 Probl. Anal. Issues Anal. Vol. 9 (27), No 2, 2020, pp. 152–162
DOI: 10.15393/j3.art.2020.6332

UDC 517.98, 517.521

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SOME IDENTITIES AND INEQUALITIES FOR G-FUSION FRAMES

Abstract. G-fusion frames, which are obtained from the combination of g-frames and fusion frames, were recently introduced for Hilbert spaces. In this paper, we present a new identity for g-frames, which was given by Najati for a special case. Also, by using the idea of this identity and the dual frames, some equalities and inequalities are presented for g-fusion frames.

Key words: *g-frame, dual g-frame, g-fusion frame, dual g-fusion frame*

2010 Mathematical Subject Classification: *Primary 42C15; Secondary 46C99, 41A58.*

1. Introduction. Recent developments in the frame theory and their applications are the result of some mathematicians' efforts in this topic (see [10], [13], [12], [3], [6], [8]). By more than half a century, this theory has got interesting applications in different branches of science, such as the filter bank theory, signal and image processing, wireless communications, atomic systems, and the Kadison-Singer problem. In 2005, Balan, Casazza, and others found some useful identities for frames by studying properties of the Parseval frames [2]. Simil ar results for fusion frames, g-frames, and K-frames are presented in [18], [21], [1]. In [22], a special kind of frames called g-fusion frames is introduced; they are combinations of g-frames and fusion frames. We present some identities for these frames.

2. Preliminaries. Throughout this paper, H and K are separable Hilbert spaces, π_V is the orthogonal projection from H onto a closed subspace $V \subset H$, and $\mathcal{B}(H, K)$ is the collection of all the bounded linear operators of H into K. If $K = H$, then $\mathcal{B}(H, H)$ will be denoted by $\mathcal{B}(H)$. Also, $\{H_j\}_{j\in\mathbb{J}}$ is a sequence of Hilbert spaces and $\Lambda_j \in \mathcal{B}(H, H_j)$ for each

$j \in \mathbb{J}$, where $\mathbb{J}$ is a subset of $\mathbb{Z}$. The following lemmas from the operator theory will be needed.

Lemma 1. [13] *Let $V \subseteq H$ be a closed subspace, and T be a linear bounded operator on H. Then*

$$\pi_V T^* = \pi_V T^* \pi_{\overline{TV}}.$$

Lemma 2. [21] *Let $u \in \mathcal{B}(H)$ be adjoint and $v := au^2 + bu + c$ where $a, b, c \in \mathbb{R}$.*

(I) If $a > 0$, then

$$\inf_{\|f\|=1} \langle vf, f \rangle \geq \frac{4ac - b^2}{4a}.$$

(II) If $a < 0$, then

$$\sup_{\|f\|=1} \langle vf, f \rangle \leq \frac{4ac - b^2}{4a}.$$

Lemma 3. [2] *If u, v are operators on H satisfying $u + v = id_H$, then $u - v = u^2 - v^2$.*

We define the space $\mathscr{H}_2 := \left(\sum_{j\in\mathbb{J}} \oplus H_j\right)_{\ell_2}$ by

$$\mathscr{H}_2 = \{\{f_j\}_{j\in\mathbb{J}} \ : \ f_j \in H_j, \ \sum_{j\in\mathbb{J}} \|f_j\|^2 < \infty\},$$

with the inner product defined by

$$\langle \{f_j\}, \{g_j\} \rangle = \sum_{j\in\mathbb{J}} \langle f_j, g_j \rangle.$$

It is clear that $\mathscr{H}_2$ is a Hilbert space with pointwise operations.

