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ON THE CONVERGENCE OF MAPPINGS WITH k-FINITE DISTORTION.

Abstract. We prove that a locally uniform limit of a sequence of homeomorphisms with finite k-distortion is also a mapping with finite k-distortion. We obtain also an estimation for the distortion coefficient of the limit mapping.

Key words: *mapping with k-finite distortion, distortion coefficient, passing to the limit, differential form.*

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1. Introduction. It is well known that the limit of a uniformly converging sequence of analytic functions is an analytic function. Reshetnyak generalized this result to mappings with bounded distortion: *the limit of a locally uniformly converging sequence of mappings with bounded distortion is a mapping with bounded distortion.*

Definition 1. [8] *A mapping $f:\Omega\to\mathbb{R}^n$ is called mapping with bounded distortion if f is continuous, $f\in W^1_{n,\mathrm{loc}}(\Omega)$, the Jacobian $J(x,f)$ does not change the sign in the domain Ω and*

$$|Df(x)|^n\leqslant K|J(x,f)|\quad \textit{for almost all}\, x\in\Omega. \tag{1}$$

The smallest constant in this inequality is called the distortion coefficient of the mapping f and is denoted by the symbol $K(f)$. It is clear that

$$K(f)=\sup\left\{\frac{|Df(x)|^n}{|J(x,f)|}:\quad x\in\Omega,\quad J(x,f)\neq 0\right\}.$$

Reshetnyak used the weak convergence of Jacobians to prove the following theorem on the limit of a sequence of mappings with bounded distortion.

Theorem 1. [8] *Let $f_m : \Omega \to \mathbb{R}^n$, $m = 1, 2, \ldots,$ be an arbitrary sequence of mappings with bounded distortion, locally converging in $L_n(\Omega)$ to a mapping $f_0 : \Omega \to \mathbb{R}^n$. Assume that the sequence of distortion coefficients $K(f_m)$, $m = 1, 2, \ldots$, is bounded. Then the limit mapping f_0 is a mapping with bounded distortion and the following inequality holds:*

$$K(f_0) \leqslant \varliminf_{k\to\infty} K(f_m). \tag{2}$$

We briefly outline the proof in the case of non-negative Jacobians. For a test function $\varphi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega |Df_0(x)|^n \varphi(x)\, dx \leqslant \varliminf_{m\to\infty} \int_\Omega |Df_m(x)|^n \varphi(x)\, dx \leqslant$$
$$\leqslant K \varliminf_{m\to\infty} \int_\Omega J(x,f_m)\varphi(x)\, dx = K \int_\Omega J(x,f_0)\varphi(x)\, dx.$$

To justify the limit in the last equality, we apply the weak convergence of Jacobians. Consequently, for the limit mapping, the point-wise inequality $|Df_0(x)|^n \leqslant KJ(x,f_0)$ holds a .e. in Ω.

More recently, research has begun on mappings with finite distortion. They are a natural generalization of mappings with bounded distortion.

Definition 2. [7] *Let a mapping $f : \Omega \to \mathbb{R}^n$ belong to the Sobolev class $W^1_{n,\mathrm{loc}}(\Omega)$ and $J(x,f) \geqslant 0$. We define the pointwise distortion coefficient $K(x, f)$ of the mapping f as a value*

$$K(x, f) = \begin{cases} \frac{|Df(x)|^n}{|J(x,f)|} & \text{if } J(x,f) > 0, \\ 1 & \text{otherwise.} \end{cases}$$

The mapping $f : \Omega \to \mathbb{R}^n$ is called mapping with finite distortion *($f \in$ $\in FD(\Omega)$) if*

$$|Df(x)|^n \leqslant K(x,f)J(x,f) \ \text{ where } \ 1 \leqslant K(x,f) < \infty \ \text{ for almost all } x \in \Omega.$$

Remark. In other words, the condition *of finite distortion* is that the partial derivatives of the mapping $f \in W^1_{n,\mathrm{loc}}(\Omega)$ vanish a.e. on the set of zeros of the Jacobian $J(x,f)$.

