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# A NOTE ON CHARACTERIZATION OF $h$-CONVEX FUNCTIONS VIA HERMITE-HADAMARD TYPE INEQUALITY

**Abstract.** A characterization of $h$-convex function via Hermite-Hadamard inequality related to the $h$-convex functions is investigated. In fact it is determined that under what conditions a function is $h$-convex, if it satisfies the $h$-convex version of Hermite-Hadamard inequality.

**Key words:** *h-convex function, Hermite-Hadamard inequality*

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**1. Introduction.** The following result is well-known in the literature:

**Theorem 1.** [6] A function $f : (a, b) \subset \mathbb{R} \to \mathbb{R}$ is convex if and only if

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{y-x}\int_a^b f(t)dt \leqslant \frac{f(x)+f(y)}{2} \tag{1}$$

holds for all $x, y \in (a, b)$ with $x \neq y$.

Inequality (1) is known as the Hermite-Hadamard integral inequality for convex functions. Note that the left-hand part and the right-hand part of (1) separately are equivalent to the convexity of $f$ (see [5, 6]).

In 2006, the concept of $h$-convex functions related to the nonnegative real functions has been introduced in [9] by S. Varošanec. This class includes a large class of nonnegative functions, such as nonnegative convex functions, Godunova-Levin functions [3], s-convex functions in the second sense [1], and P-functions [2]. In [4], A. Házy used the following definition of $h$-convex functions, which is a generalization of convexexity:

**Definition 1.** *Let $h : [0,1] \to \mathbb{R}$ be a function, such that $h \not\equiv 0$. We say that $f : (a,b) \to \mathbb{R}$ is an $h$-convex function, if for all $x, y \in (a,b)$, $\lambda \in [0,1]$, we have*

$$f\big(\lambda x + (1-\lambda)y\big) \leqslant h(\lambda)f(x) + h(1-\lambda)f(y). \tag{2}$$

We use this definition for the real functions defined on open intervals $(a,b) \subseteq \mathbb{R}$ in this paper. The $h$-convex version of the Hermite-Hadamard inequality was introduced in [8] by Sarikaya et al. as the following:

**Theorem 2.** *Let $f : I \to [0,\infty]$ be an integrable $h$-convex function. If $a, b \in I$, with $a < b$, then*

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \big[f(a)+f(b)\big]\left(\int_0^1 h(t)dt\right). \tag{3}$$

Motivated by the abovementioned works and results, we, in this paper, reply to the problem of conditions $h$-convexity of a function that satisfies (3). Since inequality (3) is double, we separate the problem to the right-hand and the left-hand versions, for the sake of convenience.

**2. Main results.** To achieve our main results about the characterization of an $h$-convex function via (3), we introduce a primary definition along with an example and then establish a basic lemma related to $h$-convex functions.

**Definition 2.** *A function $h : [0,1] \to \mathbb{R}$ is said to be self-concave if*

$$h\big(zx + (1-z)y\big) \geqslant h(z)h(x) + h(1-z)h(y),$$

*for all $z \in (0,1)$ and $x, y \in [0,1]$.*

We can find some simple functions that are self concave.

**Example.** Consider the function $h(x) = x^n$ for $n \in \mathbb{N}$ and $x \in [0,1]$. It is not hard to see that this function is self-concave. In fact, since the function $h$ is nonnegative,

$$h\big(\lambda x + (1-\lambda)y\big) = \big(\lambda x + (1-\lambda)y\big)^n = \textstyle\sum_{i=0}^n \binom{n}{i}\big(\lambda x\big)^{n-i}\big((1-\lambda)y\big)^i \geqslant$$
$$\geqslant \tbinom{n}{0}\big(\lambda x\big)^n + \tbinom{n}{n}\big((1-\lambda)y\big)^n = h(\lambda)h(x) + h(1-\lambda)h(y).$$

Now consider the function $h(x) = \tan(x)$, for $x \in (0,1)$ and $z \in (0,1)$. Expanding this function and using the self-concavity of $x^n$ for $n \in \mathbb{N}$ and