Definition 1. [23] *We call the sequence $\{\Lambda_j\}_{j\in\mathbb{J}}$ a g-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$ if there exist $0 < A \leq B < \infty$, such that for each $f \in H$*

$$A\|f\|^2 \leq \sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2 \leq B\|f\|^2. \tag{1}$$

If $A = B = 1$, we call $\{\Lambda_j\}_{j\in\mathbb{J}}$ a Parseval g-frame. The synthesis and analysis operators in g-frames are defined by

$$T : \mathscr{H}_2 \longrightarrow H \ , \qquad T^* : H \longrightarrow \mathscr{H}_2$$

$$T(\{f_j\}_{j\in\mathbb{J}}) = \sum_{j\in\mathbb{J}} \Lambda_j^* f_j \ , \qquad T^*(f) = \{\Lambda_j f\}_{j\in\mathbb{J}}.$$

Therefore, the g-frame operator is defined by

$$Sf = TT^*f = \sum_{j\in\mathbb{J}} \Lambda_j^*\Lambda_j f.$$

The operator S is bounded, positive, and invertible. If $\tilde{\Lambda}_j := \Lambda_j S^{-1}$, then $\{\tilde{\Lambda}_j\}_{j\in\mathbb{J}}$ is called a (canonical) dual g-frame of $\{\Lambda_j\}_{j\in\mathbb{J}}$, and we can write

$$f = \sum_{j\in\mathbb{J}} \tilde{\Lambda}_j^*\Lambda_j f = \sum_{j\in\mathbb{J}} \Lambda_j^*\tilde{\Lambda}_j f. \tag{2}$$

If $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-frame for H with bounds A and B, respectively, then $\{\tilde{\Lambda}_j\}_{j\in\mathbb{J}}$ is also a g-frame for H with bounds B^{-1} and A^{-1}, respectively.

Definition 2. [22] *Let $W = \{W_j\}_{j\in\mathbb{J}}$ be a family of closed subspaces of H, $\{v_j\}_{j\in\mathbb{J}}$ be a family of weights, i. e., $v_j > 0$. We say $\Lambda := (W_j, \Lambda_j, v_j)$ is a g-fusion frame for H if there exist $0 < A \leq B < \infty$, such that for each $f \in H$*

$$A\|f\|^2 \leq \sum_{j\in\mathbb{J}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 \leq B\|f\|^2. \tag{3}$$

It is easy to see that these frames are extensions of g-frames. We call Λ a Parseval g-fusion frame if $A = B = 1$. When the right-hand part of (3) holds, Λ is called a g-fusion Bessel sequence for H with the bound B. Throughout this paper, Λ is a triple (W_j, Λ_j, v_j) with $j \in \mathbb{J}$.

The synthesis and analysis operators in the g-fusion frames are defined by (for more details, we refer [22])

$$T_\Lambda : \mathscr{H}_2 \longrightarrow H, \qquad T_\Lambda^* : H \longrightarrow \mathscr{H}_2$$
$$T_\Lambda(\{f_j\}_{j\in\mathbb{J}}) = \sum_{j\in\mathbb{J}} v_j \pi_{W_j} \Lambda_j^* f_j, \qquad T_\Lambda^*(f) = \{v_j \Lambda_j \pi_{W_j} f\}_{j\in\mathbb{J}}.$$

Thus, the g-fusion frame operator is given by

$$S_\Lambda f = T_\Lambda T_\Lambda^* f = \sum_{j\in\mathbb{J}} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f.$$

Therefore,

$$A\, id_H \leq S_\Lambda \leq B\, id_H.$$

This means that S_Λ is a bounded, positive, and invertible operator (with an adjoint inverse), and we have

$$B^{-1}id_H \leq S_\Lambda^{-1} \leq A^{-1}id_H.$$

So, we have the following reconstruction formula for any $f \in H$:

$$f = \sum_{j\in\mathbb{J}} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} S_\Lambda^{-1} f = \sum_{j\in\mathbb{J}} v_j^2 S_\Lambda^{-1} \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} f. \tag{4}$$

Let $\tilde{\Lambda} := (S_\Lambda^{-1} W_j, \Lambda_j \pi_{W_j} S_\Lambda^{-1}, v_j)$. Then $\tilde{\Lambda}$ is called the *(canonical) dual g-fusion frame* of Λ. Hence, for each $f \in H$ we get

$$f = \sum_{j\in\mathbb{J}} v_j^2 \pi_{W_j} \Lambda_j^* \tilde{\Lambda}_j \pi_{\tilde{W}_j} f = \sum_{j\in\mathbb{J}} v_j^2 \pi_{\tilde{W}_j} \tilde{\Lambda}_j^* \Lambda_j \pi_{W_j} f, \tag{5}$$

where $\tilde{W}_j := S_\Lambda^{-1} W_j$, $\tilde{\Lambda}_j := \Lambda_j \pi_{W_j} S_\Lambda^{-1}$. Thus, we obtain

$$\langle S_\Lambda^{-1} f, f \rangle = \sum_{j\in\mathbb{J}} v_j^2 \|\tilde{\Lambda}_j \pi_{\tilde{W}_j} f\|^2. \tag{6}$$