For the first time, the essential properties of mappings with finite distortion were investigated in the paper [15] in the study of homeomorphisms inducing a bounded composition operator. The name "mapping with finite distortion" was proposed much later in the paper [7].

In [5] F. Gehring and T. Iwaniec showed that the limit of a weakly converging sequence of mappings with finite distortion is also a mapping with finite distortion, and they obtained also an estimation for the distortion coefficient of the limit mapping.

Theorem 2. [5] *Let $f_m : \Omega \to \mathbb{R}^n$, $m = 1, 2, \dots$, be an arbitrary sequence of mappings with finite distortion converging weakly in $W^1_{n,\mathrm{loc}}(\Omega)$ to a mapping $f_0 : \Omega \to \mathbb{R}^n$. Assume that*

$$K(x, f_m) \leqslant M(x) < \infty, \quad m = 1, 2, \dots,$$

where $\Omega \ni x \mapsto M(x) \in [1,\infty]$ is a measurable function. Then the limit mapping f_0 is a mapping with finite distortion and the inequality

$$K(x, f_0) \leqslant M(x)$$

holds.

More precisely, in this paper existence of a subsequence f_{m_k} such that

$$K(x, f_0) \leqslant b * \lim_{k\to\infty} K(x, f_{m_k})$$

was shown. Here the limit is understood in the sense of so-called biting convergence.

Definition 3. [2] *Let h and h_k, $k \in \mathbb{N}$, be Lebesgue measurable functions defined on a set $E \subset \mathbb{R}^n$. The sequence h_k is said to converge in the biting sense on E to h if there exists an increasing sequence E_ν of measurable subsets of E,*

$$\bigcup_\nu E_\nu = E,$$

such that

$$\lim_{k\to\infty} \int_{E_\nu} \varphi h_k \, dx = \int_{E_\nu} \varphi h \, dx$$

for any function $\varphi \in L_\infty(E_\nu)$.

In the paper [4] the limit of a sequence of homeomorphisms with finite distortion converging weakly in W_1^1 was shown to have, also, a finite distortion under the condition that the limit mapping is a homeomorphism.

Theorem 3. [4] *Let Ω, Ω' be bounded domains in $\mathbb{R}^n$, $f_j : \Omega \to \Omega'$, $j \in \mathbb{N}$, $f : \Omega \to \Omega'$, be homeomorpisms belonging to $W_1^1(\Omega)$, and $f_j \to f$ weakly in $W_1^1(\Omega)$. Assume that*

$$|Df_j(x)|^n \leqslant K(x,f_j)J(x,f_j) \qquad \text{for almost all } x \in \Omega,$$

where $K(x,f_j) : \Omega \to [1,\infty)$ are Borel functions for all j, and the sequence $K(x,f_j)$ converges in the biting sense to $K(x)$ as $j \to \infty$. Then the limit mapping f is a mapping with finite distortion and the inequality $K(x,f) \leqslant K(x)$ holds for almost all $x \in \Omega$.

In the paper [1] mappings with bounded (q,p)-distortion ($n-1 < q \leqslant p < \infty$) were defined and investigated; these mappings coincide with the class of mappings with bounded distortion if $q = p = n$.

In the paper [17] a locally uniform limit of a sequence of mappings with bounded $(\theta,1)$-weighted (q,p)-distortion was shown to be, also, a mapping with a bounded $(\theta,1)$-weighted (q,p)-distortion and an estimation similar to (2) was established. The proofs of the theorems like those in the article [17] and in this work are based on the method developed in [11] for extending Reshetnyak's result to Carnot groups.

In this paper we extend the above-mentioned assertions to the class of mappings with *k-finite distortion*, which arise naturally in the problem of operating with differential forms of degree k (see [12]).

2. Preliminaries. Let U be a domain in $\mathbb{R}^n$. We consider the Banach space $\mathcal{L}_p(U,\Lambda^k)$ of differential forms ω of degree k, $k = 1,\dots,n$, with measurable coefficients, which have the following finite norm: $\|\omega\|_p = \left(\int_U |\omega|^p\,dx\right)^{1/p}$.

