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MULTIVARIATE IYENGAR TYPE INEQUALITIES FOR RADIAL FUNCTIONS

Abstract. Here we present a variety of multivariate Iyengar type inequalities for radial functions defined on the shell and ball. Our approach is based on the polar coordinates in $\mathbb{R}^N$, $N \geqslant 2$, and the related multivariate polar integration formula. Via this method we transfer well-known univariate Iyengar type inequalities and univariate author's related results into multivariate Iyengar inequalities.

Key words: *Iyengar inequality, Polar coordinates, radial function, Shell, Ball.*

2010 Mathematical Subject Classification: *26D10, 26D15.*

1. Background. In the year 1938, Iyengar [5] proved the following interesting inequality.

Theorem 1. *Let f be a differentiable function on $[a,b]$ and $|f'(x)| \leqslant M_1$. Then*

$$\left| \int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) \right| \leqslant \frac{M_1(b-a)^2}{4} - \frac{(f(b)-f(a))^2}{4M_1}. \tag{1}$$

In 2001, X.-L. Cheng [4] proved that

Theorem 2. *Let $f \in C^2([a,b])$ and $|f''(x)| \leqslant M_2$. Then*

$$\left| \int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{1}{8}(b-a)^2(f'(b)-f'(a)) \right| \leqslant$$

$$\leqslant \frac{M_2}{24}(b-a)^3 - \frac{(b-a)}{16M_2}\Delta_1^2, \tag{2}$$

where

$$\Delta_1 = f'(a) - \frac{2(f(b) - f(a))}{(b-a)} + f'(b).$$

In 1996, Agarwal and Dragomir [1] obtained a generalization of (1):

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function such that for all $x \in [a,b]$ with $M > m$ we have $m \leqslant f'(x) \leqslant M$. Then*

$$\left| \int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) \right| \leqslant$$

$$\leqslant \frac{(f(b) - f(a) - m(b-a))(M(b-a) - f(b) + f(a))}{2(M-m)}. \tag{3}$$

In [7], Qi proved

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable function such that for all $x \in [a,b]$ with $M > 0$ we have $|f''(x)| \leqslant M$. Then*

$$\left| \int_a^b f(x)\,dx - \frac{(f(a)+f(b))}{2}(b-a) + \frac{(1+Q^2)}{8}(f'(b) - f'(a))(b-a)^2 \right| \leqslant$$

$$\leqslant \frac{M(b-a)^3}{24}\left(1 - 3Q^2\right), \tag{4}$$

where

$$Q^2 = \frac{\left(f'(a) + f'(b) - 2\left(\frac{f(b)-f(a)}{b-a}\right)\right)^2}{M^2(b-a)^2 - (f'(b) - f'(a))^2}. \tag{5}$$

In 2005, Zheng Liu, [6], proved the following:

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function such that f' is integrable on $[a,b]$ and for all $x \in [a,b]$ with $M > m$ we have*

$$m \leqslant \frac{f'(x) - f'(a)}{x-a} \leqslant M \quad \text{and } m \leqslant \frac{f'(b) - f'(x)}{b-x} \leqslant M. \tag{6}$$

Then

$$\left| \int_a^b f(x)\,dx - \frac{(f(a)+f(b))}{2}(b-a) + \left(\frac{1+P^2}{8}\right)(f'(b) - f'(a))(b-a)^2 - \right.$$

$$-\left(\frac{1+3P^2}{48}\right)(m+M)(b-a)^3\Bigg| \leqslant \frac{(M-m)(b-a)^3}{48}\left(1-3P^2\right), \quad (7)$$

where

$$P^2 = \frac{\left(f'(a)+f'(b)-2\left(\frac{f(b)-f(a)}{b-a}\right)\right)^2}{\left(\frac{M-m}{2}\right)^2(b-a)^2-\left(f'(b)-f'(a)-\left(\frac{m+M}{2}\right)(b-a)\right)^2}. \quad (8)$$

We need

Remark. *We define the ball* $B(0,R) = \{x \in \mathbb{R}^N : |x| < R\} \subseteq \mathbb{R}^N$, $N \geqslant 2$, $R > 0$, *and the sphere*

$$S^{N-1} := \left\{x \in \mathbb{R}^N : |x| = 1\right\},$$

where $|\cdot|$ *is the Euclidean norm. Let* $d\omega$ *be the element of surface measure on* S^{N-1} *and*

$$\omega_N = \int_{S^{N-1}} d\omega = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}$$

be the area of S^{N-1}.

For $x \in \mathbb{R}^N - \{0\}$ *we can write uniquely* $x = r\omega$, *where* $r = |x| > 0$ *and* $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$. *Note that* $\int_{B(0,R)} dy = \frac{\omega_N R^N}{N}$ *is the Lebesgue measure on the ball, that is the volume of* $B(0,R)$, *which exactly is* $Vol(B(0,R)) = \frac{\pi^{\frac{N}{2}} R^N}{\Gamma\left(\frac{N}{2}+1\right)}$.

Following [8, pp. 149–150, exercise 6], and [9, pp. 87–88, Theorem 5.2.2] we can write for $F : \overline{B(0,R)} \to \mathbb{R}$ *a Lebesgue integrable function that*

$$\int_{B(0,R)} F(x)\,dx = \int_{S^{N-1}} \left(\int_0^R F(r\omega)\, r^{N-1} dr\right) d\omega, \quad (9)$$

and we use this formula a lot.

Typically here the function $f : \overline{B(0,R)} \to \mathbb{R}$ *is radial; that is, there exists a function* g *such that* $f(x) = g(r)$, *where* $r = |x|$, $r \in [0,R]$, $\forall x \in \overline{B(0,R)}$.

Remark. *Let the spherical shell* $A := B(0,R_2) - \overline{B(0,R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geqslant 2$, $x \in \overline{A}$. *Consider that* $f : \overline{A} \to \mathbb{R}$ *is radial; that is, there* *exists g such that* $f(x) = g(r)$, $r = |x|$, $r \in [R_1, R_2]$, $\forall\ x \in \overline{A}$. *Here* x *can be written uniquely as* $x = r\omega$, *where* $r = |x| > 0$ *and* $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$, *see ([8, p. 149–150], and [2, p.421]), furthermore for* $F : \overline{A} \to \mathbb{R}$ *a Lebesgue integrable function we have that*

$$\int_A F(x)\,dx = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} F(r\omega)\, r^{N-1} dr \right) d\omega. \tag{10}$$

Here

$$Vol(A) = \frac{\omega_N \left(R_2^N - R_1^N\right)}{N} = \frac{\pi^{\frac{N}{2}} \left(R_2^N - R_1^N\right)}{\Gamma\left(\frac{N}{2}+1\right)}. \tag{11}$$

In this article we derive multivariate Iyengar type inequalities on the shell and ball of $\mathbb{R}^N$, $N \geqslant 2$, for radial functions. Our results are based on Theorem 1 – Theorem 5 and several other results by the author.

2. Main Results. We present the following multivariate Iyengar type inequalities on the shell and the ball:

We start with

Theorem 6. *Let the spherical shell* $A := B(0, R_2) - \overline{B(0, R_1)}$, $A \subseteq \mathbb{R}^N$, $N \geqslant 2$, $0 < R_1 < R_2$. *Consider* $f : \overline{A} \to \mathbb{R}$ *that is radial, that is, there exists g such that* $f(x) = g(r)$, $r = |x|$, $r \in [R_1, R_2]$, $\forall x \in \overline{A}$; $x = r\omega$, $\omega \in S^{N-1}$. *We assume that* $g \in C^1([R_1, R_2])$.

