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SOLVABILITY HOMOGENEOUS RIEMANN-HILBERT BOUNDARY VALUE PROBLEM WITH SEVERAL POINTS OF TURBULENCE

Abstract. We consider the so called Hilbert boundary value problem with infinite index in the unit disk. Its coefficient is assumed to be Hölder-continuous everywhere on the unit circle excluding a finite set of points. At these points its argument has power discontinuities of orders less than one. We obtain formulas for the general solution and describe completely the solvability picture in a special functional class. Our technique is based on the theory of entire functions and the geometric theory of functions.

Key words: *Riemann–Hilbert problem, maximum principle, infinite index, entire functions*

2010 Mathematical Subject Classification: *30E25*

1. Statement of the problem. Let D be the unit disk in the plane of complex variable z, $L = \partial D$. We consider the Riemann–Hilbert boundary value problem for analytic in D functions with the boundary condition

$$a(t)\operatorname{Re}\Phi(t) - b(t)\operatorname{Im}\Phi(t) = 0,\ t \in L.$$

The given coefficients a, b satisfy the Hölder condition everywhere on the unit circle except vicinities of a finite set of singular points $t_j = e^{i\theta_j}$, $j = \overline{1, n}$, where their arguments $(\arg[a(t) - ib(t)])$ have diskontinuities of the second kind. Thus, the problem under consideration belongs to the class of Hilbert boundary value problems with undefined index. A great body of publications on this subject beginning from the fundamental N. V. Govorov's investigations [3] (see also [14], [12], [5]) are dealing with the case, where the problem is stated on the real axis and the turbulence

(i. e., the essential discontinuity of argument of coefficients) is located in infinity.

A boundary value problem with infinite index and more complicated asymptotics of argument of the coefficient is investigated by V. N. Monahov, E. V. Semenko [6]. In the same monograph the problem with infinite index is firstly considered on Riemann surfaces. M. I. Zhuravleva [15], [16] studied the problems with infinite sets of zeros, poles and jumps of the coefficient on a half of the real axis.

The Hilbert problem for the half-plane with two-sided turbulence of power order less than one is considered by I. E. Sandrygaylo [11]. He obtained formulas for its general solution and described the main cases of solvability by means of the N. I. Muskhelishvili method and results of [12]. R. B. Salimov and P. L. Shabalin [9] applied a more successful formula for the general solution, deduced in [8] by means of the F. D. Gahov regularizing multiplier. As a result, they obtained the full picture of solvability of the problem. Alehno A. G. [1] used the same technique for solving the problem with the logarithmic turbulence. Sevruk A. B. [13] solved the Riemann–Hilbert problem for piecewise analytic functions with many-sided turbulence at the infinity point.

The Hilbert boundary value problem with several points of turbulence was formulated first by R. B. Salimov, A. Kh. Fatykhov and P. L. Shabalin in [10]. The authors described the general solution, existence and uniqueness of solutions, and the set of solutions in the case of non-uniqueness. This article continues this investigation. We define more accurately the general solution and clarify the conditions of solvability. Our results are of interest for the research of boundary-value problems for generalized Cauchy–Riemann system of equations with singular manifolds. For instance, A. B. Rasulov [7] found out that solvability of certain boundary-value problem for generalized analytic functions is immediately connected with solvability of corresponding boundary value problems for analytic functions with finite set of turbulences.

We study the customary statement of the Riemann–Hilbert problem in the form

$$\mathrm{Re}[e^{-i\nu(\theta)}\Phi(t)] = 0, \quad t = e^{i\theta}, t \neq t_j, \ j = \overline{1,n}. \tag{1}$$