$x \in [0,1]$, we get

$$\tan\big(\lambda x + (1-\lambda)y\big) = \big(\lambda x + (1-\lambda)y\big) + \frac{1}{3}\big(\lambda x + (1-\lambda)y\big)^3 +$$

$$+\frac{2}{15}\big(\lambda x+(1-\lambda)y\big)^5+\frac{17}{315}\big(\lambda x+(1-\lambda)y\big)^7+\frac{62}{2835}\big(\lambda x+(1-\lambda)y\big)^9+\ldots \geqslant$$

$$\geqslant \lambda x + \frac{1}{3}(\lambda x)^3 + \frac{2}{15}(\lambda x)^5 + \frac{17}{315}(\lambda x)^7 + \frac{62}{2835}(\lambda x)^9 + \cdots + \big((1-\lambda)y\big)$$

$$+\frac{1}{3}\big((1-\lambda)y\big)^3+\frac{2}{15}\big((1-\lambda)y\big)^5+\frac{17}{315}\big((1-\lambda)y\big)^7+\frac{62}{2835}\big((1-\lambda)y\big)^9+\cdots =$$

$$= \tan(\lambda x) + \tan((1-\lambda)y) > \tan(\lambda)\tan(x) + \tan(1-\lambda)\tan(y),$$

which implies the self-concavity of $h(x) = \tan(x)$ on $(0,1)$. Note that we have used the fact that $\tan(xy) > \tan(x)\tan(y)$ for all $x, y \in (0,1)$.

The following lemma plays an important role in obtaining our expected results.

**Lemma 1**. *Let $f:(a,b)\to\mathbb{R}$ be a continuous function and $h:[0,1]\to\mathbb{R}$ be a continuous self-concave function. Suppose that for any $x, y \in (a,b)$ with $x \neq y$ there is a $\lambda \in (0,1)$ such that $f\big(\lambda x + (1-\lambda)y\big) \leqslant h(\lambda)f(x) + h(1-\lambda)f(y)$. Then $f$ is $h$-convex on $(a,b)$.*

**Proof.** Without loss of generality, consider $x,y \in (a,b)$ with $x < y$. Define

$$M_{x,y} = \Big\{\lambda \in [0,1]; f\big(\lambda x + (1-\lambda)y\big) \leqslant h(\lambda)f(x) + h(1-\lambda)f(y)\Big\}.$$

It is obvious that $M_{x,y}$ is nonempty. Since $f$ and $h$ are continuous on their domains, $M_{x,y}$ is closed in $[0,1]$. We prove that $M_{x,y} = [0,1]$. On the contrary, suppose that $M_{x,y}$ is a proper subset of $[0,1]$; then we can find $\alpha, \beta \in M_{x,y}$ such that $(\alpha,\beta) \subset [0,1] \setminus M_{x,y}$. Set

$$w = \alpha x + (1-\alpha)y \quad , \quad z = \beta x + (1-\beta)y. \tag{4}$$

From the assumption, there is a $\lambda \in (0,1)$ such that

$$f\big(\lambda w + (1-\lambda)z\big) \leqslant h(\lambda)f(w) + h(1-\lambda)f(z). \tag{5}$$

Also

$$\begin{cases} f(w) = f\big(\alpha x + (1-\alpha)y\big) \leqslant h(\alpha)f(x) + h(1-\alpha)f(y), \\ f(z) = f\big(\beta x + (1-\beta)y\big) \leqslant h(\beta)f(x) + h(1-\beta)f(y). \end{cases} \tag{6}$$

Set $t = \lambda\alpha + (1-\lambda)\beta$. It is clear that $t \in (\alpha, \beta)$ and $t \notin M_{x,y}$. Therefore, from the self-concavity of $h$ and relations (4)-(6), we have

$$f\big(tx + (1-t)y\big) > h(t)f(x) + h(1-t)f(y) =$$

$$= h\big(\lambda\alpha + (1-\lambda)\beta\big)f(x) + h\big(1 - (\lambda\alpha + (1-\lambda)\beta)\big)f(y) =$$

$$= h\big(\lambda\alpha + (1-\lambda)\beta\big)f(x) + h\big(\lambda(1-\alpha) + (1-\lambda)(1-\beta)\big)f(y) \geqslant$$