3. The Main Results. Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a g-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$ with bounds A, B and $\{\tilde{\Lambda}_j\}_{j\in\mathbb{J}}$ be a (canonical) dual g-frame of $\{\Lambda_j\}_{j\in\mathbb{J}}$. Suppose that $\mathbb{I} \subseteq \mathbb{J}$ and let

$$S_\mathbb{I} : H \to H$$
$$S_\mathbb{I} f := \sum_{j\in\mathbb{I}} \Lambda_j^* \tilde{\Lambda}_j f.$$

This is a general case of the operator S_J presented in [21]. We have

$$\|S_\mathbb{I} f\|^2 = \Big(\sup_{\|h\|=1} |\langle S_\mathbb{I} f, h \rangle| \Big)^2 = \sup_{\|h\|=1} \Big(\sum_j |\langle \tilde{\Lambda}_j f, \Lambda_j h \rangle| \Big)^2 \leq$$
$$\leq \sum_j \|\tilde{\Lambda}_j f\|^2 \times \sup_{\|h\|=1} \sum_j \|\Lambda_j h\|^2 \leq BA^{-1} \|f\|^2.$$

Thus, $S_\mathbb{I} \in \mathcal{B}(H)$ and is positive. From (2) we obtain that $S_\mathbb{I} + S_{\mathbb{I}^c} = id_H$.

Theorem 1. *For $f \in H$, we have*

$$\sum_{j\in\mathbb{I}} \langle \tilde{\Lambda}_j f, \Lambda_j f \rangle - \|S_\mathbb{I} f\|^2 = \sum_{j\in\mathbb{I}^c} \overline{\langle \tilde{\Lambda}_j f, \Lambda_j f \rangle} - \|S_{\mathbb{I}^c} f\|^2$$

where $\mathbb{I}^c$ is the complement of $\mathbb{I}$.

Proof. For each $f \in H$, we have

$$\sum_{j\in\mathbb{I}}\langle\tilde{\Lambda}_j f, \Lambda_j f\rangle - \Big\|\sum_{j\in\mathbb{I}}\Lambda_j^*\tilde{\Lambda}_j f\Big\|^2 = \langle S_{\mathbb{I}}f, f\rangle - \|S_{\mathbb{I}}f\|^2 =$$
$$= \langle S_{\mathbb{I}}f, f\rangle - \langle S_{\mathbb{I}}^* S_{\mathbb{I}}f, f\rangle = \langle (id_H - S_{\mathbb{I}})^* S_{\mathbb{I}}f, f\rangle = \langle S_{\mathbb{I}^c}^*(id_H - S_{\mathbb{I}^c})f, f\rangle =$$
$$= \langle S_{\mathbb{I}^c}^* f, f\rangle - \langle S_{\mathbb{I}^c}^* S_{\mathbb{I}^c} f, f\rangle = \langle f, S_{\mathbb{I}^c} f\rangle - \langle S_{\mathbb{I}^c} f, S_{\mathbb{I}^c} f\rangle =$$
$$= \sum_{j\in\mathbb{I}^c}\langle\Lambda_j f, \tilde{\Lambda}_j f\rangle - \|\sum_{j\in\mathbb{I}^c}\Lambda_j^*\tilde{\Lambda}_j f\|^2 = \sum_{j\in\mathbb{I}^c}\overline{\langle\tilde{\Lambda}_j f, \Lambda_j f\rangle} - \|\sum_{j\in\mathbb{I}^c}\Lambda_j^*\tilde{\Lambda}_j f\|^2$$

and the proof is complete. □

Now, if $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a Parseval g-frame, then $\tilde{\Lambda}_j = \Lambda_j$, and we obtain the following famous formula presented in [21]:

$$\sum_{j\in\mathbb{I}}\|\Lambda_j f\|^2 - \|S_{\mathbb{I}}f\|^2 = \sum_{j\in\mathbb{I}^c}\|\Lambda_j f\|^2 - \|S_{\mathbb{I}^c}f\|^2,$$

where $S_{\mathbb{I}}f = \sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f$.