Then

$$\left| \int_A f(y)\,dy - (R_2 - R_1)\left[g(R_1) R_1^{N-1} + g(R_2) R_2^{N-1}\right] \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leqslant$$

$$\leqslant \frac{\pi^{\frac{N}{2}}}{2\Gamma\left(\frac{N}{2}\right)} \Bigg[\left\| \left(g(s)\, s^{N-1}\right)' \right\|_{\infty, [R_1, R_2]} (R_2 - R_1)^2 -$$

$$- \frac{\left(g(R_2) R_2^{N-1} - g(R_1) R_1^{N-1}\right)^2}{\left\| \left(g(s)\, s^{n-1}\right)' \right\|_{\infty, [R_1, R_2]}} \Bigg]. \tag{12}$$

Proof. Here $g \in C^1([R_1, R_2])$ and clearly $h(s) := g(s) s^{N-1} \in C^1([R_1, R_2])$, $N \geqslant 2$. We set $\|h'\|_{\infty, [R_1, R_2]} = M$. By (1) we get

$$\left| \int_{R_1}^{R_2} h(x)\,dx - \frac{1}{2}(R_2 - R_1)(h(R_1) + h(R_2)) \right| \leqslant$$

$$\leqslant \frac{M\left(R_{2}-R_{1}\right)^{2}}{4}-\frac{\left(h\left(R_{2}\right)-h\left(R_{1}\right)\right)^{2}}{4M}=$$

$$=\frac{\left\|\left(g\left(s\right)s^{N-1}\right)'\right\|_{\infty,\,[R_{1},R_{2}]}\left(R_{2}-R_{1}\right)^{2}}{4}-$$

$$-\frac{\left(g\left(R_{2}\right)R_{2}^{N-1}-g\left(R_{1}\right)R_{1}^{N-1}\right)^{2}}{4\left\|\left(g\left(s\right)s^{N-1}\right)'\right\|_{\infty,\,[R_{1},R_{2}]}}=:\lambda. \quad (13)$$

Equivalently, we have

$$-\lambda\leqslant\int_{R_{1}}^{R_{2}}g\left(s\right)s^{N-1}ds-\frac{1}{2}\left(R_{2}-R_{1}\right)\left(g\left(R_{1}\right)R_{1}^{N-1}+g\left(R_{2}\right)R_{2}^{N-1}\right)\leqslant\lambda, \quad (14)$$

$$-\lambda\leqslant\int_{R_{1}}^{R_{2}}f\left(s\omega\right)s^{N-1}ds-\left(\frac{R_{2}-R_{1}}{2}\right)\left(g\left(R_{1}\right)R_{1}^{N-1}+g\left(R_{2}\right)R_{2}^{N-1}\right)\leqslant\lambda. \quad (15)$$

Hence it holds

$$-\lambda\int_{S^{N-1}}d\omega\leqslant\int_{S^{N-1}}\left(\int_{R_{1}}^{R_{2}}f(s\omega)s^{N-1}ds\right)d\omega-$$

$$-\left(\frac{R_{2}-R_{1}}{2}\right)\left(g\left(R_{1}\right)R_{1}^{N-1}+g\left(R_{2}\right)R_{2}^{N-1}\right)\int_{S^{N-1}}d\omega\leqslant\lambda\int_{S^{N-1}}d\omega, \quad (16)$$

that is (by (10))

$$-\lambda\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\leqslant\int_{A}f\left(y\right)dy-\left(\frac{R_{2}-R_{1}}{2}\right)\left[g\left(R_{1}\right)R_{1}^{N-1}+\right.$$

$$\left.+g\left(R_{2}\right)R_{2}^{N-1}\right]\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\leqslant\lambda\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}. \quad (17)$$

Therefore we get

$$\left|\int_{A}f\left(y\right)dy-\left(R_{2}-R_{1}\right)\left[g\left(R_{1}\right)R_{1}^{N-1}+g\left(R_{2}\right)R_{2}^{N-1}\right]\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leqslant\lambda\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}. \quad (18)$$

The theorem is proved. □

We give

Corollary. (to Theorem 6) *Let $f : \overline{B(0,R)} \to \mathbb{R}$ be radial, that is, there exists a function g such that $f(x) = g(r)$, where $r = |x|$, $r \in [0,R]$, $R > 0$, $\forall\, x \in \overline{B(0,R)}$; $N \geqslant 2$. We assume that $g \in C^1([0,R])$. Then*

$$\left| \int\limits_{B(0,R)} f(y)dy - R^N g(R) \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leqslant$$

$$\leqslant \frac{\pi^{\frac{N}{2}}}{2\Gamma\left(\frac{N}{2}\right)} \left[\left\| \left(g(s)s^{N-1}\right)' \right\|_{\infty,\,[0,\,R]} R^2 - \frac{g^2(R)\, R^{2(N-1)}}{\left\| \left(g(s)\, s^{N-1}\right)' \right\|_{\infty,\,[0,\,R]}} \right]. \tag{19}$$

Proof. Similar to Theorem 6, use of (9). □

We also give

Theorem 7. *Let $f : \overline{A} \to \mathbb{R}$ be radial; that is, there exists g such that $f(x) = g(r)$, $r = |x|$, $r \in [R_1, R_2]$, $\forall\, x \in \overline{A}$; $x = r\omega$, $\omega \in S^{N-1}$, $N \geqslant 2$. We assume that $g \in C^2([R_1, R_2])$.*

Then

$$\left| \int\limits_A f(y)\, dy - \left[(R_2 - R_1)\left(g(R_1)\, R_1^{N-1} + g(R_2)\, R_2^{N-1}\right) + \right.\right.$$

$$\left.\left. + \frac{(R_2 - R_1)^2}{4} \left(\left(g(s)\, s^{N-1}\right)'(R_2) - \left(g(s)\, s^{N-1}\right)'(R_1) \right) \right] \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leqslant$$

$$\leqslant \left[\frac{\left\| \left(g(s)\, s^{N-1}\right)'' \right\|_{\infty,\,[R_1,R_2]}}{12} (R_2 - R_1)^3 - \right.$$

$$\left. - \left(\frac{(R_2 - R_1)}{8 \left\| \left(g(s)\, s^{N-1}\right)'' \right\|_{\infty,\,[R_1,\,R_2]}} \right) \Delta_1^2 \right] \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}, \tag{20}$$

where

$$\Delta_1 := \left(g(s)\, s^{N-1}\right)'(R_1) - \frac{2\left(g(R_2)\, R_2^{N-1} - g(R_1)\, R_1^{N-1}\right)}{(R_2 - R_1)} + \left(g(s)s^{N-1}\right)'(R_2). \tag{21}$$

Proof. Here $g \in C^2([R_1, R_2])$ and clearly $h(s) := g(s)s^{N-1} \in C^2([R_1, R_2])$, $N \geqslant 2$. We set $\|h''\|_{\infty,\,[R_1,R_2]} = M$. By (2) we get

$$\Bigg| \int_{R_1}^{R_2} h(s)\,ds - \frac{1}{2}(R_2 - R_1)(h(R_1) + h(R_2)) + \frac{1}{8}(R_2 -$$

$$- R_1)^2 (h'(R_2) - h'(R_1)) \Bigg| \leqslant \frac{M}{24}(R_2 - R_1)^3 - \frac{(R_2 - R_1)}{16M}\Delta_1^2, \quad (22)$$

where

$$\Delta_1 = h'(R_1) - \frac{2(h(R_2) - h(R_1))}{(R_2 - R_1)} + h'(R_2). \quad (23)$$