A mapping $f : U \to \mathbb{R}^n$ is said to be *approximate differentiable* at a point $x \in U$ [3], if there exists a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\lim_{r\to 0} \frac{|\{y \in B(x,r) : |f(y) - f(x) - L(y-x)| > \varepsilon\}|}{r^n} = 0 \tag{3}$$

for any $\varepsilon > 0$. Here the symbol $|\cdot|$ denotes the Lebesgue measure. It is well known that the approximate differential is unique [3] if x is a density point. In what follows, it is denoted by the symbol $\operatorname{ap} Df(x)$.

In our paper we consider the mappings belonging to some Sobolev class. Such mappings are unconditionally approximately differentiable.

Let $\omega = \sum \omega_\beta \, dy^\beta$ be any k-form, $k = 1, \dots, n$, in W with continuous coefficients $\omega_\beta : W \to \mathbb{R}$, where the summation is over all k-dimensional ordered multi-indices $\beta = (\beta_1, \dots, \beta_k)$, $1 \leqslant \beta_1 < \dots < \beta_k \leqslant n$, and $dy^\beta = dy_{\beta_1} \wedge dy_{\beta_2} \wedge \dots \wedge dy_{\beta_k}$. Let a mapping $f = (f_1, \dots, f_n) : U \to W$ of Euclidean domains U, $W \subset \mathbb{R}^n$ be approximately differentiable almost everywhere in U. We write the pull-back of the k-form ω in the following way:

$$f^*\omega(x) = \sum_\beta \omega_\beta(f(x))\, df_{\beta_1} \wedge df_{\beta_2} \wedge \dots \wedge df_{\beta_k} = \sum_\alpha \sum_\beta \omega_\beta(f(x)) M_\alpha^\beta(x)\, dx^\alpha.$$

In other words, it is a k-form with measurable coefficients, which are defined for almost all $x \in U$ (here $df_{\beta_k} = \sum\limits_{i=1}^{n} \frac{\partial f_{\beta_k}}{\partial x_i}\, dx_i$ and the partial derivatives are understood in the approximate sense, $M_\alpha^\beta(x)$ are $(k \times k)$-minors of the matrix $\operatorname{ap} Df(x) = \left(\frac{\partial f_j}{\partial x_i}\right)$, $i, j = 1, \dots, n$, with ordered lines and columns).

We recall that the approximate differential $\operatorname{ap} Df(x) : T_x U \to T_{f(x)} W$ is defined a.e. in U. It generates canonically the linear mapping $\Lambda_k f(x) : \Lambda_k T_x U \to \Lambda_k T_{f(x)} W$ of the spaces of k-vectors, and the pullback operation f^* of k-forms. We denote the norm of the last linear mapping by the symbol $|\Lambda^k f(x)|$.

The minimal analytic and geometric properties of the mapping f were obtained in [12] for generating a bounded pullback operator

$$f^* : \mathcal{L}_p(W,\Lambda^k) \to \mathcal{L}_q(U,\Lambda^k), \quad 1 \leqslant q \leqslant p \leqslant \infty, \tag{4}$$

of differential forms of degree $k = 1, \dots, n$.

We say that an approximately differentiable mapping $f : U \to W$ has *k-finite distortion*, $1 \leqslant k \leqslant n$, (shortly $f \in \mathcal{CD}^k(U;W)$) if $\operatorname{rank} \operatorname{ap} Df(x) < k$ almost everywhere on a set Z. (Hereinafter $Z = \{x \in U : \det \operatorname{ap} Df(x) = 0\}$.) For $k = 1$ ($k = n-1$) and $f \in W^1_{1,\mathrm{loc}}(U)$ ($f \in W^1_{n-1,\mathrm{loc}}(U)$), this notion is well-known in literature: it is just the class of Sobolev mappings with finite distortion (codistortion), which is characterized by the property: $\operatorname{ap} Df(x) = 0$ ($\operatorname{adj} Df(x) = 0$) almost everywhere on Z (see [10], [14] for the second notion).

