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ON APPROXIMATION OF THE RATIONAL FUNCTIONS, WHOSE INTEGRAL IS SINGLE-VALUED ON $\mathbb{C}$, BY DIFFERENCES OF SIMPLEST FRACTIONS

Abstract. We study a uniform approximation by differences $\Theta_1 - \Theta_2$ of simplest fractions (s.f.'s), i. e., by logarithmic derivatives of rational functions on continua K of the class Ω_r, $r > 0$ (i. e., any points $z_0, z_1 \in K$ can be joined by a rectifiable curve in K of length $\leqslant r$). We prove that for any proper one-pole fraction T of degree m with a zero residue there are such s.f.'s Θ_1,Θ_2 of order $\leqslant (m-1)n$ that $\|T + \Theta_1 - \Theta_2\|_K \leqslant 2r^{-1}A^{2n+1}n!^2/(2n)!^2$, where the constant A depends on r, T and K. Hence, the rate of approximation of any fixed individual rational function R, whose integral is single-valued on $\mathbb{C}$, has the same order. This result improves the famous estimate $\|R + \Theta_1 - \Theta_2\|_{C(K)} \leqslant 2e^r r^n/n!$, given by Danchenko for the case $\|R\|_{C(K)} \leqslant 1$.

Key words: *difference of simplest fractions, rate of uniform approximation, logarithmic derivative of rational function*

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1. Introduction. By a *simplest fraction* (s.f.) *of order* n, $n \in \mathbb{N}$, we mean a logarithmic derivative of polynomial of degree n:

$$\Theta(z) = \sum_{j=1}^{n} \frac{1}{z - z_j}, \qquad z_j \in \mathbb{C} \qquad (n \geqslant 1).$$

The function $\Theta(z) \equiv 0$ is the s.f. of order $n = 0$.

The approximation properties of s.f.'s have become an object of intensive study after the paper [5] was published. It turned out, for example, that the rate of the approximation by s.f.'s for a wide class of functions and sets has the same order as for the polynomial approximation [5], [9].

The first result on approximation by *differences of s.f.'s*, i.e., by *logarithmic derivatives of rational functions*, was also proved in [5] (Theorem A below). Let $\mathcal{R}_n^*$ be the class of rational functions of degree $\leqslant n$, whose integral is single-valued on $\mathbb{C}$. We say that a set $K \subset \mathbb{C}$ is of the class Ω_r, $r > 0$, if any points z_0,$z_1 \in K$ can be joined by a rectifiable curve in K of length $\leqslant r$. Let $\|\cdot\|_K$ be a sup-norm over K.

Theorem A. [5] *Let $K \in \Omega_r$, $R \in \mathcal{R}_N^*$, $\|R\|_K \leqslant 1$. There are s.f.'s Θ_1, Θ_2 of order $\leqslant (N+1)n$ such that*

$$\|R + \Theta_1 - \Theta_2\|_K \leqslant 2e^r r^n/n! \qquad (n \geqslant 5r). \tag{1}$$

The author has proved the following much more strong estimate in the case where $R = M$ is a polynomial [7], [8] (hereinafter $n_0(x) = 14 + ex^2/4$):

Theorem B. [8] *Let $M \not\equiv 0$ be a polynomial of degree $N \geqslant 0$, $K \in \Omega_r$, $\|M\|_K \leqslant c$. There are s.f.'s Θ_1, Θ_2 of order $(N+1)n$, such that*

$$\|M + \Theta_1 - \Theta_2\|_K \leqslant \frac{2}{r}(cr)^{2n+1}\frac{n!^2}{(2n)!^2} \qquad (n \geqslant n_0(cr)). \tag{2}$$

In this paper we prove that the approximation of any given function $R \in \mathcal{R}_N^*$ has the same order. The crucial point is the following theorem on approximation of a one-pole fraction.

Denote by K^∞ the unbounded component of the complement of continuum K, and let $K^0 = \mathbb{C}\backslash\{K \cup K^\infty\}$.

