DOI: 10.15393/j3.art.2019.6350

UDC 517.38, 517.51

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FURTHER RESULTS ON JENSEN-TYPE INEQUALITIES

Abstract. In this paper, we establish some Jensen-type inequalities for continuous functions of self-adjoint operators on complex Hilbert spaces. Furthermore, using the Cartesian decomposition of an operator, we improve the known result due to Mond and Pečarić. Some refinements of the Hölder-McCarthy inequality are given as well.

Key words: *Jensen's inequality, convex function, synchronous (asynchronous) function, self-adjoint operator*

2010 Mathematical Subject Classification: *Primary 47A63, Secondary 26A51, 26D15, 26B25, 39B62*

1. Introduction and Preliminaries. Let $\mathcal{B}(\mathcal{H})$ be the C^*- algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. As customary, we reserve m, M for scalars and $\mathbf{1}_{\mathcal{H}}$ for the identity operator on $\mathcal{H}$. A self-adjoint operator A is said to be positive (written $A \geqslant 0$) if $\langle Ax, x\rangle \geqslant 0$ holds for all $x \in \mathcal{H}$; also an operator A is said to be strictly positive (denoted by $A > 0$) if A is positive and invertible. If A and B are self-adjoint, we write $B \geqslant A$ in case $B - A \geqslant 0$. The Gelfand map $f(t) \mapsto f(A)$ is an isometrical $*$ - isomorphism between the C^*- algebra $C(sp(A))$ of continuous functions on the spectrum $sp(A)$ of a self-adjoint operator A and the C^*- algebra generated by A and the identity operator $\mathbf{1}_{\mathcal{H}}$. If $f, g \in C(sp(A))$, then $f(t) \geqslant g(t)$ $(t \in sp(A))$ implies that $f(A) \geqslant g(A)$.

For $A, B \in \mathcal{B}(\mathcal{H})$, $A \oplus B$ is the operator defined on $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ by $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. A linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is positive if $\Phi(A) \geqslant 0$ whenever $A \geqslant 0$. It is said to be unital if $\Phi(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{K}}$. A continuous function f defined on the interval J is called an operator convex function if $f((1-v)A + vB) \leqslant (1-v) f(A) + v f(B)$ for every $0 < v < 1$ and for every pair of bounded self-adjoint operators A and B whose spectra are both in J. For instance, the function $f(t) = t^p$ is operator convex on $(0,\infty)$ if either $1 \leqslant p \leqslant 2$ or $-1 \leqslant p \leqslant 0$.

The well known operator Jensen inequality (sometimes called the Choi–Davis–Jensen inequality) states:

$$f(\Phi(A)) \leqslant \Phi(f(A)). \tag{1}$$

It holds for every operator convex $f : J \to \mathbb{R}$, self-adjoint operator A with the spectra in J, and unital positive linear map Φ [2,3].

Hansen et al. [7] gave a general formulation of (1). The discrete version of their result reads as follows: If $f : J \to \mathbb{R}$ is an operator convex function, $A_1, \ldots, A_n \in \mathcal{B}(\mathcal{H})$ are self-adjoint operators with the spectra in J, and $\Phi_1, \ldots, \Phi_n : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ are positive linear mappings such that $\sum\limits_{i=1}^{n} \Phi_i(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{K}}$, then

$$f\Big(\sum_{i=1}^{n} \Phi_i(A_i)\Big) \leqslant \sum_{i=1}^{n} \Phi_i(f(A_i)). \tag{2}$$

Though, in the case of convex function inequality (2) does not hold in general (see [2, Remark 2.6]); we have the following estimate [4, Theorem 1]:

$$f\Big(\Big\langle \sum_{i=1}^{n} \Phi_i(A_i)x, x\Big\rangle\Big) \leqslant \Big\langle \sum_{i=1}^{n} \Phi_i(f(A_i))x, x\Big\rangle \tag{3}$$

for any unit vector $x \in \mathcal{K}$. For recent results on the Jensen operator inequality, we refer the reader to [8–10].