Besides of the property of k-finite distortion we consider mappings with a certain behavior of some characteristics of the distortion containing in itself the ratio $\frac{|\Lambda^k f(x)|^q}{|J(x,f)|}$, where $J(x,f) = \det \operatorname{ap} Df(x)$ [12]: *operator* (4) *is bounded if and only if the mapping* $f \in \mathcal{CD}^k(U;W)$ *and the distortion function* $\mathbb{W} \ni y \mapsto H_{k,q}(y) =$

$$= \begin{cases} \Bigl(\sum\limits_{x\in f^{-1}(y)\setminus(\Sigma\cup Z)} \frac{|\Lambda^k f(x)|^q}{|J(x,f)|}\Bigr)^{\frac{1}{q}} & \text{if} \quad f^{-1}(y)\setminus(\Sigma\cup Z)\neq\emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

belongs to $L_\varkappa(W)$ *where* $\frac{1}{\varkappa} = \frac{1}{q} - \frac{1}{p}$ *if* $q < p$, *and* $\varkappa = \infty$ $(\varkappa = q)$ *if* $q = p$ $(p = \infty)$. *Moreover, the norm of the operator* f^* *is comparable with the value* $\|H_{k,q}(\cdot) \mid L_\varkappa(W)\|$:

$$\alpha_{q,p}\bigl\|H_{k,q}(\cdot) \mid L_\varkappa(\Omega)\bigr\| \leqslant \|f^*\| \leqslant \bigl\|H_{k,q}(\cdot) \mid L_\varkappa(\Omega)\bigr\|$$

where $\alpha_{q,p}$ *is some constant.*

Hereinafter $\Sigma \subset U$ is a set of measure zero outside of which the mapping f has the Luzin property $\mathcal{N}$.

For homeomorphic mappings one can use a simpler characteristic.

Corollary 1. [12] *Let* $f : U \to W$ *be an approximate differentiable homeomorphism. The operator* $f^* : \mathcal{L}_p(W,\Lambda^k) \to \mathcal{L}_q(U,\Lambda^k)$, $1 \leqslant q \leqslant p \leqslant \infty$, $k = 1,\dots,n$, *is bounded if and only if the following conditions are satisfied:*

1) $f : U \to W$ *has the* k*-finite distortion;*
2) *the function* $K_{k,p}(x,f) = \begin{cases} \frac{|\Lambda^k f(x)|}{|J(x,f)|^{1/p}} & \text{if } J(x,f) \neq 0, \\ 0 & \text{otherwise,} \end{cases}$ *belongs to* $L_\varkappa(U)$, *where* $\frac{1}{\varkappa} = \frac{1}{q} - \frac{1}{p}$ *if* $q < p$, *and* $\varkappa = \infty$ $(\varkappa = q)$ *if* $q = p$ $(p = \infty)$.

In this case, the norm of the operator f^* *is comparable with*

$$\|K_{k,p}(\cdot,f) \,|\, L_\varkappa(U)\| : \alpha_{q,p}\bigl\|K_{k,p}(\cdot,f) \,|\, L_\varkappa(\Omega)\bigr\| \leqslant \|f^*\| \leqslant \bigl\|K_{k,p}(\cdot,f) \,|\, L_\varkappa(\Omega)\bigr\|,$$

where $\alpha_{q,p}$ *is some constant.*

3. Main results.

Definition 4. [12] *An approximately differentiable homeomorphism* $f : U \to W$ *belongs to the class* $\mathcal{CD}^k_{q,p}(U;W)$ *if the following conditions hold*

1) $f \in \mathcal{CD}^k(U;W)$;

2) $K_{k,p}(\cdot,f) \in L_{\varkappa}(U)$ *where* $\frac{1}{\varkappa} = \frac{1}{q} - \frac{1}{p}$, $1 \leqslant q \leqslant p < \infty$.