That is

$$\Bigg| \int_{R_1}^{R_2} g(s)\, s^{N-1} ds - \frac{(R_2 - R_1)}{2}\left(g(R_1) R_1^{N-1} + g(R_2) R_2^{N-1}\right) +$$

$$+ \frac{(R_2 - R_1)^2}{8}\left(\left(g(s)\, s^{N-1}\right)'(R_2) - \left(g(s)\, s^{N-1}\right)'(R_1)\right) \Bigg| \leqslant$$

$$\leqslant \frac{M}{24}(R_2 - R_1)^3 - \frac{(R_2 - R_1)}{16M}\Delta_1^2 =: \psi, \quad (24)$$

where

$$\Delta_1 := \left(g(s)\, s^{N-1}\right)'(R_1) - \frac{2\left(g(R_2) R_2^{N-1} - g(R_1) R_1^{N-1}\right)}{(R_2 - R_1)} +$$

$$+ \left(g(s)\, s^{N-1}\right)'(R_2). \quad (25)$$

Equivalently, we have

$$-\psi \leqslant \int_{R_1}^{R_2} f(s\omega)\, s^{N-1} ds - \frac{(R_2 - R_1)}{2}\left(g(R_1) R_1^{N-1} + g(R_2) R_2^{N-1}\right) +$$

$$+ \frac{(R_2 - R_1)^2}{8}\left(\left(g(s)\, s^{N-1}\right)'(R_2) - \left(g(s)\, s^{N-1}\right)'(R_1)\right) \leqslant \psi. \quad (26)$$

Hence it holds

$$- \psi \int_{S^{N-1}} d\omega \leqslant \int_{S^{N-1}} \left(\int_{R_1}^{R_2} f(s\omega)\, s^{N-1} ds\right) d\omega -$$

$$-\left[\frac{(R_2-R_1)}{2}\Big(g(R_1)\,R_1^{N-1}+g(R_2)\,R_2^{N-1}\Big)+ + \frac{(R_2-R_1)^2}{8}\Big(\left(g(s)s^{N-1}\right)'(R_2)- - (g(s)s^{N-1})'(R_1)\Big)\right]\int\limits_{S^{N-1}} d\omega \leqslant \psi \int\limits_{S^{N-1}} d\omega, \quad (27)$$

that is (by (10))

$$-\psi\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \leqslant \int\limits_A f(y)\,dy - \left[\frac{(R_2-R_1)}{2}\left(g(R_1)\,R_1^{N-1}+g(R_2)\,R_2^{N-1}\right)+ + \frac{(R_2-R_1)^2}{8}\left(\left(g(s)s^{N-1}\right)'(R_2)-\left(g(s)\,s^{N-1}\right)'(R_1)\right)\right]\frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \leqslant \psi\frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}. \quad (28)$$

Therefore we get

$$\Bigg|\int\limits_A f(y)dy - \Big[(R_2-R_1)\big(g(R_1)R_1^{N-1}+g(R_2)R_2^{N-1}\big)+ + \frac{(R_2-R_1)^2}{4}\Big(\left(g(s)s^{N-1}\right)'(R_2)- - \left(g(s)s^{N-1}\right)'(R_1)\Big)\Big]\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\Bigg| \leqslant \psi\frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}. \quad (29)$$

The theorem is proved. □

We give

Corollary. (to Theorem 7) *Let $f:\overline{B(0,R)}\to\mathbb{R}$ be radial, that is, there exists a function g such that $f(x)=g(r)$, where $r=|x|$, $r\in[0,R]$, $R>0$, $\forall\, x\in\overline{B(0,R)}$; $N\geqslant 2$. We assume that $g\in C^2([0,R])$. Then*

$$\Bigg|\int\limits_{B(0,R)} f(y)\,dy - \Big[R^N g(R)+\frac{R^2}{4}\Big(\left(g(s)\,s^{N-1}\right)'(R)- - \left(g(s)\,s^{N-1}\right)'(0)\Big)\Big]\frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\Bigg| \leqslant \Bigg[\frac{\left\|\left(g(s)\,s^{N-1}\right)''\right\|_{\infty,[0,R]}}{12}R^3-$$

$$- \frac{R}{8\left\| \left(g(s)s^{N-1}\right)'' \right\|_{\infty, [0, R]}} \Delta_1^{*2} \Bigg] \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}, \quad (30)$$

where

$$\Delta_1^* := \left(g(s)\, s^{N-1}\right)'(0) - 2g(R)\, R^{N-2} + \left(g(s)\, s^{N-1}\right)'(R). \quad (31)$$

If $N > 2$, *then* $\left(g(s)\, s^{N-1}\right)'(0) = 0$.

Proof. Similar to Theorem 7, use of (9). □

We present

Theorem 8. *Consider* $f : \overline{A} \to \mathbb{R}$ *that is radial; that is, there exists* g *such that* $f(x) = g(r)$, $r = |x|$, $r \in [R_1, R_2]$, $\forall\, x \in \overline{A}$; $x = r\omega$, $\omega \in S^{N-1}$, $N \geqslant 2$. *We assume that* $g \in C^1([R_1, R_2])$.

Then

$$\left| \int_A f(y)\, dy - (R_2 - R_1)\left[g(R_1)\, R_1^{N-1} + g(R_2)\, R_2^{N-1}\right] \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leqslant$$

$$\leqslant \left(g(R_2)\, R_2^{N-1} - g(R_1)\, R_1^{N-1} - m(R_2 - R_1)\right) \times$$

$$\times \frac{\left(M(R_2 - R_1) - g(R_2)\, R_2^{N-1} + g(R_1)\, R_1^{N-1}\right)}{(M - m)} \left(\frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\right), \quad (32)$$

where $M > m$ *with*

$$m \leqslant \left(g(s)\, s^{N-1}\right)' \leqslant M, \quad \forall\, s \in [R_1, R_2]. \quad (33)$$

Proof. Here $g \in C^1([R_1, R_2])$ and clearly $h(s) := g(s)\, s^{N-1} \in C^1([R_1, R_2])$, $N \geqslant 2$. We assume here $m \leqslant h'(s) \leqslant M$, $\forall\, s \in [R_1, R_2]$ with $M > m$. By (3) we get

$$\left| \int_{R_1}^{R_2} h(s)\, ds - \frac{1}{2}(R_2 - R_1)(h(R_1) + h(R_2)) \right| \leqslant \quad (34)$$

$$\leqslant \frac{(h(R_2) - h(R_1) - m(R_2 - R_1))(M(R_2 - R_1) - h(R_2) + h(R_1))}{2(M - m)}.$$

That is

$$\left| \int_{R_1}^{R_2} g(s)\, s^{N-1} ds - \frac{1}{2}(R_2 - R_1)\left(g(R_1)\, R_1^{N-1} + g(R_2)\, R_2^{N-1}\right) \right| \leqslant$$

$$\leqslant \left(g\left(R_2\right)R_2^{N-1}-g\left(R_1\right)R_1^{N-1}-m\left(R_2-R_1\right)\right)\times$$
$$\times\frac{\left(M\left(R_2-R_1\right)-g\left(R_2\right)R_2^{N-1}+g\left(R_1\right)R_1^{N-1}\right)}{2\left(M-m\right)}=:\rho. \tag{35}$$

