$x_1(t_k)$	$x_2(t_k)$	k	t_k	$x_1(t_k)$	$x_2(t_k)$
0	0	0.581421	∞	27	-0.54	0.671495	1.047944
1	-0.02	0.584192	2.607677	28	-0.56	0.675680	1.041549
2	-0.04	0.586996	2.095431	29	-0.58	0.679954	1.035639
3	-0.06	0.589833	1.849825	30	-0.6	0.684325	1.030184
4	-0.08	0.592706	1.696659	31	-0.62	0.688797	1.025157
5	-0.1	0.595614	1.588941	32	-0.64	0.693377	1.020539
6	-0.12	0.598559	1.507710	33	-0.66	0.698072	1.016313
7	-0.14	0.601542	1.443586	34	-0.68	0.702890	1.012468
8	-0.16	0.604564	1.391304	35	-0.7	0.707842	1.008999
9	-0.18	0.607626	1.347643	36	-0.72	0.712936	1.005911
10	-0.2	0.610729	1.310499	37	-0.74	0.718185	1.003228
11	-0.22	0.613876	1.278433	38	-0.76	0.723604	1.001015
12	-0.24	0.617066	1.250421	39	-0.78	0.729208	0.999457
13	-0.26	0.620302	1.225709	40	-0.8	0.735017	0.997390
14	-0.28	0.623585	1.203729	41	-0.82	0.741053	0.994797
15	-0.3	0.626917	1.184045	42	-0.84	0.747345	0.991762
16	-0.32	0.630299	1.166313	43	-0.86	0.753926	0.988295
17	-0.34	0.633734	1.150260	44	-0.88	0.760838	0.984381
18	-0.36	0.637223	1.135664	45	-0.9	0.768138	0.979982
19	-0.38	0.640770	1.122345	46	-0.92	0.775896	0.975038
20	-0.4	0.644375	1.110153	47	-0.94	0.784212	0.969461
21	-0.42	0.648041	1.098962	48	-0.96	0.793228	0.963114
22	-0.44	0.651772	1.088668	49	-0.98	0.803162	0.955787
23	-0.46	0.655569	1.079182	50	-1	0.814378	0.947120
24	-0.48	0.659437	1.070427	51	-1.02	0.827585	0.936407
25	-0.5	0.663378	1.062338	52	-1.04	0.844608	0.921828
26	-0.52	0.667396	1.054860	53	-1.06	0.884406	0.884406

Table 1: Two solutions of the equation $\Psi'(x) = t_k$, $k = \overline{1,53}$

From Table 1, for an arbitrary $n \geqslant 7$, the following inequality holds:

$$(n-1)x_1(t_k) + x_2(t_{k+1}) > nx_1(t_k) + (x_2(t_{k+1}) - x_1(t_k)) > 0.58n,$$

since $x_1(t_k) \geqslant 0.5830$ and $x_2(t_{k+1}) - x_1(t_k) \geqslant 0$. Using the condition

$$(n-1)x_1(t) + x_2(t) = 2\sqrt{\gamma_n},$$

we assume that $2\sqrt{\gamma_n} = 0.58n$. Thus, $\gamma_n = 0.084n^2$, that is, when $\gamma \in (0; 0.084n^2]$ then the sum $(n-1)x_1(t) + x_2(t)$ is less than $0.58n$. Thus, for $n \geqslant 7$ and $\gamma \in (0,\gamma_n]$, we obtain

$$J_n(\gamma) \leqslant \left(\frac{2}{\sqrt{\gamma}}\right)^n \left[S\left(\frac{2\sqrt{\gamma}}{n}\right)\right]^{n/2}.$$