Theorem 4. *Let* $f_m \in \mathcal{CD}^k_{q,p}(U;W)$, $m \in \mathbb{N}$, *be a sequence of homeomorphisms of the Sobolev class* $W^1_{l,\mathrm{loc}}(U)$ *with* $k < l$, $q \leqslant l/k$, $1 < q \leqslant p < \infty$. *Suppose that the sequence* f_m *is locally bounded in* $W^1_l(U)$, *and locally uniformly converges to a homeomorphism* $f : U \to W$ *as* $m \to \infty$. *Assume also that there exists a sequence of functions* $U \ni x \mapsto M_m(x)$, *belonging to* $L_{\varkappa}(U)$, *that is bounded in* $L_{\varkappa}(U)$, $\frac{1}{\varkappa} = \frac{1}{q} - \frac{1}{p}$, *for which the inequality*

$$K_{k,p}(x,f_m) \leqslant M_m(x) \text{ for almost all } x \in U \tag{5}$$

is true.

Then there exists a function $U \ni x \mapsto M(x)$ *of* $L_{\varkappa}(U)$ *such that some subsequence*

(i) *in the case* $1 < q < p < \infty$: *of functions* $\{M_m(x)^{\varkappa}\}_{m\in\mathbb{N}}$ *converges in the biting sense to* $M(x)^{\varkappa}$;

(ii) *in the case* $1 < q = p < \infty$: *of numbers* $\{\|M_m \mid L_\infty(U)\|\}_{m\in\mathbb{N}}$ *converges to* $M = \varliminf\limits_{m\to\infty} \|M_m \mid L_\infty(U)\|$;

the limit mapping f *belongs to* $\mathcal{CD}^k_{q,p}(U;W)$ *and* $K_{k,p}(\cdot,f) \in L_{\varkappa}(U)$, *where* $\frac{1}{\varkappa} = \frac{1}{q} - \frac{1}{p}$.

Moreover, the inequalities

$$\begin{cases} K_{k,p}(x,f) \leqslant M(x) & \text{in the case } q < p, \\ K_{k,p}(x,f) \leqslant M & \text{in the case } q = p, \end{cases}$$

hold for almost all $x \in U$.

In the proof we use some arguments from the paper [17], where Theorem 4 is proved for $k = 1$.

Proof. It follows from the conditions of the theorem that $f \in W^1_{l,\mathrm{loc}}(U)$. First, we show that the limit mapping f belongs to $\mathcal{CD}^k(U;W)$. For doing this, we show that the mapping f induces a bounded operator $f^* : \mathcal{L}_p(W,\Lambda^k) \to \mathcal{L}_q(U,\Lambda^k)$, $1 < q \leqslant p < \infty$. Since every mapping $f_m \in \mathcal{CD}^k_{q,p}(U;W)$, it follows from Corollary 1 that the homeomorphism $f_m : U \to W$ induces the bounded operator $f^*_m : \mathcal{L}_p(W,\Lambda^k) \to \mathcal{L}_q(U,\Lambda^k)$, $1 < q \leqslant p < \infty$, $m \in \mathbb{N}$. Moreover, the norms of the operators f^*_m are totally bounded

$$\|f^*_m\| \leqslant \|K_{k,p}(\cdot,f_m) \mid L_{\varkappa}(U)\| \leqslant \|M_m(\cdot) \mid L_{\varkappa}(U)\| \leqslant \tilde{M} < \infty.$$

Take a k-form $\omega \in \mathcal{L}_p(W,\Lambda^k) \cap \mathcal{C}(W,\Lambda^k)$ and set $\sigma_m = f_m^*(\omega)$. Since $\|f_m^*\| \leqslant \tilde{M}$, the sequence of forms σ_m is bounded in $\mathcal{L}_q(U,\Lambda^k)$. Therefore, we can extract a weakly converging subsequence. We assume that the sequence σ_m converges weakly in $\mathcal{L}_q(U,\Lambda^k)$ to a form σ_0. The weak convergence of forms means that coefficients of the forms σ_m converge weakly in $L_q(U)$ to the corresponding coefficients of the form σ_0. Since the sequence σ_m converges weakly in $\mathcal{L}_q(U,\Lambda^k)$ to σ_0 as $m \to \infty$, we have

$$\|\sigma_0 \mid L_q(U)\| \leqslant \varliminf_{m\to\infty} \|\sigma_m \mid L_q(U)\| = \varliminf_{m\to\infty} \|f_m^*\omega \mid L_q(U)\| \leqslant \\ \leqslant \varliminf_{k\to\infty} \|f_m^*\| \cdot \|\omega \mid L_p(U)\| \leqslant \tilde{M} \cdot \|\omega \mid L_p(U)\|. \quad (6)$$