Equivalently, we have

$$-\rho\leqslant\int_{R_1}^{R_2}g\left(s\right)s^{N-1}ds-\left(\frac{R_2-R_1}{2}\right)\left(g\left(R_1\right)R_1^{N-1}+g\left(R_2\right)R_2^{N-1}\right)\leqslant\rho, \tag{36}$$

$$-\rho\leqslant\int_{R_1}^{R_2}f\left(s\omega\right)s^{N-1}ds-\left(\frac{R_2-R_1}{2}\right)\left(g\left(R_1\right)R_1^{N-1}+g\left(R_2\right)R_2^{N-1}\right)\leqslant\rho. \tag{37}$$

Hence it holds

$$-\rho\int_{S^{N-1}}d\omega\leqslant\int_{S^{N-1}}\left(\int_{R_1}^{R_2}f\left(s\omega\right)s^{N-1}ds\right)d\omega-$$
$$-\left(\frac{R_2-R_1}{2}\right)\left(g\left(R_1\right)R_1^{N-1}+g\left(R_2\right)R_2^{N-1}\right)\int_{S^{N-1}}d\omega\leqslant\rho\int_{S^{N-1}}d\omega, \tag{38}$$

that is (by (10))

$$-\rho\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\leqslant\int_A f\left(y\right)dy-\left(\frac{R_2-R_1}{2}\right)\Big(g(R_1)R_1^{N-1}+$$
$$+g\left(R_2\right)R_2^{N-1}\Big)\frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\leqslant\rho\frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}. \tag{39}$$

Therefore we get

$$\left|\int_A f\left(y\right)dy-\left(R_2-R_1\right)\left(g\left(R_1\right)R_1^{N-1}+g\left(R_2\right)R_2^{N-1}\right)\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leqslant\rho\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}. \tag{40}$$

The theorem is proved. □

We give

Corollary. (to Theorem 8) *Let $f : \overline{B(0,R)} \to \mathbb{R}$ be radial, that is, there exists a function g such that $f(x) = g(r)$, where $r = |x|$, $r \in [0,R]$, $R > 0$, $\forall\, x \in \overline{B(0,R)}$; $N \geqslant 2$. We assume that $g \in C^1([0,R])$. Then*

$$\left| \int_A f(y)\,dy - R^N g(R) \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leqslant$$

$$\leqslant \frac{\left(g(R) R^{N-1} - mR\right)\left(MR - g(R) R^{N-1}\right)}{(M-m)} \left(\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right), \tag{41}$$

where $M > m$ with

$$m \leqslant \left(g(s) s^{N-1}\right)' \leqslant M, \ \ \forall\, s \in [0,R]. \tag{42}$$

Proof. Similar to Theorem 8, use of (9). □

We continue with

Theorem 9. *Let $f : \overline{A} \to \mathbb{R}$ be radial; that is, there exists g such that $f(x) = g(r)$, $r = |x|$, $r \in [R_1, R_2]$, $\forall\, x \in \overline{A}$; $x = r\omega$, $\omega \in S^{N-1}$, $N \geqslant 2$. We assume that $g \in C^2([R_1,R_2])$. We call $M_1 := \left\| \left(g(s) s^{N-1}\right)'' \right\|_{\infty, [R_1, R_2]}$.*

Then

$$\Bigg| \int_A f(y)dy - \Big[\left(g(R_1)R_1^{N-1} + g(R_2)R_2^{N-1}\right)(R_2 - R_1) + \frac{(1+Q_1^2)}{4} \times$$

$$\times \left(\left(g(s) s^{N-1}\right)'(R_2) - \left(g(s) s^{N-1}\right)'(R_1) \right)(R_2 - R_1)^2 \Big] \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \Bigg| \leqslant$$

$$\leqslant \frac{M_1 (R_2 - R_1)^3}{12} \left(1 - 3Q_1^2\right) \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}, \tag{43}$$

where

$$Q_1^2 := \frac{\left[\left(g(s) s^{N-1}\right)'(R_1) + \left(g(s) s^{N-1}\right)'(R_2) - 2\left(\frac{g(R_2)R_2^{N-1} - g(R_1)R_1^{N-1}}{R_2 - R_1}\right) \right]^2}{M_1^2 (R_2 - R_1)^2 - \left(\left(g(s) s^{N-1}\right)'(R_2) - \left(g(s) s^{N-1}\right)'(R_1) \right)^2}. \tag{44}$$

Proof. Similar to the proof of Theorem 7 by the use of Theorem 4. □

We give

Corollary. (to Theorem 9) *Let $f : \overline{B(0,R)} \to \mathbb{R}$ be radial, that is, there exists a function g such that $f(x) = g(r)$, where $r = |x|$, $r \in [0,R]$, $R > 0$, $\forall\, x \in \overline{B(0,R)}$; $N \geqslant 2$. We assume that $g \in C^2([0,R])$. We call $M_2 := \left\| \left(g(s)\, s^{N-1}\right)'' \right\|_{\infty,\,[0,R]}$. Then*

$$\left| \int_A f(y)dy - \left[g(R)R^N + \frac{(1+Q_2^2)}{4} \left[\left(g(s)\, s^{N-1}\right)'(R) - \right.\right.\right.$$
$$\left.\left.\left. - \left(g(s)\, s^{N-1}\right)'(0) \right] R^2 \right] \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leqslant \frac{M_2}{12} R^3 \left(1 - 3Q_2^2\right) \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}, \quad (45)$$

where

$$Q_2^2 := \frac{\left[\left(g(s)\, s^{N-1}\right)'(0) + \left(g(s)\, s^{N-1}\right)'(R) - 2g(R)\, R^{N-2}\right]^2}{M_2^2 R^2 - \left(\left(g(s)\, s^{N-1}\right)'(R) - \left(g(s)\, s^{N-1}\right)'(0)\right)^2}. \quad (46)$$

Proof. Similar to Corollary to Theorem 7. □

We present

Theorem 10. *Here all as in Theorem 6 and $M_1 > m_1$. Assume that*

$$m_1 \leqslant \frac{\left(g(s)\, s^{N-1}\right)'(s) - \left(g(s)\, s^{N-1}\right)'(R_1)}{s - R_1} \leqslant M_1 \quad (47)$$

and

$$m_1 \leqslant \frac{\left(g(s)\, s^{N-1}\right)'(R_2) - \left(g(s)\, s^{N-1}\right)'(s)}{R_2 - s} \leqslant M_1, \quad (48)$$

for all $s \in [R_1, R_2]$. Then

$$\left| \int_A f(y)\, dy - \left[\left(g(R_1)\, R_1^{N-1} + g(R_2)\, R_2^{N-1}\right)(R_2 - R_1) + \right.\right.$$
$$+ \left(\frac{1+P_1^2}{4}\right) \left(\left(g(s)\, s^{N-1}\right)'(R_2) - \left(g(s)\, s^{N-1}\right)'(R_1) \right) (R_2 - R_1)^2 -$$
$$\left.\left. - \left(\frac{1+3P_1^2}{24}\right) (m_1 + M_1)(R_2 - R_1)^3 \right] \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leqslant$$
$$\leqslant \frac{(M_1 - m_1)(R_2 - R_1)^3}{24} \left(1 - 3P_1^2\right) \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}, \quad (49)$$

where

$$P_1^2 = \frac{\left[\left(g(s)s^{N-1}\right)'(R_1) + \left(g(s)s^{N-1}\right)'(R_2) - 2\left(\frac{g(R_2)R_2^{N-1} - g(R_1)R_1^{N-1}}{R_2 - R_1}\right)\right]^2}{\left(\frac{M_1 - m_1}{2}\right)^2 (R_2 - R_1)^2 - \left[(g(s)s^{N-1})'(R_2) - (g(s)s^{N-1})'(R_1) - \left(\frac{m_1 + M_1}{2}\right)(R_2 - R_1)\right]^2} \tag{50}$$