Theorem 1. *Let $K \in \Omega_r$, $a \in \mathbb{C}\backslash K$, $\delta = \mathrm{dist}(a, K) > 0$,*

$$T(z) = \sum_{j=2}^{m} \frac{c_j}{(z-a)^j}, \qquad m \geqslant 2, \tag{3}$$

and $C = \|T(z)(z-a)^2\|_K$. There are s.f.'s Θ_1,Θ_2 of order $\leqslant (m-1)n$ such that

$$\Delta := \|T + \Theta_1 - \Theta_2\|_K \leqslant \frac{2}{r}\left(\frac{Cr}{\delta^2}\right)^{2n+1}\frac{n!^2}{(2n)!^2} \qquad (n \geqslant n_0(Cr/\delta^2)). \tag{4}$$

If $|c_m| \geqslant 1$ and $a \in K^0$, then

$$\Delta \leqslant \frac{2}{r}\left(16r^3\|T\|_K^{1+2/m}\right)^{2n+1}\frac{n!^2}{(2n)!^2} \qquad (n \geqslant n_0(16r^3\|T\|_K^{1+2/m})). \tag{5}$$

In the case where $|c_m| \geqslant 1$, $\delta \leqslant (\operatorname{diam} K)/6$ *and* $a \in K^\infty$, *the estimate* (5) *is also true, but the factor* 16 *must be replaced by* $(28/3)^2$.

Theorem 1 is proved in Section 4. In Section 2 we consider the general case where $R \in \mathcal{R}_N^*$. The following example shows that the conditions $R \in \mathcal{R}_N^*$ and $T \in \mathcal{R}_m^*$ are essential for Theorem A and Theorem 1, respectively.

Denote by $d_n = d_n(f)$ the best approximation of the function

$$f(x) = \frac{1}{2(x+a)}, \qquad a \in \mathbb{R}, \quad a > 1,$$

over $x \in [-1,1]$ by all differences of s.f.'s of order at most n.

Proposition 1. *If* $a =: \frac{1}{2}(\rho + \rho^{-1}) \geqslant \frac{3}{2}$ $(\rho \geqslant \frac{3+\sqrt{5}}{2})$, *then*

$$d_n(f) > \mu_n(1 + o(1)), \qquad \mu_n := 2^{1-2n}(\rho + \sqrt{\rho^2 - 1} - \lambda\rho^{-1})^{-2n-1}$$

as $n \to \infty$ *for some* $\lambda \in [-\frac{1}{2}, \frac{1}{2}]$.

Proof. Set $\|\cdot\| = \|\cdot\|_{[-1,1]}$. There is a difference $D(x)$ of s.f.'s of order $\leqslant n$, such that $\|D - f\| = d_n \cdot (1 + o(1))$ as $n \to \infty$ ($\|\cdot\| := \|\cdot\|_{[-1,1]}$). Let $R(x)$ be the rational function of degree at most n such that $R(0) = \sqrt{a}$ and $D = R'/R$.

Set $I(x) = \int_0^x (D(t) - f(t))\, dt$. Obviously, $\|I\| \leqslant d_n \cdot (1 + o(1))$,

$$(e^{I(x)} - 1)\sqrt{x+a} = R(x) - \sqrt{x+a}, \qquad -1 \leqslant x \leqslant 1.$$

Since $d_n \to 0$ as $n \to \infty$, we have

$$\mu = \mu(R) := \|(R(x) - \sqrt{x+a})/\sqrt{x+a}\| \leqslant e^{\|I\|} - 1 < d_n \cdot (1 + o(1)).$$

But $\inf_R \mu = \mu_n(1 + o(1))$ (over all rationals $R(x)$ of degree $\leqslant n$) [1]. □

In Section 5 we study the approximation of arbitrary rational functions by the *quotients between two differences of s.f.'s*. This useful method for the calculation of values of rational functions and polynomials was introduced in [2]. Recall, that the Horner scheme is usually applied for this. However, if the values of arguments and coefficients of these functions are large, using this scheme may lead to loss of accuracy because of multiple multiplications (see examples in [4, § 3]).