The following lemma is proved in the book [8, Chapter 2, § 4].

Lemma 1. [8] *Suppose U is an open subset in $\mathbb{R}^n$, and suppose that $\varphi_m = (\varphi_{m1}, \varphi_{m2}, \ldots, \varphi_{mk})$, $1 \leqslant k \leqslant n$, $m = 1, 2, \ldots$, is a sequence of vector-functions of $W^1_{l,\mathrm{loc}}(U)$, $k \leqslant l$, locally bounded in $W^1_l(U)$. Assume that, as $m \to \infty$, the functions φ_m converge in $L_{1,\mathrm{loc}}$ to a vector function $\varphi_0 = (\varphi_{01}, \varphi_{02}, \ldots, \varphi_{0k})$, and set $\omega_m = d\varphi_{m1} \wedge d\varphi_{m2} \wedge \ldots \wedge d\varphi_{mk}$. Then the sequence of forms ω_m weakly converges in $L_{l/k,\mathrm{loc}}(U)$ to a form ω_0*[1].

Since the homeomorphisms f_m locally uniformly converge to f and the form ω has continuous coefficients, the functions $\omega_\beta(f_m(x))$ converge locally uniformly to $\omega_\beta(f(x))$ as $m \to \infty$. Lemma 1 implies that the minors of the matrices Df_m converge weakly in $L_{l/k,\mathrm{loc}}(U)$ to minors of the matrix Df. Therefore, the forms σ_m converge weakly in $L_{l/k,\mathrm{loc}}(U)$ to $f^*(\omega)$.

It is not hard to see that both limits σ_0 and $f^*(\omega)$ coincide: $\sigma_0 = f^*(\omega)$. In view of (6) the mapping f induces a bounded operator $f^* : \mathcal{L}_p(W,\Lambda^k) \to \mathcal{L}_q(U,\Lambda^k)$, $1 < q \leqslant p < \infty$. By Corollary 1, $f \in \mathcal{CD}^k(U;W)$.

First, we consider the case $q < p$. The following lemma is valid.

Lemma 2. [2] *Every sequence of mappings h_m, $m = 1, 2, \ldots$, that is bounded in $L_1(U)$, contains a subsequence, converging in the biting sense to some function $h \in L_1(U)$.*

This lemma implies existence of a function $U \ni x \mapsto M(x)$ of $L_\varkappa(U)$ such that some subsequence of the function $h_m = M_m(x)^\varkappa$ converges

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[1] It means that the sequence of forms ω_m converges weakly in $L_{l/k}(D)$ to a form ω_0 on every subdomain $D \Subset U$.

in the biting sense to $h = M(x)^{\varkappa}$. We assume that the given sequence $M_m(x)^{\varkappa}$ converges in the biting sense to the function $M(x)^{\varkappa}$ (the set of the Definition 3 is denoted by E_ν).

Now we estimate the distortion coefficient of the limit mapping f. For this, we consider estimates on a set E_ν. Let Z_m be the set of zeros of the Jacobian of the mapping f_m. Since the rank of the matrix Df_m on the set Z_m is less then k, it follows that all k-th-order minors are equal to zero on the set Z_m.