Proof. Similar to Theorem 6 by the use of Theorem 5. □

We give

Corollary. (to Theorem 10) *Here all as in Corollary to Theorem 6 and $M_2 > m_2$. Assume that*

$$m_2 \leqslant \frac{\left(g(s)s^{N-1}\right)'(s) - \left(g(s)s^{N-1}\right)'(0)}{s} \leqslant M_2 \tag{51}$$

and

$$m_2 \leqslant \frac{\left(g(s)s^{N-1}\right)'(R) - \left(g(s)s^{N-1}\right)'(s)}{R - s} \leqslant M_2, \tag{52}$$

for all $s \in [0, R]$. Then

$$\Bigg| \int_A f(y)\,dy - \Bigg[g(R)R^N + \left(\frac{1 + P_2^2}{4}\right)\Big(\left(g(s)s^{N-1}\right)'(R) - \\ - \left(g(s)s^{N-1}\right)'(0)\Big)R^2 - \left(\frac{1 + 3P_2^2}{24}\right)(m_2 + M_2)R^3\Bigg] \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \Bigg| \leqslant \\ \leqslant \frac{(M_2 - m_2)R^3}{24}(1 - 3P_2^2)\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}, \tag{53}$$

where

$$P_2^2 = \frac{\left[\left(g(s)s^{N-1}\right)'(0) + \left(g(s)s^{N-1}\right)'(R) - 2g(R)R^{N-2}\right]^2}{\left(\frac{M_2 - m_2}{2}\right)^2 R^2 - \left[\left(g(s)s^{N-1}\right)'(R) - \left(g(s)s^{N-1}\right)'(0) - \left(\frac{m_2 + M_2}{2}\right)R\right]^2}. \tag{54}$$

Proof. Similar to Corollary to Theorem 6, based on Theorem 5. □

We continue with some author's results to be used later in this article:

Theorem 11. [3] *Let $n \in \mathbb{N}$, $f \in AC^n([a,b])$ (i.e. $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions). We assume that $f^{(n)} \in L_\infty([a,b])$. Then* *i)* $\forall\, t \in [a,b]$

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leqslant$$

$$\leqslant \frac{\left\| f^{(n)} \right\|_{L_\infty([a,b])}}{(n+1)!} \left[(t-a)^{n+1} + (b-t)^{n+1} \right], \tag{55}$$

ii) at $t = \frac{a+b}{2}$, *the right hand side of* (55) *is minimized, and we get:*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leqslant$$

$$\leqslant \frac{\left\| f^{(n)} \right\|_{L_\infty([a,b])}}{(n+1)!} \frac{(b-a)^{n+1}}{2^n}, \tag{56}$$

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, *for all* $k = 0, 1, \ldots, n-1$, *we obtain*

$$\left| \int_a^b f(x)\,dx \right| \leqslant \frac{\left\| f^{(n)} \right\|_{L_\infty([a,b])}}{(n+1)!} \frac{(b-a)^{n+1}}{2^n}, \tag{57}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, \ldots, N \in \mathbb{N}$, *it holds*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + \right. \right.$$

$$\left. \left. + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right| \leqslant \frac{\left\| f^{(n)} \right\|_{L_\infty([a,b])}}{(n+1)!} \left(\frac{b-a}{N} \right)^{n+1} \times$$

$$\times \left[j^{n+1} + (N-j)^{n+1} \right], \tag{58}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, \ldots, n-1$, *from* (58) *we get:*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N} \right) \left[j f(a) + (N-j) f(b) \right] \right| \leqslant$$

$$\leqslant \frac{\left\|f^{(n)}\right\|_{L_\infty([a,\,b])}}{(n+1)!}\left(\frac{b-a}{N}\right)^{n+1}\left[j^{n+1}+(N-j)^{n+1}\right], \tag{59}$$

for $j = 0, 1, 2, \ldots, N \in \mathbb{N}$,

vi) when $N = 2$ *and* $j = 1$, (59) *turns to*

$$\left|\int_a^b f(x)\,dx - \left(\frac{b-a}{2}\right)(f(a)+f(b))\right| \leqslant$$

$$\leqslant \frac{\left\|f^{(n)}\right\|_{L_\infty([a,\,b])}}{(n+1)!}\frac{(b-a)^{n+1}}{2^n}, \tag{60}$$

vii) when $n = 1$ *(without any boundary conditions), we get from* (60) *that*

$$\left|\int_a^b f(x)\,dx - \left(\frac{b-a}{2}\right)(f(a)+f(b))\right| \leqslant \|f'\|_{\infty,\,[a,\,b]}\frac{(b-a)^2}{4}, \tag{61}$$

a similar to Iyengar inequality (1).

Theorem 12. [3] *Let* $f \in AC^n([a,b])$, $n \in \mathbb{N}$. *Then*

i) $\forall\ t \in [a,\,b]$

$$\left|\int_a^b f(x)\,dx - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[f^{(k)}(a)\,(t-a)^{k+1} + \right.\right.$$

$$\left.\left. + (-1)^k f^{(k)}(b)\,(b-t)^{k+1}\right]\right| \leqslant \frac{\left\|f^{(n)}\right\|_{L_1([a,\,b])}}{n!}\left[(t-a)^n + (b-t)^n\right], \tag{62}$$

ii) at $t = \frac{a+b}{2}$, *the right hand side of* (62) *is minimized, and we get:*

$$\left|\int_a^b f(x)\,dx - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left[f^{(k)}(a) + (-1)^k f^{(k)}(b)\right]\right| \leqslant$$

$$\leqslant \frac{\left\|f^{(n)}\right\|_{L_1([a,\,b])}}{n!}\frac{(b-a)^n}{2^{n-1}}, \tag{63}$$

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, for all $k = 0, 1, \ldots, n-1$, we obtain

$$\left|\int_a^b f(x)\,dx\right| \leqslant \frac{\left\|f^{(n)}\right\|_{L_1([a,b])}}{n!}\frac{(b-a)^n}{2^{n-1}}, \tag{64}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, \ldots, N \in \mathbb{N}$, it holds

$$\left|\int_a^b f(x)dx - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{N}\right)^{k+1}\left[j^{k+1}f^{(k)}(a)+\right.\right.$$
$$\left.\left.+(-1)^k(N-j)^{k+1}f^{(k)}(b)\right]\right| \leqslant \frac{\left\|f^{(n)}\right\|_{L_1([a,b])}}{n!}\left(\frac{b-a}{N}\right)^n\left[j^n+(N-j)^n\right], \tag{65}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, \ldots, n-1$, from (65) we get:

$$\left|\int_a^b f(x)dx - \left(\frac{b-a}{N}\right)\left[jf(a) + (N-j)f(b)\right]\right| \leqslant$$
$$\leqslant \frac{\left\|f^{(n)}\right\|_{L_1([a,b])}}{n!}\left(\frac{b-a}{N}\right)^n\left[j^n + (N-j)^n\right], \tag{66}$$

for $j = 0, 1, 2, \ldots, N \in \mathbb{N}$,

vi) when $N = 2$ and $j = 1$, (66) turns to

$$\left|\int_a^b f(x)dx - \frac{(b-a)}{2}\left(f(a) + f(b)\right)\right| \leqslant$$
$$\leqslant \frac{\left\|f^{(n)}\right\|_{L_1([a,b])}}{n!}\frac{(b-a)^n}{2^{n-1}}, \tag{67}$$

vii) when $n = 1$ (without any boundary conditions), we get from (67) that

$$\left|\int_a^b f(x)dx - \left(\frac{b-a}{2}\right)\left(f(a) + f(b)\right)\right| \leqslant \left\|f'\right\|_{L_1([a,b])}(b-a). \tag{68}$$