$$\geqslant \Big[h(\lambda)h(\alpha) + h(1-\lambda)h(\beta)\Big]f(x) + \Big[h(\lambda)h(1-\alpha) + h(1-\lambda)h(1-\beta)\Big]f(y) =$$

$$= h(\lambda)\Big[h(\alpha)f(x) + h(1-\alpha)f(y)\Big] + h(1-\lambda)\Big[h(\beta)f(x) + h(1-\beta)f(y)\Big] \geqslant$$

$$\geqslant h(\lambda)f(w) + h(1-\lambda)f(z) \geqslant f\big(\lambda w + (1-\lambda)z\big).$$

On the other hand,

$$\lambda w + (1-\lambda)z = \lambda\big(\alpha x + (1-\alpha)y\big) + (1-\lambda)\big(\beta x + (1-\beta)y\big) =$$

$$= \Big[\lambda\alpha + (1-\lambda)\beta\Big]x + \Big[\lambda(1-\alpha) + (1-\lambda)(1-\beta)\Big]y =$$

$$= \Big[\lambda\alpha + (1-\lambda)\beta\Big]x + \Big[1 - \big(\lambda\alpha + (1-\lambda)\beta\big)\Big]y = tx + (1-t)y.$$

So,

$$f\big(tx + (1-t)y\big) = f\big(\lambda w + (1-\lambda)z\big) < f\big(tx + (1-t)y\big),$$

which is a contradiction. It follows that $M_{x,y}$ is not a proper subset of $[0,1]$ and hence $M_{x,y} = [0,1]$. Since this happens for any $x, y \in (a,b)$ with $x < y$, we conclude that $f$ is $h$-convex on $(a,b)$. □

**Theorem 3**. *Let $f : (a,b) \to \mathbb{R}$ be a continuous function. Also suppose that $h : [0,1] \to \mathbb{R}$ is a continuous self-concave function, such that*

$$\frac{1}{y-x}\int_x^y f(t)dt \leqslant [f(x) + f(y)]\left(\int_0^1 h(t)dt\right),$$

*for all $x, y \in (a,b)$ with $x \neq y$. Then $f$ is $h$-convex on $(a,b)$.*

**Proof.** Suppose that $f$ is not $h$-convex on $(a,b)$. Then, by Lemma 1, there are $x, y \in (a,b)$ with $x < y$ such that

$$f(tx + (1-t)y) > h(t)f(x) + h(1-t)f(y) \quad \forall t \in (0,1).$$

For such $x$ and $y$,

$$\frac{1}{y-x}\int_x^y f(t)dt=\int_0^1 f(tx+(1-t)y)dt > \int_0^1 [h(t)f(x)+h(1-t)f(y)]dt =$$

$$=\Big(\int_0^1 h(t)dt\Big)f(x)+\Big(\int_0^1 h(1-t)dt\Big)f(y)=\big[f(x)+f(y)\big]\Big(\int_0^1 h(t)dt\Big).$$

This is a contradiction. Hence, $f$ is $h$-convex on $(a,b)$. □

The following lemma, along with Lemma 1, are the base for characterization of a $h$-convex function via the left-hand side of (3).

**Lemma 2**. (Also see Theorem 1.1.4 in [5].) Suppose that $\varphi:[a,b]\to\mathbb{R}$ is a continuous function such that $\varphi(a)=\varphi(b)=0$ and $\varphi(t)>0$ for some $t\in(a,b)$. Then there exists an $x\in(a,b)$ such that

$$\varphi(x)=\max_{a\leqslant y\leqslant b}\varphi(y) \text{ and } \varphi(x)>\varphi(y) \text{ for all } a\leqslant y<x.$$