Applying the Hölder inequality, and taking into account that $\frac{q}{\varkappa}+\frac{q}{p}=1$ on each intersection $E_\nu \cap B(x_0,r)$, where $x_0 \in E_\nu$, $B(x_0,r) \Subset U$, in view of (5) we have

$$\begin{aligned}
\int\limits_{E_\nu\cap B(x_0,r)} |\Lambda^k f_m(x)|^q\,dx &= \int\limits_{(E_\nu\cap B(x_0,r))\setminus Z_m} \frac{|\Lambda^k f_m(x)|^q}{|J(x,f_m)|^{\frac{q}{p}}}|J(x,f_m)|^{\frac{q}{p}}\,dx \leqslant \\
&\leqslant \Bigg(\int\limits_{(E_\nu\cap B(x_0,r))\setminus Z_m} \frac{|\Lambda^k f_m(x)|^{q\frac{\varkappa}{q}}}{|J(x,f_m)|^{\frac{q}{p}\frac{\varkappa}{q}}}\,dx\Bigg)^{\frac{q}{\varkappa}}\Bigg(\int\limits_{E_\nu\cap B(x_0,r)} |J(x,f_m)|^{\frac{q}{p}\frac{p}{q}}\,dx\Bigg)^{\frac{q}{p}} = \\
&= \Bigg(\int\limits_{E_\nu\cap B(x_0,r)} (K_{k,p}(f_m))^{\varkappa}(x)\,dx\Bigg)^{\frac{q}{\varkappa}}\Bigg(\int\limits_{E_\nu\cap B(x_0,r)} |J(x,f_m)|\,dx\Bigg)^{\frac{q}{p}} \leqslant \\
&\leqslant \Bigg(\int\limits_{E_\nu\cap B(x_0,r)} M_m^{\varkappa}(x)\,dx\Bigg)^{\frac{q}{\varkappa}}\Bigg(\int\limits_{B(x_0,r)} |J(x,f_m)|\,dx\Bigg)^{\frac{q}{p}}. \qquad (7)
\end{aligned}$$

Elements of the matrix $\Lambda^k(f_m)(x)$ are the k-th-order minors of ap $D_m f(x)$. In view of Lemma 1, they converge weakly in $L_{l/k,\mathrm{loc}}(U)$ to elements of the matrix $\Lambda^k(f)(x)$. Since $q \leqslant l/k$, $\Lambda^k(f_m)(x)$ converges weakly in $L_{q,\mathrm{loc}}(U)$ to $\Lambda^k(f)(x)$. Since the norm is semicontinuous in the Banach space L_q, the left-hand side of the inequality can be estimated as

$$\int\limits_{E_\nu\cap B(x_0,r)} |\Lambda^k f(x)|^q\,dx \leqslant \varliminf_{m\to\infty} \int\limits_{E_\nu\cap B(x_0,r)} |\Lambda^k f_m(x)|^q\,dx.$$

We have also $\int\limits_{B(x_0,r)} |J(x,f_m)|\,dx \leqslant |f_m(B(x_0,r)|$.

Since $|f(B(x_0,r))| < \infty$ and the mapping f is a homeomorphism, the images $f(S(x_0,r))$ of the spheres $S(x_0,r)$ do not intersect under different

r. It follows that the n-measure of the image of any sphere is zero for almost all r: $|f(S(x_0,r))| = 0$. We fix r so that

$$|f(S(x_0,r))| = 0$$

and surround the image of the sphere $f(S(x_0,r))$ by an ε-neighborhood U_ε. Since the mappings f_m converge locally uniformly to the mapping f, it follows that, starting from a number m_0, the images of the spheres $f_m(S(x_0,r))$, $m \geqslant m_0$, are contained in this ε-neighborhood. It is clear that $|U_\varepsilon| \to 0$ as $\varepsilon \to 0$, and hence $|f_m(B(x_0,r)| \to |f(B(x_0,r)|$ as $m \to \infty$.