Theorem 13. [3] *Let $f \in AC^n([a,b])$, $n \in \mathbb{N}$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $f^{(n)} \in L_q([a,b])$. Then* *i)* $\forall\ t \in [a,b]$

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leqslant$$

$$\leqslant \frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)! \left(n+\frac{1}{p}\right) (p(n-1)+1)^{\frac{1}{p}}} \left[(t-a)^{n+\frac{1}{p}} + (b-t)^{n+\frac{1}{p}} \right], \quad (69)$$

ii) at $t = \frac{a+b}{2}$, *the right hand side of* (69) *is minimized, and we get:*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leqslant$$

$$\leqslant \frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)! \left(n+\frac{1}{p}\right) (p(n-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{n+\frac{1}{p}}}{2^{n-\frac{1}{q}}}, \quad (70)$$

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, *for all* $k = 0, 1, \ldots, n-1$, *we obtain*

$$\left| \int_a^b f(x)\,dx \right| \leqslant \frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)! \left(n+\frac{1}{p}\right) (p(n-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{n+\frac{1}{p}}}{2^{n-\frac{1}{q}}}, \quad (71)$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, \ldots, N \in \mathbb{N}$, *it holds*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + \right.\right.$$

$$\left.\left. + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right| \leqslant \frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)! \left(n+\frac{1}{p}\right) (p(n-1)+1)^{\frac{1}{p}}} \times$$

$$\times \left(\frac{b-a}{N} \right)^{n+\frac{1}{p}} \left[j^{n+\frac{1}{p}} + (N-j)^{n+\frac{1}{p}} \right], \quad (72)$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, \ldots, n-1$, *from* (72) *we get:*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N} \right) [j f(a) + (N-j) f(b)] \right| \leqslant$$

$$\leqslant \frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}} \left(\frac{b-a}{N}\right)^{n+\frac{1}{p}} \left[j^{n+\frac{1}{p}} + (N-j)^{n+\frac{1}{p}}\right], \tag{73}$$

for $j = 0, 1, 2, \ldots, N \in \mathbb{N}$,

vi) when $N = 2$ and $j = 1$, (73) turns to

$$\left| \int_a^b f(x)\,dx - \frac{(b-a)}{2}(f(a)+f(b)) \right| \leqslant$$

$$\leqslant \frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{n+\frac{1}{p}}}{2^{n-\frac{1}{q}}}, \tag{74}$$

vii) when $n = 1$ (without any boundary conditions), we get from (74) that

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2}\right)(f(a)+f(b)) \right| \leqslant \frac{\|f'\|_{L_q([a,b])}}{\left(1+\frac{1}{p}\right)} \frac{(b-a)^{1+\frac{1}{p}}}{2^{\frac{1}{p}}}. \tag{75}$$

Next, we extend Theorems 11–13 to the multivariate case over shells and balls for radial functions. The proving method is the same as in our earlier results of this article, as such we omit these next proofs.

We present (use of Theorem 11)

Theorem 14. *Consider $f : \overline{A} \to \mathbb{R}$ which is radial; that is, there exists g such that $f(x) = g(r)$, $r = |x|$, $r \in [R_1, R_2]$, $\forall\, x \in \overline{A}$; $x = r\omega$, $\omega \in S^{N-1}$, $N \geqslant 2$. We assume that $g(s)\,s^{N-1} \in AC^n([R_1, R_2])$ and $\left(g(s)\,s^{N-1}\right)^{(n)} \in L_\infty([R_1, R_2])$, $n \in \mathbb{N}$. Then*

i) $\forall\, t \in [R_1, R_2]$

$$\left| \int_A f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[\left(g(s)\,s^{N-1}\right)^{(k)}(R_1)(t-R_1)^{k+1} + \right.\right.\right.$$

$$\left.\left.\left. + (-1)^k \left(g(s)\,s^{N-1}\right)^{(k)}(R_2)(R_2-t)^{k+1} \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leqslant$$

$$\leqslant \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\left\| \left(g(s)\,s^{N-1}\right)^{(n)} \right\|_{L_\infty([R_1,R_2])}}{(n+1)!} \left[(t-R_1)^{n+1} + (R_2-t)^{n+1}\right], \tag{76}$$

ii) at $t = \frac{R_1+R_2}{2}$, *the right hand side of* (76) *is minimized, and we get:*

$$\left| \int_A f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(R_2-R_1)^{k+1}}{2^k} \left[\left(g(s)\,s^{N-1}\right)^{(k)}(R_1) + \right.\right.\right.$$

$$\left.\left.\left. + (-1)^k \left(g(s)\,s^{N-1}\right)^{(k)}(R_2) \right] \right\} \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leqslant$$

$$\leqslant \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\left\| \left(g(s)\,s^{N-1}\right)^{(n)} \right\|_{L_\infty([R_1,R_2])}}{(n+1)!} \frac{(R_2-R_1)^{n+1}}{2^{n-1}}, \tag{77}$$

iii) if $(g(s)s^{N-1})^{(k)}(R_1) = (g(s)s^{N-1})^{(k)}(R_2) = 0$, *for all* $k = 0,1,\ldots,n-1$, *we obtain*

$$\left| \int_A f(y)\,dy \right| \leqslant \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\left\| \left(g(s)\,s^{N-1}\right)^{(n)} \right\|_{L_\infty([R_1,R_2])}}{(n+1)!} \frac{(R_2-R_1)^{n+1}}{2^{n-1}}, \tag{78}$$

which is a sharp inequality,

iv) more generally, for $j = 0,1,2,\ldots,N \in \mathbb{N}$, *it holds*

$$\left| \int_A f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{R_2-R_1}{N} \right)^{k+1} \left[j^{k+1} \left(g(s)\,s^{N-1}\right)^{(k)}(R_1) + \right.\right.\right.$$

$$\left.\left.\left. + (-1)^k (N-j)^{k+1} \left(g(s)\,s^{N-1}\right)^{(k)}(R_2) \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leqslant$$

$$\leqslant \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\left\| \left(g(s)\,s^{N-1}\right)^{(n)} \right\|_{L_\infty([R_1,R_2])}}{(n+1)!} \left(\frac{R_2-R_1}{N} \right)^{n+1} \left(j^{n+1} + (N-j)^{n+1} \right), \tag{79}$$

v) if $\left(g(s)\,s^{N-1}\right)^{(k)}(R_1) = \left(g(s)\,s^{N-1}\right)^{(k)}(R_2) = 0$, $k = 1,\ldots,n-1$, *from* (79) *we get:*

$$\left| \int_A f(y)\,dy - \left\{ \left(\frac{R_2-R_1}{N} \right) \left[j g(R_1)\, R_1^{N-1} + \right.\right.\right.$$