Here $G(t) = a(t) - ib(t)$, and $a^2(t) + b^2(t) \neq 0$ on L, and the function $\nu(\theta) = \arg G(t)$, $t = e^{i\theta}$, satisfies the Hölder condition everywhere on L except vicinities of points t_j, where it has essential discontinuities. We assume that

$$\nu(\theta) = \sum_{j=1}^{n} \nu_j(\theta) + \widetilde{\nu}(\theta), \quad \nu_j(\theta) = \begin{cases} \dfrac{\nu_j^-}{|\sin((\theta - \theta_j)/2)|^{\rho_j}}, & 0 \leqslant \theta < \theta_j, \\[3mm] \dfrac{\nu_j^+}{|\sin((\theta - \theta_j)/2)|^{\rho_j}}, & \theta_j < \theta \leqslant 2\pi, \end{cases} \tag{2}$$

where ν_j^+, ν_j^-, ρ_j, are known values $0 < \rho_j < 1$, and the function $\widetilde{\nu}(\theta)$ satisfies the Hölder condition on L. We seek an analytic and bounded in the unit disk D function $\Phi(z)$ satisfying the boundary condition (1).

2. Solution of the homogeneous problem. In order to isolate the essential discontinuities (2) at the points t_j, we introduce in the unit disk D the function

$$P_j(z) + iQ_j(z) = \frac{l_j e^{i\alpha_j}}{(z - t_j)^{\rho_j}}, \quad j = \overline{1, n}.$$

Its boundary values on the circle L are

$$P_j(e^{i\theta}) + iQ_j(e^{i\theta}) = \frac{l_j\Big[\cos(\alpha_j - \gamma_j(\theta)\rho_j) + i\sin(\alpha_j - \gamma_j(\theta)\rho_j)\Big]}{2^{\rho_j}|\sin((\theta - \theta_j)/2)|^{\rho_j}}, \tag{3}$$

where

$$\gamma_j(\theta) := \arg(e^{i\theta} - e^{i\theta_j}) = \begin{cases} (3\pi + \theta + \theta_j)/2, & 0 \leqslant \theta < \theta_j, \\[2mm] (\pi + \theta + \theta_j)/2, & \theta_j < \theta \leqslant 2\pi. \end{cases}$$

Let us fix the values $l_j > 0$ and $\alpha_j \in [0, 2\pi]$ such that the condition

$$\begin{cases} \dfrac{l_j}{2^{\rho_j}} \cos\left(\alpha_j - \dfrac{3\pi}{2}\rho_j - \theta_j\rho_j\right) = \nu_j^-, \\[4mm] \dfrac{l_j}{2^{\rho_j}} \cos\left(\alpha_j - \dfrac{\pi}{2}\rho_j - \theta_j\rho_j\right) = \nu_j^+ \end{cases}$$

fulfils. We separate its imaginary and real parts of formula (3), and obtain the representations

$$P_j(e^{i\theta}) = \begin{cases} \dfrac{\nu_j^+ \sin\left(\frac{\theta_j - \theta}{2}\rho_j\right) + \nu_j^- \sin\left(\left(\pi - \frac{\theta_j - \theta}{2}\right)\rho_j\right)}{\left|\sin\frac{\theta - \theta_j}{2}\right|^{\rho_j} \sin(\pi\rho_j)}, & 0 \leqslant \theta < \theta_j, \\[6mm] \dfrac{\nu_j^- \sin\left(\frac{\theta - \theta_j}{2}\rho_j\right) + \nu_j^+ \sin\left(\left(\pi - \frac{\theta - \theta_j}{2}\right)\rho_j\right)}{\left|\sin\frac{\theta - \theta_j}{2}\right|^{\rho_j} \sin(\pi\rho_j)}, & \theta_j < \theta \leqslant 2\pi, \end{cases}$$

$$Q_j(e^{i\theta}) = \begin{cases} \dfrac{\nu_j^- \cos\left(\left(\pi - \frac{\theta_j-\theta}{2}\right)\rho_j\right) - \nu_j^+ \cos\left(\frac{\theta_j-\theta}{2}\rho_j\right)}{\left|\sin\frac{\theta-\theta_j}{2}\right|^{\rho_j} \sin(\pi\rho_j)}, & 0 \leqslant \theta < \theta_j, \\[4mm] \dfrac{\nu_j^- \cos\left(\frac{\theta-\theta_j}{2}\rho_j\right) - \nu_j^+ \cos\left(\left(\pi - \frac{\theta-\theta_j}{2}\right)\rho_j\right)}{\left|\sin\frac{\theta-\theta_j}{2}\right|^{\rho_j} \sin(\pi\rho_j)}, & \theta_j < \theta \leqslant 2\pi. \end{cases}$$