The same can be obtained for g-fusion frames. Let Λ be a g-fusion frame for H with a (canonical) dual g-fusion frame $\tilde{\Lambda} = (\tilde{W}_j, \tilde{\Lambda}_j, v_j)$, where $\tilde{W}_j = S_\Lambda W_j$ and $\tilde{\Lambda}_j = \Lambda_j \pi_{W_j} S_\Lambda^{-1}$. For simplicity, we denote the following operator with the same symbol $S_{\mathbb{I}}$, where, again, $\mathbb{I}$ is a finite subset of $\mathbb{J}$:

$$S_{\mathbb{I}}f = \sum_{j\in\mathbb{I}} v_j^2 \pi_{W_j}\Lambda_j^*\tilde{\Lambda}_j\pi_{\tilde{W}_j} f, \qquad (\forall f \in H). \tag{7}$$

It is easy to check that $S_{\mathbb{I}} \in \mathcal{B}(H)$ and positive. Again, we have

$$S_{\mathbb{I}} + S_{\mathbb{I}^c} = id_H.$$

Remark 1. *Let Λ be a Parseval g-fusion frame for H. Since $\mathcal{B}(H)$ is a C^*-algebra and $S_{\mathbb{I}}$ is positive, so $r(S_{\mathbb{I}}) = \|S_{\mathbb{I}}\|$, where r is the spectral radius. Thus*

$$\max_{\lambda\in\sigma(S_{\mathbb{I}})}|\lambda| = r(S_{\mathbb{I}}) \leq 1$$

and we conclude that $\sigma(S_{\mathbb{I}}) \in [0, 1]$.

Theorem 2. *Let $f \in H$; then*

$$\sum_{j\in\mathbb{I}} v_j^2\langle\tilde{\Lambda}_j\pi_{\tilde{W}_j} f, \Lambda_j\pi_{W_j} f\rangle - \|S_{\mathbb{I}}f\|^2 = \sum_{j\in\mathbb{I}^c} v_j^2\overline{\langle\tilde{\Lambda}_j\pi_{\tilde{W}_j} f, \Lambda_j\pi_{W_j} f\rangle} - \|S_{\mathbb{I}^c}f\|^2.$$

Proof. The proof follows a similar argument as in the proof of Theorem 1. □

Corollary 1. *Let Λ be a Parseval g-fusion frame for H. Then*

$$\sum_{j\in\mathbb{I}} v_j^2\|\Lambda_j\pi_{W_j}f\|^2 - \Big\|\sum_{j\in\mathbb{I}} v_j^2\pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j}f\Big\|^2 = \\ = \sum_{j\in\mathbb{I}^c} v_j^2\|\Lambda_j\pi_{W_j}f\|^2 - \Big\|\sum_{j\in\mathbb{I}^c} v_j^2\pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j}f\Big\|^2.$$

Moreover,

$$\sum_{j\in\mathbb{I}} v_j^2\|\Lambda_j\pi_{W_j}f\|^2 + \Big\|\sum_{j\in\mathbb{I}^c} v_j^2\pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j}f\Big\|^2 \geq \frac{3}{4}\|f\|^2.$$

Proof. If $f \in H$, we obtain

$$\sum_{j\in\mathbb{I}} v_j^2\|\Lambda_j\pi_{W_j}f\|^2 + \|S_{\mathbb{I}^c}f\|^2 = \langle(S_{\mathbb{I}} + S_{\mathbb{I}^c}^2)f, f\rangle = \\ = \langle(S_{\mathbb{I}} + id_H - 2S_{\mathbb{I}} + S_{\mathbb{I}}^2)f, f\rangle = \langle(id_H - S_{\mathbb{I}} + S_{\mathbb{I}}^2)f, f\rangle.$$