$$\left.\left.\left. + (N-j)\, g(R_2)\, R_2^{N-1} \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leqslant \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \times$$

$$\times \frac{\left\| \left(g\left(s\right) s^{N-1}\right)^{(n)} \right\|_{L_\infty([R_1, R_2])}}{(n+1)!} \left(\frac{R_2 - R_1}{N}\right)^{n+1} \left(j^{n+1} + (N-j)^{n+1}\right), \tag{80}$$

for $j = 0, 1, 2, \ldots, N \in \mathbb{N}$,

vi) when $N = 2$ *and* $j = 1$, (80) *turns to*

$$\left| \int_A f\left(y\right) dy - \left(R_2 - R_1\right) \left(g\left(R_1\right) R_1^{N-1} + g\left(R_2\right) R_2^{N-1}\right) \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leqslant$$

$$\leqslant \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\left\| \left(g\left(s\right) s^{N-1}\right)^{(n)} \right\|_{L_\infty([R_1, R_2])}}{(n+1)!} \frac{\left(R_2 - R_1\right)^{n+1}}{2^{n-1}}, \tag{81}$$

vii) when $n = 1$ *(without any boundary conditions), we get from* (81) *that*

$$\left| \int_A f\left(y\right) dy - \left(R_2 - R_1\right) \left(g\left(R_1\right) R_1^{N-1} + g\left(R_2\right) R_2^{N-1}\right) \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leqslant$$

$$\leqslant \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\| \left(g\left(s\right) s^{N-1}\right)' \right\|_{L_\infty([R_1, R_2])} \frac{\left(R_2 - R_1\right)^2}{2}, \tag{82}$$

which is related to (12).

We present (use of Theorem 12)

Theorem 15. *Consider* $f : \overline{A} \to \mathbb{R}$ *which is radial; that is, there exists* g *such that* $f\left(x\right) = g\left(r\right)$, $r = |x|$, $r \in [R_1, R_2]$, $\forall\, x \in \overline{A}$; $x = r\omega$, $\omega \in S^{N-1}$, $N \geqslant 2$. *We assume that* $g\left(s\right) s^{N-1} \in AC^n\left([R_1, R_2]\right)$, $n \in \mathbb{N}$. *Then*

i) $\forall\, t \in [R_1, R_2]$

$$\left| \int_A f(y)dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[\left(g\left(s\right) s^{N-1}\right)^{(k)} \left(R_1\right) \left(t - R_1\right)^{k+1} + \right.\right.\right.$$

$$\left.\left.\left. + \left(-1\right)^k \left(g\left(s\right) s^{N-1}\right)^{(k)} \left(R_2\right) \left(R_2 - t\right)^{k+1} \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leqslant$$

$$\leqslant \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\left\| \left(g\left(s\right) s^{N-1}\right)^{(n)} \right\|_{L_1([R_1, R_2])}}{n!} \left[\left(t - R_1\right)^n + \left(R_2 - t\right)^n\right], \tag{83}$$

ii) at $t=\frac{R_1+R_2}{2}$*, the right hand side of* (83) *is minimized, and we get:*

$$\left|\int_A f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(R_2-R_1)^{k+1}}{2^k}\left[\left(g(s)\,s^{N-1}\right)^{(k)}(R_1)+\right.\right.\right.$$

$$\left.\left.\left.+(-1)^k\left(g(s)\,s^{N-1}\right)^{(k)}(R_2)\right]\right\}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leqslant$$

$$\leqslant\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\left\|\left(g(s)\,s^{N-1}\right)^{(n)}\right\|_{L_1([R_1,R_2])}}{n!}\frac{(R_2-R_1)^n}{2^{n-2}},\tag{84}$$

iii) if $\left(g(s)\,s^{N-1}\right)^{(k)}(R_1)=\left(g(s)\,s^{N-1}\right)^{(k)}(R_2)=0$*, for all* $k=0,1,\ldots,n-1$*, we obtain*

$$\left|\int_A f(y)\,dy\right|\leqslant\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\left\|\left(g(s)\,s^{N-1}\right)^{(n)}\right\|_{L_1([R_1,R_2])}}{n!}\frac{(R_2-R_1)^n}{2^{n-2}},\tag{85}$$

which is a sharp inequality,

iv) more generally, for $j=0,1,2,\ldots,N\in\mathbb{N}$*, it holds*

$$\left|\int_A f(y)\,dy-\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{R_2-R_1}{N}\right)^{k+1}\left[j^{k+1}\left(g(s)\,s^{N-1}\right)^{(k)}(R_1)+\right.\right.\right.$$

$$\left.\left.\left.+(-1)^k(N-j)^{k+1}\left(g(s)\,s^{N-1}\right)^{(k)}(R_2)\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leqslant$$

$$\leqslant\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{\left\|\left(g(s)\,s^{N-1}\right)^{(n)}\right\|_{L_1([R_1,R_2])}}{n!}\left(\frac{R_2-R_1}{N}\right)^n\left(j^n+(N-j)^n\right),\tag{86}$$

v) if $\left(g(s)\,s^{N-1}\right)^{(k)}(R_1)=\left(g(s)\,s^{N-1}\right)^{(k)}(R_2)=0$*,* $k=1,\ldots,n-1$*, from* (86) *we get:*

$$\left|\int_A f(y)\,dy-\left\{\left(\frac{R_2-R_1}{N}\right)\left[jg(R_1)\,R_1^{N-1}+\right.\right.\right.$$

$$\left.\left.\left.+(N-j)\,g(R_2)\,R_2^{N-1}\right]\right\}\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\right|\leqslant\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\times$$

$$\times \frac{\left\| \left(g\left(s\right)s^{N-1}\right)^{(n)} \right\|_{L_1([R_1,\, R_2])}}{n!} \left(\frac{R_2 - R_1}{N}\right)^n \left(j^n + \left(N - j\right)^n\right), \tag{87}$$

for $j = 0, 1, 2, \ldots, N \in \mathbb{N}$,

vi) when $N = 2$ *and* $j = 1$, (87) *turns to*

$$\left| \int_A f\left(y\right) dy - \left(R_2 - R_1\right) \left(g\left(R_1\right) R_1^{N-1} + g\left(R_2\right) R_2^{N-1}\right) \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leqslant$$

$$\leqslant \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\left\| \left(g\left(s\right)s^{N-1}\right)^{(n)} \right\|_{L_1([R_1,R_2])}}{n!} \frac{\left(R_2 - R_1\right)^n}{2^{n-2}}, \tag{88}$$

vii) when $n = 1$ *(without any boundary conditions), we get from* (88) *that*

$$\left| \int_A f\left(y\right) dy - \left(R_2 - R_1\right) \left(g\left(R_1\right) R_1^{N-1} + g\left(R_2\right) R_2^{N-1}\right) \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leqslant$$

$$\leqslant \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\| \left(g\left(s\right)s^{N-1}\right)' \right\|_{L_1([R_1,R_2])} \left(R_2 - R_1\right). \tag{89}$$

We present (use of Theorem 13)