These relations imply the asymptotic formulas

$$P_j(e^{i\theta}) = \begin{cases} \dfrac{\nu_j^-}{|\sin(\theta-\theta_j)/2|^{\rho_j}} + O\left(|\theta - \theta_j|^{1-\rho_j}\right), & \theta \to \theta_j - 0, \\[4mm] \dfrac{\nu_j^+}{|\sin(\theta-\theta_j)/2|^{\rho_j}} + O\left(|\theta - \theta_j|^{1-\rho_j}\right), & \theta \to \theta_j + 0, \end{cases} \tag{4}$$

$$Q_j(e^{i\theta}) = \begin{cases} \dfrac{\nu_j^- \cos(\pi\rho_j) - \nu_j^+}{|\sin(\theta-\theta_j)/2|^{\rho_j} \sin(\pi\rho_j)} + O\left(|\theta - \theta_j|^{1-\rho_j}\right), & \theta \to \theta_j - 0, \\[4mm] \dfrac{\nu_j^- - \nu_j^+ \cos(\pi\rho_j)}{|\sin(\theta-\theta_j)/2|^{\rho_j} \sin(\pi\rho_j)} + O\left(|\theta - \theta_j|^{1-\rho_j}\right), & \theta \to \theta_j + 0. \end{cases}$$

The function $\hat{\nu}(e^{i\theta}) = \nu(e^{i\theta}) - \sum_{j=1}^{n} P_j(e^{i\theta})$ is continuous by virtue of (4). We introduce the Cauchy type integral

$$\Gamma(z) = \frac{1}{2\pi} \int\limits_0^{2\pi} \hat{\nu}(\theta) \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta,$$

and rewrite the boundary condition of homogeneous problem (1), (2) as

$$Re\left[e^{-i\Gamma(t)} \prod_{j=1}^{n} \exp\left\{ -i \frac{l_j e^{i\alpha_j}}{(t - t_j)^{\rho_j}} \right\} \Phi(t) \right] = 0. \tag{5}$$

Consider the analytic in the disk D function

$$F(z) = i e^{-i\Gamma(z)} \prod_{j=1}^{n} \exp\left\{ -i \frac{l_j e^{i\alpha_j}}{(z - t_j)^{\rho_j}} \right\} \Phi(z). \tag{6}$$

Its boundary values, by virtue of (5), satisfy the relation

$$\operatorname{Im} F(t) = 0, \quad t \in L, \quad t \neq t_j. \tag{7}$$

We express the desired function from equation (6):

$$\Phi(z) = -ie^{i\Gamma(z)} \prod_{j=1}^{n} \exp\left\{ i\frac{l_j e^{i\alpha_j}}{(z-t_j)^{\rho_j}} \right\} F(z). \qquad (8)$$

If function (8) is a bounded solution of problem (1), then, clearly, there exists an analytic in unit disc D function (6) growing near t_j no faster than $C_1 e^{C_2/|t_j-z|^{\rho_j}}$, $0 < \rho_j < 1$, i.e., it satisfies the inequalities

$$|F(z)| \leqslant C_1 e^{C_2/|t_j-z|^{\rho_j}}, \; z \to t_j, \, z \in D, \, 0 < \rho_j < 1, \, j = \overline{1,n}, \qquad (9)$$

and its boundary values satisfy condition (7) and the inequality

$$|F(t)| \leqslant C \exp\left\{ \sum_{j=1}^{n} Q_j(t) \right\}, \quad C = \text{const}, \, t \in L. \qquad (10)$$

In order to prove the converse proposition, we need the following version of the maximum principle for analytical functions ([4], pp.456, 457). One can find its proof in [10].