2. Corollaries of Theorem 1. Set

$$R(z)=\sum_{k=1}^{p}T_k(z),\qquad T_k(z):=\sum_{j=2}^{m_k}\frac{c_{k,j}}{(z-z_k)^j},\qquad m_k\geqslant 2,\qquad (6)$$

$$z_k\neq z_j\quad (k\neq j),\qquad m_1+\dots+m_p=N,\qquad p\geqslant 1.$$

Corollary 1. *Let R be a function (6), $m=\max_k m_k$, $c=\max_{k,j}|c_{k,j}|$, $\delta_k=\operatorname{dist}(z_k,K)$. If $K\in\Omega_r$, $\delta:=\min_k\delta_k>0$ and $A:=c\sum_{j=2}^{m}\delta^{-j}$, then there are s.f.'s Θ_1,Θ_2 of order $\leqslant(N-p)n$ such that*

$$\Delta_1:=\|R+\Theta_1-\Theta_2\|_K\leqslant\frac{2p}{r}(Ar)^{2n+1}\frac{n!^2}{(2n)!^2}\qquad (n\geqslant n_0(Ar)).\qquad (7)$$

This assertion follows from (4), because R is the sum of p functions T_k of the form (3), $\sum_{k=1}^{p}(m_k-1)n=(N-p)n$ and

$$\frac{C_k}{\delta_k^2}\leqslant\sum_{j=2}^{m_k}\frac{c}{\delta_k^j}\leqslant\sum_{j=2}^{m}\frac{c}{\delta^j}=A\qquad (C_k:=\|T_k(z)(z-z_k)^2\|_K,\ 1\leqslant k\leqslant p).$$

The estimate (7) is better than (1) for any fixed individual function R of the form (6). On the other hand, (1) is a universal estimate (i. e., (1) only depends on $\|R\|_K$ and r), whereas (7) depends on the norms $\|T_k\|_K$ of all p components of the function $R=\sum T_k$, and it is easy to construct such a fraction $R=T_1+T_2$ that $\|T_k\|_K\gg 1$ while $\|R\|_K\leqslant 1$.

We now consider the case where the set K has special form and in this case we get new estimates of Δ_1 of the same order as in (7) but with more universal constants. Let R be a function of the form (6). We write $K\in\Omega_r^*(R)$ if $K\in\Omega_r$ and all poles $z_k\in K^0$, and every bounded component K_j^0 of the complement of the set K ($\bigcup K_j^0=K^0$) contains at most one of the poles z_k, i. e., "poles of $R(z)$ are separated by K".

Corollary 2. *If $K\in\Omega_r^*(R)$ and $\|R\|_K\leqslant 1$, then (see (7))*

$$\Delta_1\leqslant\frac{2p}{r}\left(\frac{50mr^3}{\delta^2}\right)^{2n+1}\frac{n!^2}{(2n)!^2}\qquad (n\geqslant n_0(50mr^3/\delta^2)).$$

If, in addition, $|c_{k,m_k}|\geqslant 1$ for all $1\leqslant k\leqslant p$ (see (6)), then

$$\Delta_1\leqslant\frac{2p}{r}\left(16\cdot 10^4r^3\right)^{2n+1}\frac{n!^2}{(2n)!^2}\qquad (n\geqslant n_0(16\cdot 10^4r^3)).$$

Indeed, because of $K \in \Omega_r^*(R)$, the singularities of the function $R = \sum T_k$ are separated [3]: $\|T_k\|_K \leqslant 50 m_k \|R\|_K$, $1 \leqslant k \leqslant p$. Thus, the assertion follows from the estimates (4), (5) and $\|z-a\|_K \leqslant r$. To prove the last estimate of Δ_1 we also use the fact that the function $(50x)^{1+2/x}$ is decreasing for $x \geqslant 2$, and hence (see (5)),

$$\max_k \|T_k\|_K{}^{1+2/m_k} \leqslant \max_k (50 m_k)^{1+2/m_k} \leqslant (50\cdot 2)^2 = 10^4.$$