Now, by Lemma 2 for $a = 1$, $b = -1$, and $c = 1$, the inequality holds. □

Corollary 2. *Let Λ be a Parseval g-fusion frame for H. Then*

$$0 \leq S_{\mathbb{I}} - S_{\mathbb{I}}^2 \leq \frac{1}{4} id_H$$

Proof. We have $S_{\mathbb{I}}S_{\mathbb{I}^c} = S_{\mathbb{I}^c}S_{\mathbb{I}}$. Then $0 \leq S_{\mathbb{I}}S_{\mathbb{I}^c} = S_{\mathbb{I}} - S_{\mathbb{I}}^2$. Also, by Lemma 2, we get

$$S_{\mathbb{I}} - S_{\mathbb{I}}^2 \leq \frac{1}{4} id_H.$$

The proof is complete. □

Theorem 3. *Let Λ be a g-fusion frame with the g-fusion frame operator S_Λ. If $\mathbb{I} \subseteq \mathbb{J}$ and $f \in H$, then*

$$\sum_{j\in\mathbb{I}} v_j^2\|\Lambda_j\pi_{W_j}f\|^2 + \|S_\Lambda^{-\frac{1}{2}}S_{\mathbb{I}^c}f\|^2 = \sum_{j\in\mathbb{I}^c} v_j^2\|\Lambda_j\pi_{W_j}f\|^2 + \|S_\Lambda^{-\frac{1}{2}}S_{\mathbb{I}}f\|^2.$$

Proof. Let $\Theta_j := \Lambda_j\pi_{W_j}S_\Lambda^{-\frac{1}{2}}$ and $X_j := S_\Lambda^{-\frac{1}{2}}W_j$. Example 2.2 [22] shows that (X_j, Θ_j, v_j) is a Parseval g-fusion frame for H. Then, by Corollary 1, we have

$$\sum_{j\in\mathbb{I}} \upsilon_j^2\|\Theta_j\pi_{X_j}f\|^2 - \Big\|\sum_{j\in\mathbb{I}} \upsilon_j^2\pi_{X_j}\Theta_j^*\Theta_j\pi_{X_j}f\Big\|^2 =$$

$$= \sum_{j\in\mathbb{I}^c} \upsilon_j^2\|\Theta_j\pi_{X_j}f\|^2 - \Big\|\sum_{j\in\mathbb{I}^c} \upsilon_j^2\pi_{X_j}\Theta_j^*\Theta_j\pi_{X_j}f\Big\|^2.$$

By replacing f by $S_\Lambda^{\frac{1}{2}}f$ and the fact that

$$\sum_{j\in\mathbb{I}} \upsilon_j^2\pi_{X_j}\Theta_j^*\Theta_j\pi_{X_j}f = \sum_{j\in\mathbb{I}} \upsilon_j^2(\Theta_j\pi_{X_j})^*\Theta_j\pi_{X_j}f =$$

$$= \sum_{j\in\mathbb{I}} \upsilon_j^2(\Lambda_j\pi_{W_j}S_\Lambda^{-\frac{1}{2}}\pi_{X_j})^*\Lambda_j\pi_{W_j}S_\Lambda^{-\frac{1}{2}}\pi_{X_j}f =$$

$$= \sum_{j\in\mathbb{I}} \upsilon_j^2 S_\Lambda^{-\frac{1}{2}}\pi_{W_j}\Lambda_j^*\Lambda_j\pi_{W_j}S_\Lambda^{-\frac{1}{2}}f =$$

$$= S_\Lambda^{-\frac{1}{2}}S_\mathbb{I}S_\Lambda^{-\frac{1}{2}}f,$$

the proof is complete. □

Corollary 1. *Let Λ be a g-fusion frame with the g-fusion frame operator S_Λ. If $\mathbb{I}\subseteq\mathbb{J}$, then*

$$0 \leq S_\mathbb{I} - S_\mathbb{I}S_\Lambda^{-1}S_\mathbb{I} \leq \frac{1}{4}S_\Lambda.$$

Proof. In the proof of Theorem 3, we showed that

$$\sum_{j\in\mathbb{I}} \upsilon_j^2\pi_{X_j}\Theta_j^*\Theta_j\pi_{X_j}f = S_\Lambda^{-\frac{1}{2}}S_\mathbb{I}S_\Lambda^{-\frac{1}{2}}f.$$