Taking into account that $M_m(x)^\varkappa$ converge in the biting sense to $M(x)^\varkappa$, we pass to the lower limit in (7) as $m \to \infty$. We get

$$\int\limits_{B(x_0,r)} |\Lambda^k f(x)|^q \chi_{E_\nu}(x)\, dx \leqslant \Bigg(\int\limits_{B(x_0,r)} M^\varkappa(x) \chi_{E_\nu}(x)\, dx \Bigg)^{\frac{q}{\varkappa}} |f(B(x_0,r)|^{\frac{q}{p}}.$$

Dividing both sides of this inequality by the measure of the ball $B(x_0,r)$, we obtain the following inequality

$$\frac{1}{|B(x_0,r)|} \int\limits_{B(x_0,r)} |\Lambda^k f(x)|^q \chi_{E_\nu}(x)\, dx \leqslant$$

$$\leqslant \Bigg(\frac{1}{|B(x_0,r)|} \int\limits_{B(x_0,r)} M^\varkappa(x) \chi_{E_\nu}(x)\, dx \Bigg)^{\frac{q}{\varkappa}} \bigg(\frac{|f(B(x_0,r)|}{|B(x_0,r)|} \bigg)^{\frac{q}{p}}. \qquad (8)$$

Since the homeomorphism f is Sobolev differentiable, then by [13, Section 2.3, formula (2.5)] we have

$$\frac{|f(B(x_0,r)|}{|B(x_0,r)|} \to |J(x_0,f)| \text{ as } r \to 0 \text{ for almost all } x_0 \in E_\nu.$$

Hence, by the Lebesgue differentiability theorem, letting r go to 0 we obtain that

$$|\Lambda^k f(x)|^q \leqslant M^q(x) |J(x,f)|^{\frac{q}{p}} \quad \text{for almost all } x \in E_\nu. \qquad (9)$$

As $U = \bigcup\limits_\nu E_\nu$, the point-wise inequality (9) holds in U almost everywhere.

In the case $1 < q = p < \infty$ we can assume that a sequence of numbers $\{\|M_m \mid L_\infty(U)\|\}_{m\in\mathbb{N}}$ converges to $M = \varliminf\limits_{m\to\infty} \|M_m \mid L_\infty(U)\| \in \mathbb{R}$. In this case, instead of (7), for any $\varepsilon > 0$ there exists m_1 such that for all $m \geqslant m_1$ we have

$$\int\limits_{B(x_0,r)} |\Lambda^k f_m(x)|^q\, dx \leqslant (M+\varepsilon)\Bigg(\int\limits_{B(x_0,r)} |J(x,f_m)|\, dx\Bigg).$$

Further, proceeding as in the case $q < p$, we obtain the estimation

$$K_{k,p}(x,f) \leqslant (M+\varepsilon) \text{ for almost all } x \in U.$$

Since $\varepsilon > 0$ is an arbitrary number, we get the desired estimation.

By the Corollary 1, we have proved that the limit mapping f belongs to $\mathcal{CD}^k_{q,p}(U;W)$. □

As a straightforward consequence of Theorem 4 we get the following

Corollary. [16] *Let $f_m \in \mathcal{CD}^k_{q,p}(U;W)$, $m \in \mathbb{N}$, be a sequence of homeomorphisms of the Sobolev class $W^1_{l,\mathrm{loc}}(U)$ with $k < l$, $q \leqslant l/k$, $1 < q \leqslant \leqslant p < \infty$. Suppose that the sequence f_m is locally bounded in $W^1_l(U)$, and converge locally uniformly to a homeomorphism $f : U \to W$ as $m \to \infty$. Assume also that there exists a function $M(x) \in L_\varkappa(U)$, $\frac{1}{\varkappa} = \frac{1}{q} - \frac{1}{p}$, such that*

$$K_{k,p}(\cdot,f_m)(x) \leqslant M(x) \text{ for all } m \in \mathbb{N}$$

in U almost everywhere. Then the limit mapping f belongs to $\mathcal{CD}^k_{q,p}(U;W)$ and $K_{k,p}(\cdot,f) \in L_\varkappa(U)$, where $\frac{1}{\varkappa} = \frac{1}{q} - \frac{1}{p}$.

Moreover, the inequality $K_{k,p}(x,f) \leqslant M(x)$ holds almost everywhere.

Remark. There exists a misprint in the paper [16]: in the statement of the main result the condition $q \leqslant l/k$ is missing.

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