**Proof.** From Theorem 4.16 in [7], $\varphi$ attains its maximum $\alpha$ in $[a,b]$. From the assumption, we have $\alpha\geqslant\varphi(t)>0$. Set $M=\{y\in[a,b];\varphi(y)=\alpha\}$. Since $\varphi$ is continuous, $M$ is a nonempty compact subset of $[a,b]$, such that $a,b\notin M$. If we put $x=\inf\{y;y\in M\}$, then

$$\varphi(x)=\alpha=\max_{a\leqslant y\leqslant b}\varphi(y),$$

and $f(y)<f(x)$ for all $a\leqslant y<x$. □

In what follows, we assume that the function $h:[0,1]\to\mathbb{R}$ satisfies the conditions

$$\begin{cases} h(\lambda)+h(1-\lambda)=1 \text{ for all } \lambda\in(0,1),\\ h(0)=0. \end{cases} \tag{7}$$

**Lemma 3**. *Let $h:[0,1]\to\mathbb{R}$ be a continuous self-concave function. Suppose that $f:(a,b)\to\mathbb{R}$ is a continuous function and for any $x\in(a,b)$, $\varepsilon>0$, there exist $y,z\in(a,b)\cap(x-\varepsilon,x+\varepsilon)$ with $y<x<z$ such that*

$$f(x)=f\big(\lambda y+(1-\lambda)z\big)\leqslant h(\lambda)f(y)+h(1-\lambda)f(z) \text{ for some } \lambda\in(0,1).$$

*Then $f$ is $h$-convex on $(a,b)$.*

**Proof.** If $f$ is not $h$-convex, then by Lemma 1, there are $x_1, x_2 \in (a,b)$ with $x_1 \neq x_2$ (assume that $x_1 < x_2$) such that

$$f\big(\lambda x_1 + (1-\lambda)x_2\big) > h(\lambda)f(x_1) + h(1-\lambda)f(x_2) \text{ for all } \lambda \in (0,1). \quad (8)$$

Consider the function $g : [x_1, x_2] \to \mathbb{R}$ defined by

$$g(y) = g\big(\lambda x_1 + (1-\lambda)x_2\big) =$$

$$= f\big(\lambda x_1 + (1-\lambda)x_2\big) - f(x_1) - \frac{f(x_2)-f(x_1)}{x_2-x_1}\Big(h(\lambda)x_1 + h(1-\lambda)x_2 - x_1\Big).$$

It is clear that $g$ is continuous on $[x_1, x_2]$ and $g(x_1) = g(x_2) = 0$. Also, from (7) and (8), we get

$$g\big(\lambda x_1 + (1-\lambda)x_2\big) = f\big(\lambda x_1 + (1-\lambda)x_2\big) - f(x_1) - \quad (9)$$

$$-\frac{f(x_2)-f(x_1)}{x_2-x_1}\Big((1-h(\lambda))x_2 - (1-h(\lambda))x_1\Big) =$$

$$= f\big(\lambda x_1 + (1-\lambda)x_2\big) - h(\lambda)f(x_1) - h(1-\lambda)f(x_2) > 0.$$

Lemma 2 and (9) imply that there is an $x \in (x_1, x_2)$ such that

$$g(x) = \max_{x_1 \leqslant y \leqslant x_2} g(y) \text{ and } g(x) > g(y) \text{ for } x_1 \leqslant y < x. \quad (10)$$

Hence, $x = tx_1 + (1-t)x_2$ for some $0 < t < 1$. Now choose $x_0, y_0 \in [x_1, x_2]$ such that $x_1 \leqslant x_0 < x < y_0 \leqslant x_2$. Therefore, from (10) for any $\lambda \in (0,1)$,

$$g(x) = [h(\lambda) + h(1-\lambda)]g(x) > h(\lambda)g(x_0) + h(1-\lambda)g(y_0). \quad (11)$$

$$f(x) - f(x_1) - \frac{f(x_2)-f(x_1)}{x_2-x_1}\Big(h(\lambda)x_0 + h(1-\lambda)y_0 - x_1\Big) > \quad (12)$$

$$> h(\lambda)\Big[f(x_0) - f(x_1) - \frac{f(x_2)-f(x_1)}{x_2-x_1}(x_0 - x_1)\Big] +$$

$$+ h(1-\lambda)\Big[f(y_0) - f(x_1) - \frac{f(x_2)-f(x_1)}{x_2-x_1}(y_0 - x_1)\Big].$$

From (7) we deduce, simplifying (12):

$$f(x) > h(\lambda)f(x_0) + h(1-\lambda)f(y_0) \text{ for all } \lambda \in (0,1). \quad (13)$$

Since $x_0$, $y_0$, and $\lambda$ are arbitrary, (13) contradicts the assumption. Hence, $f$ is an $h$-convex function on $(a, b)$. □

Using Lemma 3, as an immediate consequence we have two following lemmas. For more details about this kind of results related to the convex functions, see [6].