Remark 1. *Let $\tilde{R}(z) = M(z) + R(z)$, where M be a polynomial and R be a fraction of the form (6). Let $\tilde{c} := \|M\|_K > 0$. Under the assumptions of Corollary 1 we have the following assertion: there are s.f.'s Θ_1, Θ_2 of order at most $(\deg M + 1 + N - p)n$ such that*

$$\|\tilde{R} + \Theta_1 - \Theta_2\|_K \leqslant 2r^{2n}(\tilde{c}^{2n+1} + pA^{2n+1})\frac{n!^2}{(2n)!^2} \quad (n \geqslant n_0(\max\{A,\tilde{c}\}r)).$$

3. Auxiliary results. Our first lemma is trivial:

Lemma 1. *Let $B(z) \not\equiv 0$ be a polynomial of degree $N_1 > 0$, $H(v) \not\equiv 0$ be a polynomial of degree $N_2 \geqslant 0$,*

$$F(z) = H\left(\frac{1}{B(z)}\right)\frac{B'(z)}{B^2(z)}. \tag{8}$$

Let $q_1(v)$, $q_2(v)$ be polynomials of degree $(N_2+1)n > 0$. Then the functions $S_j(z) := (B(z))^{(N_2+1)n} q_j(1/B(z))$, $j = 1, 2$, are polynomials of degree at most $N_1(N_2+1)n$, and the following identity holds:

$$F(z) + \frac{S_1{}'(z)}{S_1(z)} - \frac{S_2{}'(z)}{S_2(z)} \equiv \frac{B'(z)}{B^2(z)}\left(H(v) - \frac{q_1{}'(v)}{q_1(v)} + \frac{q_2{}'(v)}{q_2(v)}\right), \quad v = \frac{1}{B(z)}.$$

Let K and a be an arbitrary fixed set and a point in $\mathbb{C}$. Put

$$K_a = \{v : \ v = (z-a)^{-1}, \ z \in K\}.$$

Lemma 2. *If $K \in \Omega_r$, $a \in \mathbb{C}\setminus K$ and $\delta := \operatorname{dist}(a,K) > 0$, then $K_a \in \Omega_{r_a}$, where $r_a := r\delta^{-2}$.*

Proof. For any fixed points $v_0, v_1 \in K_a$ we put $z_j = a + v_j^{-1}$, $j = 0, 1$. Since $K \in \Omega_r$, there is a rectifiable curve $z(s)$, $0 \leqslant s \leqslant s_1$ ($z(0) = z_0$, $z(s_1) = z_1$) in K of the length $\int_0^{s_1} |z'(s)|ds \leqslant r$ (s is a natural parameter).

Then the curve $v(s) = (z(s) - a)^{-1}$, $0 \leqslant s \leqslant s_1$ ($v(0) = v_0$, $v(s_1) = v_1$) belongs to K_a, and the length of this curve

$$\int_0^{s_1} |v'(s)| ds = \int_0^{s_1} \frac{|z'(s)|}{|z(s) - a|^2} ds \leqslant \frac{1}{\delta^2} \int_0^{s_1} |z'(s)| ds \leqslant r\delta^{-2}.$$

Thus, the lemma is proved. □

Lemma 3. *Let K be a continuum in $\mathbb{C}$, $T(z)$ be a function of the form* (3). *If $\delta := \operatorname{dist}(a,K) > 0$ and $c_m \neq 0$, then*

$$\frac{1}{\delta} \leqslant 4\nu \sqrt[m]{\|T\|_K / |c_m|}, \quad \nu := \begin{cases} 1, & a \in K^0, \\ 2, & a \in K^\infty \text{ and } \delta \leqslant (\operatorname{diam} K)/6. \end{cases} \tag{9}$$

Proof. Put $v = 1/(z - a)$, $T(z)/c_m \equiv t_m(v)$,

$$t_m(v) = \tilde{c}_2 v^2 + \cdots + \tilde{c}_{m-1} v^{m-1} + v^m, \qquad \tilde{c}_j = c_j / c_m.$$

Let $\tau(K_a)$ be the transfinite diameter of the set K_a. We have the following estimate [6]: $\tau(K_a) \leqslant \sqrt[m]{\|t_m\|_{K_a}} \equiv \sqrt[m]{\|T\|_K / |c_m|}$. But K is a continuum, therefore [6], $\operatorname{diam} K_a \leqslant 4\tau(K_a) \leqslant 4 \sqrt[m]{\|T\|_K / |c_m|}$.