Theorem 16. *Consider* $f : \overline{A} \to \mathbb{R}$ *which is radial; that is, there exists* g *such that* $f\left(x\right) = g\left(r\right)$, $r = |x|$, $r \in [R_1, R_2]$, $\forall\ x \in \overline{A}$; $x = r\omega$, $\omega \in S^{N-1}$, $N \geqslant 2$. *We assume that* $g\left(s\right)s^{N-1} \in AC^n\left([R_1, R_2]\right)$ *and* $\left(g\left(s\right)s^{N-1}\right)^{(n)} \in L_q\left([R_1, R_2]\right)$, *where* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, *and* $n \in \mathbb{N}$. *Then*

i) $\forall\ t \in [R_1, R_2]$

$$\left| \int_A f\left(y\right) dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[\left(g\left(s\right)s^{N-1}\right)^{(k)} \left(R_1\right) \left(t - R_1\right)^{k+1} + \right.\right.\right.$$

$$\left.\left.\left. + \left(-1\right)^k \left(g\left(s\right)s^{N-1}\right)^{(k)} \left(R_2\right) \left(R_2 - t\right)^{k+1} \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leqslant$$

$$\leqslant \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\left\| \left(g\left(s\right)s^{N-1}\right)^{(n)} \right\|_{L_q([R_1,\, R_2])}}{(n-1)! \left(n + \frac{1}{p}\right) \left(p\left(n-1\right) + 1\right)^{\frac{1}{p}}} \left[\left(t - R_1\right)^{n+\frac{1}{p}} + \left(R_2 - t\right)^{n+\frac{1}{p}} \right], \tag{90}$$

ii) at $t = \frac{R_1+R_2}{2}$*, the right hand side of* (90) *is minimized, and we get:*

$$\left| \int_A f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(R_2-R_1)^{k+1}}{2^k} \left[\left(g(s)\,s^{N-1}\right)^{(k)}(R_1) + \right.\right.\right.$$

$$\left.\left.\left. + (-1)^k \left(g(s)\,s^{N-1}\right)^{(k)}(R_2) \right] \right\} \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leqslant$$

$$\leqslant \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\left\| \left(g(s)\,s^{N-1}\right)^{(n)} \right\|_{L_q([R_1,R_2])}}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}} \frac{(R_2-R_1)^{n+\frac{1}{p}}}{2^{n-1-\frac{1}{q}}}, \tag{91}$$

iii) if $\left(g(s)s^{N-1}\right)^{(k)}(R_1) = \left(g(s)s^{N-1}\right)^{(k)}(R_2) = 0$*, for all* $k = 0,1,\ldots,$ $n-1$*, we obtain*

$$\left| \int_A f(y)\,dy \right| \leqslant \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\left\| \left(g(s)\,s^{N-1}\right)^{(n)} \right\|_{L_q([R_1,R_2])}}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}} \frac{(R_2-R_1)^{n+\frac{1}{p}}}{2^{n-1-\frac{1}{q}}}, \tag{92}$$

which is a sharp inequality,

iv) more generally, for $j = 0,1,2,\ldots,N \in \mathbb{N}$*, it holds*

$$\left| \int_A f(y)\,dy - \left\{ \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{R_2-R_1}{N}\right)^{k+1} \left[j^{k+1} \left(g(s)\,s^{N-1}\right)^{(k)}(R_1) + \right.\right.\right.$$

$$\left.\left.\left. + (-1)^k (N-j)^{k+1} \left(g(s)\,s^{N-1}\right)^{(k)}(R_2) \right] \right\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leqslant$$

$$\leqslant \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\left\| \left(g(s)\,s^{N-1}\right)^{(n)} \right\|_{L_q([R_1,R_2])}}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}} \left(\frac{R_2-R_1}{N}\right)^{n+\frac{1}{p}} \times$$

$$\times \left(j^{n+\frac{1}{p}} + (N-j)^{n+\frac{1}{p}} \right), \tag{93}$$

v) if $\left(g(s)\,s^{N-1}\right)^{(k)}(R_1) = \left(g(s)\,s^{N-1}\right)^{(k)}(R_2) = 0$, $k = 1,\ldots,n-1$, *from* (93) *we get:*

$$\left| \int_A f(y)\,dy - \left\{ \left(\frac{R_2-R_1}{N}\right) \left[j g(R_1) R_1^{N-1} + \right.\right.\right.$$

$$+ (N-j)\, g(R_2)\, R_2^{N-1}\big]\Big\} \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\Bigg| \leqslant \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \times$$

$$\times \frac{\left\| \left(g(s)\, s^{N-1}\right)^{(n)} \right\|_{L_q([R_1,R_2])}}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}} \left(\frac{R_2-R_1}{N}\right)^{n+\frac{1}{p}} \left(j^{n+\frac{1}{p}} + (N-j)^{n+\frac{1}{p}}\right), \tag{94}$$

for $j = 0, 1, 2, \ldots, N \in \mathbb{N}$,

vi) when $N = 2$ *and* $j = 1$, (94) *turns to*

$$\left| \int_A f(y)\, dy - (R_2 - R_1)\left(g(R_1)\, R_1^{N-1} + g(R_2)\, R_2^{N-1}\right) \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leqslant$$

$$\leqslant \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\left\| \left(g(s)\, s^{N-1}\right)^{(n)} \right\|_{L_q([R_1,\, R_2])}}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}} \frac{(R_2-R_1)^{n+\frac{1}{p}}}{2^{n-1-\frac{1}{q}}}, \tag{95}$$

vii) when $n = 1$ *(without any boundary conditions), we get from* (95) *that*

$$\left| \int_A f(y)\, dy - (R_2 - R_1)\left(g(R_1)\, R_1^{N-1} + g(R_2)\, R_2^{N-1}\right) \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \right| \leqslant$$

$$\leqslant \frac{2^{\frac{1}{q}}\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{\left\| \left(g(s)\, s^{N-1}\right)' \right\|_{L_q([R_1,R_2])}}{\left(1+\frac{1}{p}\right)} (R_2-R_1)^{1+\frac{1}{p}}. \tag{96}$$

We continue with

Remark. *Theorems 14 – 16 can easily be converted to results for the ball* $B(0,R)$, $R > 0$. *Their corresponding same assumptions will be for* $f : \overline{B(0,R)} \to \mathbb{R}$ *which is radial. All we need to do then is set* $R_1 = 0$ *and* $R_2 = R$, *and we get a plethora of interesting similar results for the ball that are simpler. Due to lack of space we omit this tedious task.*

References

[1] Agarwal R. P., Dragomir S.S. *An application of Hayashi's inequality for differentiable functions.* Computers Math. Applic., 1996, no. 6, pp. 95 – 99.

[2] Anastassiou G. A. *Fractional Differentiation Inequalities.Research Monograph.* Springer, New York, 2009.

[3] Anastassiou G. A. *General Iyengar type inequalities.* J. of Computational Analysis and Applications, 2020, vol. 28, no. 5, pp. 786 – 797.

[4] Xiao-Liang Cheng. *The Iyengar-type inequality.* Applied Math. Letters, 2001, no. 14, pp. 975 – 978.

[5] Iyengar K. S. K. *Note on an inequality.* Math. Student, 1938, no. 6, pp. 75 –76.

[6] Zheng Liu. *Note on Iyengar's inequality.* Univ. Beograd Publ. Elektrotechn. Fak., Ser. Mat., 2005, no. 16, pp. 29 – 35.

[7] Qi F. *Further generalizations of inequalities for an integral.* Univ. Beograd Publ. Elektrotechn. Fak., Ser. Mat. 1997, no. 8, pp. 79 – 83.

[8] Rudin W. *Real and Complex Analysis.* International Student edition, Mc Graw Hill, London, New York, 1970.

[9] Stroock D. *A Concise Introduction to the Theory of Integration.* Third Edition, Birkhaüser, Boston, Basel, Berlin, 1999.

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