**Corollary 1.** *Let $h : [0, 1] \to \mathbb{R}$ be a continuous self-concave function. Suppose that $f : (a, b) \to \mathbb{R}$ is a continuous function and for any $x \in (a, b)$, $\varepsilon > 0$, there exists a $\delta \in (0, \varepsilon)$ such that*

$$f(x) \leqslant h(1/2)\big[f(x-\delta) + f(x+\delta)\big].$$

*Then $f$ is $h$-convex on $(a, b)$.*

**Proof.** In Lemma 3, take $y = x - \delta$, $z = x + \delta$ and $\lambda = 1/2$. □

**Lemma 4.** *Let $h : [0, 1] \to \mathbb{R}$ be a continuous self-concave function. Suppose that $f : (a, b) \to \mathbb{R}$ is a continuous function and for any $x \in (a, b)$, $\varepsilon > 0$, there exists $\delta \in (0, \varepsilon)$ such that*

$$f(x) \leqslant \frac{h(1/2)}{\delta} \int_{x-\delta}^{x+\delta} f(u)du.$$

*Then $f$ is $h$-convex on $(a, b)$.*

**Proof.** Suppose that $f$ is not $h$-convex on $(a, b)$. From Corollary 1, there are $x \in (a, b)$ and $\varepsilon > 0$ such that $a < x - \varepsilon < x + \varepsilon < b$ and

$$f(x) > h(1/2)\big[f(x-\delta) + f(x+\delta)\big] \text{ for any } 0 < \delta < \varepsilon.$$

Integrating with respect to $\delta$ in the above inequality, we get

$$\frac{1}{h(1/2)} \int_0^{\delta} f(x)dt > \int_0^{\delta} f(x-t)dt + \int_0^{\delta} f(x+t)dt =$$

$$= - \int_x^{x-\delta} f(u)du + \int_x^{x+\delta} f(u)du = \int_{x-\delta}^{x+\delta} f(u)du.$$

So,

$$f(x) \cdot \delta \leqslant h(1/2) \int_{x-\delta}^{x+\delta} f(u)du.$$

This contradicts the assumption and, hence, $f$ is $h$-convex on $(a, b)$. $\square$

Now, using Lemma 4, we can obtain a characterization-type theorem for $h$-convex functions via the left-hand side of (3).

**Theorem 4.** *Let $h : [0,1] \to \mathbb{R}$ be a continuous self-concave function. Suppose that $f : (a,b) \to \mathbb{R}$ is a continuous function and for all $y, z \in (a,b)$ with $y \neq z$ we have*

$$\frac{1}{2h(1/2)} f\left(\frac{y+z}{2}\right) \leqslant \frac{1}{z-y} \int_y^z f(u)du; \tag{14}$$

*then $f$ is $h$-convex on $(a, b)$.*

**Proof.** Suppose that $f$ is not $h$-convex on $(a,b)$. From Lemma 4, there exist $x \in (a,b)$ and $\varepsilon > 0$ such that for all $\delta \in (0, \varepsilon)$

$$f(x) > \frac{h(1/2)}{\delta} \int_{x-\delta}^{x+\delta} f(u)du.$$

Now, if we choose $\delta < \varepsilon$ and $y, z \in (a,b)$ with $y < z$ such that

$$\begin{cases} x = \frac{1}{2}y + \frac{1}{2}z, \\ x - y = z - x = \delta, \end{cases}$$

then we have

$$f\left(\frac{y+z}{2}\right) > \frac{2h(1/2)}{z-y} \int_y^z f(u)du.$$

This contradicts (14). Thus, $f$ is $h$-convex on $(a, b)$. $\square$

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