We now need to prove that $\operatorname{diam} K_a \geqslant 1/(\nu\delta)$.

In the case $a \in K^0$, the estimate $\operatorname{diam} K_a \geqslant 1/\delta$ is trivial.

Let $a \in K^\infty$ and $\delta \leqslant (\operatorname{diam} K)/6$. Let $z_1 \in K$ be a point such that $|z_1 - a| = \delta$. Then we have $\max_{z \in K} |z - a| \geqslant \max_{z \in K} |z - z_1| - \delta$ and

$$\operatorname{diam} K = \max_{z, \tilde{z} \in K} |z - \tilde{z}| \leqslant \max_{z \in K} |z - z_1| + \max_{\tilde{z} \in K} |z_1 - \tilde{z}| = 2 \max_{z \in K} |z - z_1|,$$

therefore $\max_{z \in K} |z - a| \geqslant (\operatorname{diam} K)/2 - \delta \geqslant 3\delta - \delta = 2\delta$. Thus,

$$\operatorname{diam} K_a \geqslant \frac{1}{\min_K |z - a|} - \frac{1}{\max_K |z - a|} \geqslant \frac{1}{\delta} - \frac{1}{2\delta} = \frac{1}{2\delta},$$

and the lemma follows. □

4. Proof of Theorem 1. Firstly, we prove the estimate (4).

Assume that $T(z) \not\equiv 0$ (the other case is trivial). The function (3) has the form (8) with $B(z) = z - a$, $H(v) = \sum_{j=2}^m c_j v^{j-2}$ ($\deg H(v) = m_1 - 2 \leqslant m - 2$). By Lemma 2 we have $K_a \in \Omega_{r_a}$, where $r_a = r\delta^{-2}$.

Obviously, $H(v) \equiv T(z)(z-a)^2$, therefore $\|H\|_{K_a} = C$. By Theorem B, there are s.f.'s $\theta_j(v) = q_j{}'(v)/q_j(v)$, $j = 1, 2$, of order $(m_1 - 1)n$, such that

$$\|H - \theta_1 + \theta_2\|_{K_a} \leqslant 2C(Cr_a)^{2n} n!^2/(2n)!^2 \qquad (n \geqslant n_0(Cr_a)). \tag{10}$$

Estimate (4) follows by (10), Lemma 1 and the equality $\|B'/B^2\|_K = \delta^{-2}$.

We have $C \leqslant \|T\|_K (\operatorname{diam} K)^2$ for $a \in K^0$. Thus, the estimate (5) follows by the estimates (4), (9) and $\operatorname{diam} K \leqslant r$. Similarly, in the case $a \in K^\infty$ and $\delta \leqslant (\operatorname{diam} K)/6$ we have

$$C \leqslant \|T\|_K (\delta + \operatorname{diam} K)^2 \leqslant \|T\|_K ((7/6)\operatorname{diam} K)^2,$$

and the theorem follows.

5. On approximation by special rational functions. Consider the following special fractions, introduced in [2, § 8.2]:

$$\widetilde{\Theta}(z) = \frac{\Theta_1(z) - \Theta_2(z)}{\Theta_3(z) - \Theta_4(z)}, \tag{11}$$

where Θ_j denotes a s.f. of order m_j, $j = 1, 2, 3, 4$. Fractions (11) have strong approximative properties [2]:

Theorem C. [2] *Let K be a compact set, R be a rational function of degree $N \geqslant 1$, and $r := \|R\|_K < \infty$. There is a fraction $\widetilde{\Theta}$ of the form (11) with orders $m_j \leqslant Nn$ such that*

$$\|\widetilde{\Theta} - R\|_K \leqslant 2e^r r^{n+1}/n! \qquad (n \geqslant 5r).$$