By Corollary 2, we get

$$0 \leq \sum_{j\in\mathbb{I}} \upsilon_j^2\pi_{X_j}\Theta_j^*\Theta_j\pi_{X_j}f - \Big(\sum_{j\in\mathbb{I}} \upsilon_j^2\pi_{X_j}\Theta_j^*\Theta_j\pi_{X_j}f\Big)^2 \leq \frac{1}{4}id_H.$$

Therefore,

$$0 \leq S_\Lambda^{-\frac{1}{2}}(S_\mathbb{I} - S_\mathbb{I}S_\Lambda^{-1}S_\mathbb{I})S_\Lambda^{-\frac{1}{2}} \leq \frac{1}{4}id_H$$

and the proof is complete. □

Corollary 2. *Suppose that Λ is a g-fusion frame with the g-fusion frame operator S_Λ. If $\mathbb{I}\subseteq\mathbb{J}$ and $f\in H$, then*

$$\sum_{j\in\mathbb{I}} \upsilon_j^2\|\Lambda_j\pi_{W_j}f\|^2 + \|S_\Lambda^{-\frac{1}{2}}S_{\mathbb{I}^c}f\|^2 \geq \frac{3}{4}\|S_\Lambda^{-1}\|^{-1}\|f\|^2.$$

Proof. By Theorem 3 and Corollary 1, we can write

$$\sum_{j\in\mathbb{I}} v_j^2\|\Lambda_j\pi_{W_j}f\|^2+\|S_\Lambda^{-\frac{1}{2}}S_{\mathbb{I}^c}f\|=$$

$$=\sum_{j\in\mathbb{I}} v_j^2\|\Theta_j\pi_{X_j}S_\Lambda^{\frac{1}{2}}f\|^2+\|\sum_{j\in\mathbb{I}^c} v_j^2\pi_{X_j}\Theta_j^*\Theta_j\pi_{X_j}S_\Lambda^{\frac{1}{2}}f\|^2\geq$$

$$\geq \frac{3}{4}\|S_\Lambda^{\frac{1}{2}}f\|^2=\frac{3}{4}\langle S_\Lambda f, f\rangle \geq \frac{3}{4}\|S_\Lambda^{-1}\|^{-1}\|f\|^2.$$

The poof is complete. □

Theorem 4. *Let Λ be a Parseval g-fusion frame for H and $\mathbb{I}\subseteq\mathbb{J}$. Then*

(I) $0\leq S_{\mathbb{I}}-S_{\mathbb{I}}^2\leq \frac{1}{4}id_H.$

(II) $\frac{1}{2}id_H\leq S_{\mathbb{I}}^2+S_{\mathbb{I}^c}^2\leq \frac{3}{2}id_H.$

Proof. *(I)* Since $S_{\mathbb{I}}+S_{\mathbb{I}^c}=id_H$, $S_{\mathbb{I}}S_{\mathbb{I}^c}+S_{\mathbb{I}^c}^2=S_{\mathbb{I}^c}$. Thus,

$$S_{\mathbb{I}}S_{\mathbb{I}^c}=S_{\mathbb{I}^c}-S_{\mathbb{I}^c}^2=S_{\mathbb{I}^c}(id_H-S_{\mathbb{I}^c})=S_{\mathbb{I}^c}S_{\mathbb{I}}.$$

But Λ is Parseval, so $0\leq S_{\mathbb{I}}S_{\mathbb{I}^c}=S_{\mathbb{I}}-S_{\mathbb{I}}^2$. On the other hand, by Lemma 3, we get

$$S_{\mathbb{I}}-S_{\mathbb{I}}^2\leq\frac{1}{4}id_H.$$

(II) We have seen that $S_{\mathbb{I}}S_{\mathbb{I}^c}=S_{\mathbb{I}^c}S_{\mathbb{I}}$; then, by Lemma 3,

$$S_{\mathbb{I}}^2+S_{\mathbb{I}^c}^2=id_H-2S_{\mathbb{I}}S_{\mathbb{I}^c}=2S_{\mathbb{I}}^2-2S_{\mathbb{I}}+id_H\geq\frac{1}{2}id_H.$$