Lemma 1. *If a regular in the disk D function $F(z)$ satisfies the condition*

$$\varlimsup_{z \to t} |F(z)| \leqslant C, \quad z \in D, \, t \in \partial D,$$

at all boundary points of boundary excluding a finite set of points t_j, where it grows no faster than $C_1 e^{C_2/|z-t_j|^{\rho_j}}$, $0 < \rho_j < 1$, near t_j, then $|F(z)| \leqslant C_0$, $C_0 = \text{const}$, in the whole disc D.

The following inequality

$$\varlimsup_{z \to t} |F(z)| \prod_{j=1}^{n} e^{-Q_j(z)} \leqslant C,$$

is valid everywhere on L by virtue of (10). Hence, function (8) is bounded on D by virtue of the maximum principle. Thus, we have proved

Theorem 1.   *The solution $\Phi(z)$ of the homogeneous boundary-value problem is bounded in D if and only if the function $F(z)$ in the formula of the general solution (8) satisfies the restriction on its growth (9) in the disk D and conditions (7) and (10) on its boundary.*

3. Solvability. Theorem 1 means that existence and number of solutions of problem (1), (2) depend on existence and number of analytic functions satisfying conditions (7), (10), (9). We denote by $U(Q_1, Q_2, \ldots, Q_n)$ the set (maybe, empty) of all functions that are analytic everywhere except the points $t_1, t_2, \ldots, t_n$, where they admit essential singularities satisfying conditions (7), (10), (9).

Together with the Hilbert boundary value problem (1), (2) with finite set of turbulences, we consider n boundary value problems on L with a single turbulence for each.

For every $j = \overline{1, n}$ we consider the homogeneous Hilbert boundary value problem on the circle L with a single point of two-sided turbulence

$$Re[e^{-i\mu_j(\theta)}\Phi_j(t)] = 0, \quad t = e^{i\theta}, t \neq t_j, \tag{11}$$

$$\mu_j(\theta) = \nu_j(\theta) + \widetilde{\nu}_j(\theta), \nu_j(\theta) = \begin{cases} \dfrac{\nu_j^-}{|\sin((\theta - \theta_j)/2)|^{\rho_j}}, \ 0 \leqslant \theta < \theta_j, \\[3mm] \dfrac{\nu_j^+}{|\sin((\theta - \theta_j)/2)|^{\rho_j}}, \ \theta_j < \theta \leqslant 2\pi, \end{cases} \tag{12}$$

where $\widetilde{\nu}_j(\theta)$ satisfies the Hölder condition on the circle L.

The problem (11), (12) is studied for $\theta_j = \pi$, in the article [2]. According to the results of this paper, we rewrite the general solution $\Phi_j(z)$ for our case in the form

$$\Phi_j(z) = -ie^{i\Gamma_j(z)} \exp\left\{i\frac{l_j e^{i\alpha_j}}{(z - t_j)^{\rho_j}}\right\} F_j(z). \tag{13}$$

Here $\Gamma_j(z)$ is a Cauchy type integral on L with density $\widetilde{\nu}_j(\theta)$; $\alpha_j, l_j, t_j, \rho_j$ are the same values as above, and analytical in D function $F_j(z)$ satisfies the condition

$$|F_j(z)| \leqslant C_1 e^{C_2/|t_j - z|^{\rho_j}}, \ z \in D, \ 0 < \rho_j < 1, \ C_1, C_2 = \text{const},$$

$$|F_j(t)| \leqslant Ce^{Q_j(t)}, \ C = \text{const}, \quad \text{Im } F(t) = 0, \quad t \in L. \tag{14}$$

We denote by $U(Q_j)$ the set of analytic on the whole complex plane except the point t_j functions satisfying (14) in the disk D. Then the following theorem is true.