k	t_k	$x_1(t_k)+x_2(t_{k+1})$	k	t_k	$x_1(t_k)+x_2(t_{k+1})$
0	0		27	-0,54	1,715340
1	-0,02	3,189098	28	-0,56	1,713044
2	-0,04	2,679623	29	-0,58	1,711318
3	-0,06	2,436820	30	-0,6	1,710138
4	-0,08	2,286492	31	-0,62	1,709482
5	-0,1	2,181647	32	-0,64	**1,709336**
6	-0,12	2,103324	33	-0,66	1,709690
7	-0,14	2,042145	34	-0,68	1,710540
8	-0,16	1,992846	35	-0,7	1,711889
9	-0,18	1,952207	36	-0,72	1,713753
10	-0,2	1,918125	37	-0,74	1,716163
11	-0,22	1,889163	38	-0,76	1,719200
12	-0,24	1,864297	39	-0,78	1,723061
13	-0,26	1,842775	40	-0,8	1,726598
14	-0,28	1,824031	41	-0,82	1,729814
15	-0,3	1,807630	42	-0,84	1,732815
16	-0,32	1,793230	43	-0,86	1,735640
17	-0,34	1,780559	44	-0,88	1,738307
18	-0,36	1,769398	45	-0,9	1,740820
19	-0,38	1,759569	46	-0,92	1,743176
20	-0,4	1,750923	47	-0,94	1,745356
21	-0,42	1,743337	48	-0,96	1,747326
22	-0,44	1,736709	49	-0,98	1,749015
23	-0,46	1,730953	50	-1	1,750281
24	-0,48	1,725996	51	-1,02	1,750785
25	-0,5	1,721775	52	-1,04	1,749413
26	-0,52	1,718238	53	-1,06	1,729015

Table 2: Minimum of the sum $x_1(t_k)+x_2(t_{k+1})$, $k=\overline{1{,}53}$

The equality case is straightforward to verify. Theorem 1 is proved. □

From Theorem 1, we obtain the following results.

Corollary 1. *Let $n\in\mathbb{N}$, $n\geqslant 2$, $\gamma\in(0{,}\gamma_n]$, $\gamma_2=0.7304$, $\gamma_3=1.4175$, $\gamma_4=2.2983$, $\gamma_5=3.3683$, $\gamma_6=4.6244$, and $\gamma_n=0.084\,n^2$, $n\geqslant 7$. Then for any n-radial system of points $A_n=\{a_k\}_{k=1}^{n}$ such that $\mathcal{L}\left(A_n\right)= =1$, and any system of mutually non-overlapping domains B_0, B_∞, B_k, $a_0=0\in B_0\subset\overline{\mathbb{C}}$, $\infty\in B_\infty\subset\overline{\mathbb{C}}$, $a_k\in B_k\subset\overline{\mathbb{C}}$, $k=\overline{1,n}$, the inequality (2) holds. Equality is attained in the same case as in Theorem 1.*

Corollary 2. *Under the conditions of Theorem 1, the following inequa-*

lity holds:

$$[r(B_0,0)r(B_\infty,\infty)]^\gamma\prod_{k=1}^{n} r(B_k,a_k)\leqslant\left(\frac{4}{n}\right)^n\frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{2\gamma}{n}}}{\left|1-\frac{4\gamma}{n^2}\right|^{\frac{2\gamma}{n}+\frac{n}{2}}}\left|\frac{n-2\sqrt{\gamma}}{n+2\sqrt{\gamma}}\right|^{2\sqrt{\gamma}}. \quad (10)$$

Equality in this inequality is achieved when 0, ∞, a_k *and* B_0, B_∞, B_k, $k=\overline{1,n}$, *are, respectively, poles and circular domains of the quadratic differential* (3).

k	t_k	$2x_1(t_k)+x_2(t_{k+1})$	k	t_k	$2x_1(t_k)+x_2(t_{k+1})$
0	0		27	-0.54	2.382735
1	-0.02	3.770519	28	-0.56	2.384539
2	-0.04	3.263814	29	-0.58	2.386998
3	-0.06	3.023816	30	-0.6	2.390093
4	-0.08	2.876325	31	-0.62	2.393807
5	-0.1	2.774353	32	-0.64	2.398133
6	-0.12	2.698938	33	-0.66	2.403067
7	-0.14	2.640704	34	-0.68	2.408612
8	-0.16	2.594388	35	-0.7	2.414780
9	-0.18	2.556771	36	-0.72	2.421594
10	-0.2	2.525751	37	-0.74	2.429099
11	-0.22	2.499892	38	-0.76	2.437386
12	-0.24	2.478172	39	-0.78	2.446665
13	-0.26	2.459841	40	-0.8	2.455806
14	-0.28	2.444333	41	-0.82	2.464831
15	-0.3	2.431215	42	-0.84	2.473869
16	-0.32	2.420146	43	-0.86	2.482985
17	-0.34	2.410858	44	-0.88	2.492232
18	-0.36	2.403133	45	-0.9	2.501659
19	-0.38	2.396792	46	-0.92	2.511314
20	-0.4	2.391692	47	-0.94	2.521252
21	-0.42	2.387712	48	-0.96	2.531538
22	-0.44	2.384750	49	-0.98	2.542243
23	-0.46	2.382725	50	-1	2.553443
24	-0.48	2.381565	51	-1.02	2.565162
25	-0.5	**2.381211**	52	-1.04	2.576998
26	-0.52	2.381615	53	-1.06	2.573623