On the other hand, we have, again, by Lemma 3 and $0\leq S_{\mathbb{I}}-S_{\mathbb{I}}^2$:

$$S_{\mathbb{I}}^2+S_{\mathbb{I}^c}^2\leq id_H+2S_{\mathbb{I}}-2S_{\mathbb{I}}^2\leq\frac{3}{2}id_H.$$

This completes the proof. □

Corollary 1. *Let Λ be a g-fusion frame with the g-fusion frame operator S_Λ. If $\mathbb{I}\subseteq\mathbb{J}$, then*

$$\frac{1}{2}S_\Lambda\leq S_{\mathbb{I}}S_\Lambda^{-1}S_{\mathbb{I}}-S_{\mathbb{I}^c}S_\Lambda^{-1}S_{\mathbb{I}^c}\leq\frac{3}{2}S_\Lambda.$$

Proof. We have

$$\sum_{j\in\mathbb{I}} v_j^2 \pi_{X_j}\Theta_j^*\Theta_j\pi_{X_j} f = S_\Lambda^{-\frac{1}{2}} S_{\mathbb{I}} S_\Lambda^{-\frac{1}{2}} f.$$

Therefore, similarly to the proof of Corollary 1, we get, by Theorem 4, item (II),

$$\frac{1}{2} id_H \leq (S_\Lambda^{-\frac{1}{2}} S_{\mathbb{I}} S_\Lambda^{-\frac{1}{2}})^2 + (S_\Lambda^{-\frac{1}{2}} S_{\mathbb{I}^c} S_\Lambda^{-\frac{1}{2}})^2 \leq \frac{3}{2} id_H,$$

and the proof is now evident. □

Theorem 5. *Let Λ be a g-fusion frame with the g-fusion frame operator S_Λ. If $\mathbb{I} \subseteq \mathbb{J}$, then, for any $f \in H$,*

$$\sum_{j\in\mathbb{I}} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 - \sum_{j\in\mathbb{J}} v_j^2 \|\tilde{\Lambda}_j \pi_{\tilde{W}_j} M_{\mathbb{I}} f\|^2 =$$
$$= \sum_{j\in\mathbb{I}^c} v_j^2 \|\Lambda_j \pi_{W_j} f\|^2 - \sum_{j\in\mathbb{J}} v_j^2 \|\tilde{\Lambda}_j \pi_{\tilde{W}_j} M_{\mathbb{I}^c} f\|^2,$$

where

$$M_{\mathbb{I}} f = \sum_{j\in\mathbb{I}} v_j^2 \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{\tilde{W}_j} f.$$

Proof. Via the definition of S_Λ, it is clear that $M_{\mathbb{I}} + M_{\mathbb{I}^c} = S_\Lambda$. Therefore, $S_\Lambda^{-1} M_{\mathbb{I}} + S_\Lambda^{-1} M_{\mathbb{I}^c} = id_H$. Hence, by Lemma 3

$$S_\Lambda^{-1} M_{\mathbb{I}} - S_\Lambda^{-1} M_{\mathbb{I}^c} = (S_\Lambda^{-1} M_{\mathbb{I}})^2 - (S_\Lambda^{-1} M_{\mathbb{I}^c})^2.$$

Thus, for each $f, g \in H$ we obtain

$$\langle S_\Lambda^{-1} M_{\mathbb{I}} f, g\rangle - \langle S_\Lambda^{-1} M_{\mathbb{I}} S_\Lambda^{-1} M_{\mathbb{I}} f, g\rangle = \langle S_\Lambda^{-1} M_{\mathbb{I}^c} f, g\rangle - \langle S_\Lambda^{-1} M_{\mathbb{I}^c} S_\Lambda^{-1} M_{\mathbb{I}^c} f, g\rangle.$$

We choose g to be $g = S_\Lambda f$, and we can get

$$\langle M_{\mathbb{I}} f, f\rangle - \langle S_\Lambda^{-1} M_{\mathbb{I}} f, M_{\mathbb{I}} f\rangle = \langle M_{\mathbb{I}^c} f, f\rangle - \langle S_\Lambda^{-1} M_{\mathbb{I}^c} f, M_{\mathbb{I}^c} f\rangle.$$

Finally, by (6), the proof is complete. □

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Received May 27, 2019.
In revised form, November 29, 2019.
Accepted January 11, 2020.
Published online April 29, 2020.

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