Theorem 2. *The homogeneous boundary value problem* (1), (2) *is solvable if and only if each of n problems* (11), (12) *is solvable.*

Proof. Let problem (11), (12) be solvable for any $j = \overline{1,n}$. This means that for any j there exists an analytic in D function $F_j(z)$ satisfying condition (14) and continuous on the boundary except the point t_j, where its growth does not exceed $C_1 e^{C_2/|t_j-z|^{\rho_j}}$. We introduce the function

$$F(z) := \prod_{j=1}^{n} F_j(z).$$

Clearly, by virtue of (14)), the introduced function $F(z)$ satisfies conditions (7), (10), (9). Consequently, the formula (8) is meaningful, and the problem (1), (2) has a solution.

Now let us consider a case where the problem (1), (2) has a solution, i. e., there exists a function $F(z) \in\in U(Q_1, Q_2, \ldots, Q_n)$. We prove the necessity first for $n = 2$. Assume that the problem (11), (12) has a solution for $j = 1$ and ha not have one for $j = 2$, i. e., there exist functions $F(z) \in\in U(Q_1, Q_2)$, $F_1(z) \in U(Q_1)$. But then we have $F(z) - F_1(z) \in U(Q_2)$, and the problem (11), (12) is solvable, too. Let us suppose now that the function $F(z) \in U(Q_1, Q_2)$ exists, but the problem (11), (12) is not solvable neither for $j = 1$, nor for $j = 2$. We denote by $G_1(z)$ the main part of expansion of the function $F(z)$ in the Laurent series in powers of $1/(z - t_1)$. Then the function $R_1(z) := F(z) - G_1(z)$ is regular in $\overline{D}$, and $\operatorname{Im} R_1(t) = -\operatorname{Im} G_1(t)$, $t \in L$. Let $S_1(z)$ be a solution of the Schwartz problem in the unit disc for a continuous function $\operatorname{Im} G_1(t)$. Obviously, then the function $F(z) - G_1(z) + i S_1(z) = F_2(z)$ belongs to $U(Q_2)$, and the problem (11), (12) is solvable for $j = 2$; this contradicts our assumption. Thus, Theorem 2 is proved for $n = 2$; the case $n > 2$ can be justified by means of mathematical induction. $\square$

The complete picture of solvability of the problem (11), (12) in the class of bounded in the unit disc analytic functions is described by the following two theorems [2].

Theorem 3. *Let $\rho_j < 1/2$. Then the homogeneous boundary value problem (11), (12)*
a) has no non-trivial bounded solutions if either $\nu_j^- \cos(\pi\rho_j) - \nu_j^+ < 0$ or $\nu_j^+ \cos(\pi\rho_j) - \nu_j^- > 0$;

b) has a unique solution $\Phi_j(z) = -iAe^{\Gamma(z)}e^{-l\frac{e^{i\alpha_j}}{(z-1)_j^\rho}}$, $A = \text{const}$, $\operatorname{Im} A = 0$, if

$$\begin{cases} \nu_j^- \cos(\pi\rho_j) - \nu_j^+ = 0, \\ \nu_j^+ \cos(\pi\rho_j) - \nu_j^- < 0, \end{cases} \quad \text{or} \quad \begin{cases} \nu_j^+ \cos(\pi\rho_j) - \nu_j^- = 0, \\ \nu_j^- \cos(\pi\rho_j) - \nu_j^+ > 0; \end{cases}$$

c) has infinite set of solutions of the form (13), where $F_j(z)$ is arbitrary analytic in D function satisfying there inequality (14), if $\nu^- \cos(\pi\rho) - \nu^+ > 0$ and $\nu^+ \cos(\pi\rho) - \nu^- < 0$.

Theorem 4. *If $1 > \rho \geqslant 1/2$ and either $\nu^- < 0$ or $\nu^+ > 0$, then the homogeneous boundary-value problem has only null solution; if $1 > \rho \geqslant 1/2$ and conditions $\nu^- \geqslant 0$ and $\nu^+ \leqslant 0$ hold, then the homogeneous boundary value problem has infinite set of solutions (13), where $F(z)$ is arbitrary analytic in D function satisfying condition (14).*

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