Table 3: Minimum of the sum $2x_1(t_k)+x_2(t_{k+1})$, $k=\overline{1{,}53}$

Corollary 3. *Let* $n\in\mathbb{N}$, $n\geqslant 2$, $\gamma\in(0,\gamma_n]$, $\gamma_2=0.7304$, $\gamma_3=1.4175$, $\gamma_4=2.2983$, $\gamma_5=3.3683$, $\gamma_6=4.6244$, *and* $\gamma_n=0.084\,n^2$, $n\geqslant 7$. *Then, for any other points of the unit circle* $|w|=1$ *and any set of mutually* *non-overlapping domains* B_0, B_∞, B_k, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1, n}$, *the inequality* (2) *holds. Equality is attained in the same case as in Theorem 1.*

If we consider a sufficiently strict restriction on the distribution of the angles α_k, $k = \overline{1, n}$, then we can get a stronger result.

Let $y_0 \approx 0.884414$ be a root of the equation

$$\ln \frac{y^2}{1 - y^2} = \frac{1}{y^2}. \tag{11}$$

Then the following proposition is true.

Theorem 2. *Let* $n \in \mathbb{N}$, $n \geqslant 2$, $\gamma \in (0, \gamma_n)$, $\gamma_n = \frac{1}{4} y_0^2 n^2$. *Then for any* n*-radial system of points* $A_n = \{a_k\}_{k=1}^n$ *such that* $\mathcal{L}(A_n) = 1$, $0 < \alpha_k \leqslant y_0/\sqrt{\gamma}$, *where* y_0 *is a root of equation* (11), $k = \overline{1, n}$, *and for any collection of pairwise nonoverlapping domains* B_0, B_∞, B_k, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1, n}$, *the inequality* (10) *holds. Equality is attained in the same case as in Corollary 2.*

Proof. The proof of Theorem 2 practically repeats the proof of Theorem 1, only the logarithmic convexity of the function $S(x)$ on the segment $(0, y_0]$ and relation below are used in the final stage of the proof. The reation is

$$\frac{1}{n} \sum_{k=1}^{n} \ln S(x_k) \leqslant \ln S\left(\frac{\sum\limits_{k=1}^{n} x_k}{n} \right).$$

It is equivalent to

$$\ln \left(\prod_{k=1}^{n} S(x_k) \right)^{\frac{1}{n}} \leqslant \ln \left(S\left(\frac{2}{n} \sqrt{\gamma} \right) \right).$$

Equality in this inequality is attained if

$$\tau_1 = \tau_2 = \ldots = \tau_n = \frac{2\sqrt{\gamma}}{n},$$

i. e., if $\alpha_k = \frac{2}{n}$, $k = \overline{1,n}$. In this case, relation (7) yields

$$J_n(\gamma) \leqslant J_n^0(\gamma) = \left(\frac{4}{n} \right)^n \left[(r(D_0, 0)\, r(D_\infty, \infty))^{\frac{4\gamma}{n^2}} \cdot \frac{r(D_1, -i)\, r(D_2, i)}{|(-i) - i|^2} \right]^{\frac{n}{2}},$$

where D_0, D_∞, D_1 and D_2 are the circular domains of the quadratic differential

$$Q(z)dz^2 = -\frac{\dfrac{4\gamma}{n^2}z^4 + 2\left(\dfrac{4\gamma}{n^2} - 2\right)z^2 + \dfrac{4\gamma}{n^2}}{z^2(z^2+1)^2}\,dz^2. \tag{12}$$

From whence, we have, eventually,

$$J_n(\gamma) \leqslant \left(\frac{2}{\sqrt{\gamma}}\right)^n \left[S\left(\frac{2}{n}\sqrt{\gamma}\right)\right]^{\frac{n}{2}}.$$

Using a specific formula for $S(x)$, we get the basic inequality of Theorem 2. Changing the variable in (12) by the formula $z = -iw^{\frac{n}{2}}$, we get the quadratic differential (3). The sign of equality in inequality (10) is verified directly. Theorem 2 is proved. □

Corollary 1. *Let $n \in \mathbb{N}$, $n \geqslant 2$, $\gamma \in (0, \gamma_n]$, $\gamma_n = 0.19n^2$. Then for any n-radial system of points $A_n = \{a_k\}_{k=1}^n$ such that $\mathcal{L}(A_n) = 1$, $0 < \alpha_k \leqslant y_0/\sqrt{\gamma}$, $y_0 \approx 0.88441$, $k = \overline{1,n}$, and any set of mutually non-overlapping domains B_0, B_∞, B_k, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, the inequality (2) holds. Equality is attained in the same case as in Theorem 1.*

Consider the following problem, which was formulated as an open problem in the case $\gamma = 1$ in the paper by Dubinin [7].