We now get a stronger estimate for the case $K \in \Omega_r$:

Corollary 3. *Let P, Q be polynomials of degree at most N, $K \in \Omega_r$, $\|P\|_K \leqslant 1$, $\inf_K |Q(z)| =: c_0 > 0$. Put $c_2 = \|Q\|_K$. There is a fraction $\widetilde{\Theta}$ of the form (11) with orders $m_j \leqslant (N+1)n$ such that*

$$\|\widetilde{\Theta} - P/Q\|_K \leqslant \frac{4c_2}{c_0^2} r^{2n} (1 + c_2^{2n}) \frac{n!^2}{(2n)!^2} \qquad (n \geqslant n_2).$$

Proof. Let $\Theta_1 - \Theta_2$ ($\Theta_3 - \Theta_4$) be the difference of s.f.'s of order at most $(N+1)n$ that approximates the polynomial $-P$ ($-Q$, respectively), as in Theorem B. Let n_2 be an integer such that $n \geqslant n_0(r)$, $n \geqslant n_0(c_2 r)$ and

$c_2\|\Theta_3 - \Theta_4 - Q\|_K \leqslant c_0^2/2$, if $n \geqslant n_2$. Thus, the statement follows from (2) and the identity

$$\frac{\Theta_1 - \Theta_2}{\Theta_3 - \Theta_4} - \frac{P}{Q} = \frac{(P + \Theta_1 - \Theta_2)Q - (Q + \Theta_3 - \Theta_4)P}{-Q^2 + (Q + \Theta_3 - \Theta_4)Q}.$$

Corollary 3 is proved. □

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References

[1] Braess D. *On rational approximation of the exponential and the square root function.* In: *Rational Approximation and Interpolation* (P. R. Graves-Morris, E. B. Saff, R. S. Varga, eds.). Springer, Heidelberg-New York, 1984. pp. 89 – 99. DOI: https://doi.org/10.1007/BFb0072395.

[2] Danchenko V. I. *Estimates of derivatives of simplest fractions and other questions.* Sb. Math., 2006, vol. 197, no. 4, pp. 505 – 524. DOI: https://doi.org/10.1070/SM2006v197n04ABEH003768.

[3] Danchenko V. I. *O ratsionalnykh sostavlyayuschikh meromorfnykh funktsii i ikh proizvodnykh.* Analysis Mathematica, 1990, vol. 16, no. 4, pp. 241 – 255. (In Russian) DOI: https://doi.org/10.1007/BF02630358.

[4] Danchenko V. I., Chunaev P. V. *Approximation by simple partial fractions and their generalizations.* J. Math. Sci. (N.Y.), 2011, vol. 176, no. 6, pp. 844 – 859. DOI: https://doi.org/10.1007/s10958-011-0440-5.

[5] Danchenko V. I., Danchenko D. Y. *Approximation by simplest fractions.* Math. Notes, 2001, vol. 70, no. 4, pp. 502 – 507. DOI: https://doi.org/10.1023/A:1012328819487.

[6] Goluzin G. M. *Geometric theory of functions of a complex variable.* Translations of Mathematical Monographs, vol. 26 American Mathematical Society, Providence, R.I. 1969 vi+676 pp.

[7] Komarov M. A. *On approximation by differences of simple partial fractions.* Abstracts International Conference "Complex Analysis and its Applications" (Gelendzhik–Krasnodar, June 2 – 9, 2018), Krasnodar, Kuban St. Univ., 2018, p. 65.

[8] Komarov M. A. *On approximation by special differences of simple partial fractions.* Algebra i analiz, 2018, vol. 30, no. 4, pp. 47–60. (St. Petersburg Math. J.)

[9] Kosukhin O. N. *Approximation properties of the most simple fractions.* Moscow Univ. Math. Bull., 2001, vol. 56, no. 4, pp. 36 – 40.

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