Problem 2. Find, for any fixed value of $\gamma \in (0, n]$, the maximum of the functional

$$r^\gamma(B_0, 0)\prod_{k=1}^{n} r(B_k, a_k),$$

where B_0, B_1, B_2,..., B_n, $n \geqslant 2$, is any system of pairwise non-overlapping domains in $\overline{\mathbb{C}}$, where the domains $B_1, \ldots, B_n$ have symmetry with respect to the unit circle, $a_0 = 0$, $|a_k| = 1$, $k = \overline{1,n}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{0,n}$; describe all extremals of the functional.

This problem was solved for $\gamma = 1$ and $n \geqslant 2$ by Kovalev [13, 14]. The following theorem substantially complements the results of the papers [4, 13, 14]. We obtain the following results assuming that $B_0 \subset U$ (here U denotes the unit circle).

Theorem 3. *Let $n \in \mathbb{N}$, $n \geqslant 2$, $\gamma \in (0, \gamma_n)$, $\gamma_n = \frac{1}{2}y_0^2 n^2$. Then, for any n-radial system of points $A_n = \{a_k\}_{k=1}^n$, such that $|a_k| = 1$,*

$0 < \alpha_k \leqslant y_0/\sqrt{\gamma}$, where y_0 is a root of equation (11), $k = \overline{1,n}$, and any set of mutually non-overlapping domains B_0, B_k, $a_0 = 0 \in B_0 \subset U$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, where the domains B_k have symmetry with respect to the unit circle $|w| = 1$ for all $k = \overline{1,n}$, the following inequality holds:

$$r^{\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right) \leqslant r^{\gamma}\left(\Lambda_0, 0\right) \prod_{k=1}^{n} r\left(\Lambda_k, \lambda_k\right). \tag{13}$$

Equality in (13) is attained when 0, λ_k and Λ_0, Λ_k, $k = \overline{1,n}$, are, respectively, the poles and the circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{\gamma w^{2n} + 2(n^2 - \gamma)w^n + \gamma}{w^2(w^n - 1)^2}\, dw^2. \tag{14}$$

Proof. Note (see [6, p.59]) that if the domains B_k have symmetry with respect to the unit circle $|w| = 1$ for all $k = \overline{1,n}$, and the domain $B_0 \subset U$, then the extremal problem for the functional $r^{\gamma}\left(B_0, 0\right) \prod\limits_{k=1}^{n} r\left(B_k, a_k\right)$ can be reduced, by easy transformations, to the study of the functional $r^{\gamma/2}(B_0, 0)\, r^{\gamma/2}(B_{\infty}, \infty) \prod\limits_{k=1}^{n} r(B_k, a_k)$. Thus, using this property and proofs of Theorem 1 and Theorem 2, we obtain the result of Theorem 3. □

Using Corollary 3 and proofs of Theorem 3 and Theorem 1, it is not difficult to obtain the following result.

Theorem 4. *Let $n \in \mathbb{N}$, $\gamma \in (0, \gamma_n]$, $\gamma_2 = 1.4608$, $\gamma_3 = 2.8350$, $\gamma_4 = 4.5966$, $\gamma_5 = 6.7366$, $\gamma_6 = 9.2488$, $\gamma_n = 0.168\, n^2$, $n \geqslant 7$. Then, for any other points of the unit circle $|w| = 1$ and any system of mutually non-overlapping domains B_0, B_k, $a_0 = 0 \in B_0 \subset U$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, where the domains B_k have symmetry with respect to the unit circle $|w| = 1$ for all $k = \overline{1,n}$, the following inequality holds:*

$$r^{\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right) \leqslant \left(\frac{4}{n}\right)^{n} \frac{\left(\frac{2\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left|1 - \frac{2\gamma}{n^2}\right|^{\frac{n}{2} + \frac{\gamma}{n}}} \left|\frac{n - \sqrt{2\gamma}}{n + \sqrt{2\gamma}}\right|^{\sqrt{2\gamma}}.$$

Equality in the inequality is achieved when a_k and B_k, $k = \overline{0,n}$, are, respectively, poles and circular domains of the quadratic differential (14).

References

[1] Bakhtin A. K., Bakhtina G. P., Zelinskii Yu. B. *Topological-algebraic structures and geometric methods in complex analysis.* Zb. prats of the Inst. of Math. of NASU, 2008. (in Russian) DOI: https://doi.org/10.13140/RG.2.1.1660.6242.

[2] Bakhtin A. K., Denega I. V. *Some estimates of the functionals for n-radial systems of points.* Zb. prats of the Inst. of Math. of NASU, 2011, vol. 8, no. 1, pp. 12 – 21. (in Russian)

[3] Bakhtina G. P., Dvorak I. Y., Denega I. V. *About the product of inner radii of pairwise non-overlapping domains.* Dopov. Nac. akad. nauk Ukr., 2016, no. 1, pp. 7 – 11. DOI: https://doi.org/10.15407/dopovidi2016.01.007.

[4] Bakhtina G. P. *On the conformal radii of symmetric nonoverlapping regions.* Modern issues of material and complex analysis, K.: Inst. Math. of NAS of Ukraine, 1984, pp. 21 – 27. (in Russian)

[5] Denega I. V. *Some inequalities for inner radii of partially non-overlapping domains.* Dopov. Nac. akad. nauk Ukr., 2012, no. 5, pp. 19 – 22. (in Russian)

[6] Dubinin V. N. *Separating transformation of domains and problems on extremal decomposition.* Notes scientific. sem. Leningr. Dep. of Math. Inst. AN USSR., 1988, vol. 168, pp. 48 – 66. (in Russian); translation in J. Soviet Math., 1991, vol. 53, no. 3, pp. 252 – 263. DOI: https://doi.org/10.1007/BF01303649.

[7] Dubinin V. N. *Symmetrization method in geometric function theory of complex variables.* Successes Mat. Science, 1994, vol. 49, no. 1(295), pp. 3 – 76. (in Russian); translation in Russian Math. Surveys. 1994, vol. 1, pp. 1 – 79. DOI: https://doi.org/10.1070/RM1994v049n01ABEH002002.

[8] Dubinin V. N. *Condenser capacities and symmetrization in geometric function theory.* Birkhäuser/Springer, Basel, 2014. DOI: : https://doi.org/10.1007/978-3-0348-0843-9.

[9] Emelyanov E. G. *On the Problem of Maximizing the Product of Powers of Conformal Radii Nonoverlapping Domains.* J. Math. Sci. (N.Y.), 2004, vol. 122, no. 6, pp. 3641 – 3647. DOI: https://doi.org/10.1023/B:JOTH.0000035239.55516.08.

[10] Goluzin G. M. *Geometric theory of functions of a complex variable.* Amer. Math. Soc. Providence, R.I., 1969.

[11] Jenkins J. *Univalent functions and conformal mapping.* Moscow:Publishing House of Foreign Literature, **256**, 1962. (in Russian) DOI: : https://doi.org/10.1007/978-3-642-88563-1.

[12] Kovalev L. V. *On the problem of extremal decomposition with free poles on a circle.* Dal'nevost. Mat. Zh., 1996, no. 2, pp. 96 – 98. (in Russian)

[13] Kovalev L. V. *On the inner radii of symmetric nonoverlapping domains.* Izv. Vyssh. Uchebn. Zaved. Mat., 2000, no. 6, pp. 77 – 78. (in Russian)

[14] Kovalev L. V. *On three nonoverlapping domains.* Dal'nevost. Mat. Zh., 2000, no. 1, pp. 3 – 7. (in Russian)

[15] Kuzmina G. V. *Problems on extremal decomposition of the riemann sphere. Notes scientific.* Sem. Leningr. Dep. of Math. Inst. AN USSR., 2001, vol. 276, pp. 253–275. (in Russian); translation in J. Math. Sci. (N.Y.), 2003, vol. 118, no. 1, pp. 4880 – 4894. DOI: https://doi.org/10.1023/A:1025580802209.

[16] Lavrent'ev M. A. *On the theory of conformal mappings.* Tr. Sci. Inst An USSR, 1934, vol. 5, pp. 159 – 245. (in Russian)

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