Let $f : J \to \mathbb{R}$ be a convex function, $A \in \mathcal{B}(\mathcal{H})$ self-adjoint operator with spectrum in J, and let $x \in \mathcal{H}$ be a unit vector. Then, from [11],

$$f(\langle Ax, x\rangle) \leqslant \langle f(A)x, x\rangle. \tag{4}$$

The Hölder-McCarthy inequality is a special case of (4):

$$\langle Ax, x\rangle^p \leqslant \langle A^p x, x\rangle, \ (x \in \mathcal{H};\ \|x\| = 1), \tag{5}$$

where A is a positive operator and $p > 1$. If the operator is positive and invertible, (5) is also true for $p < 0$ (see, e.g., [12,13]).

Replace A with $\Phi(A)$, where $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is a unital positive linear map, we get

$$f(\langle \Phi(A)x, x\rangle) \leqslant \langle f(\Phi(A))x, x\rangle \tag{6}$$

for any unit vector $x \in \mathcal{K}$. Assume that $A_1, \dots, A_n$ are self-adjoint operators on $\mathcal{H}$ with the spectra in J and $\Phi_1, \dots, \Phi_n : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ are positive linear maps with $\sum\limits_{i=1}^{n} \Phi_i(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{K}}$. Now apply inequality (6) to the self-adjoint operator A on the Hilbert space $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$ defined by $A = A_1 \oplus \cdots \oplus A_n$ and the positive linear map Φ defined on $\mathcal{B}(\mathcal{H} \oplus \cdots \oplus \mathcal{H})$ by $\Phi(A) = \Phi_1(A_1) \oplus \cdots \oplus \Phi_n(A_n)$. Thus,

$$f\left(\left\langle \sum_{i=1}^{n} \Phi_i(A_i)x, x\right\rangle\right) \leqslant \left\langle f\left(\sum_{i=1}^{n} \Phi_i(A_i)\right)x, x\right\rangle. \tag{7}$$

This research paper is mainly focused on the inequalities of types (3) and (7), where two functions f and g are involved. This approach nicely extends the previously known results in the literature. The new inequalities are applied to obtain refinements of Hölder-McCarthy inequality (5). Additionally, we improve inequality (4) using the Cartesian decomposition of the operator.

2. Inequalities Related to Synchronous (Asynchronous) Functions. We say that functions $f, g : J \to \mathbb{R}$ are synchronous (asynchronous) on the interval J if they satisfy the following condition:

$$(f(t) - f(s))(g(t) - g(s)) \geqslant (\leqslant)\, 0 \tag{8}$$

for each $t, s \in J$ [1, 6]. For several recent results concerning synchronous (asynchronous) functions, see [5, 14].

The first result reads as follows.

Theorem 1. *Let $f, g : J \to \mathbb{R}$ be continuous and synchronous functions, $A_1, \dots, A_n \in \mathcal{B}(\mathcal{H})$ self-adjoint operators with the spectra in J, and let $\Phi_1, \dots, \Phi_n : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be positive linear mappings such that $\sum\limits_{i=1}^{n} \Phi_i(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{K}}$. Then for any unit vector $x \in \mathcal{K}$,*

$$\left\langle \sum_{i=1}^{n} \Phi_i(f(A_i)\, g(A_i))x, x\right\rangle + f\left(\left\langle \sum_{i=1}^{n} \Phi_i(A_i)x, x\right\rangle\right) g\left(\left\langle \sum_{i=1}^{n} \Phi_i(A_i)x, x\right\rangle\right) \geqslant$$

$$\geqslant g\left(\left\langle \sum_{i=1}^{n} \Phi_i(A_i)x, x\right\rangle\right)\left\langle \sum_{i=1}^{n} \Phi_i(f(A_i))x, x\right\rangle +$$

$$+ f\left(\left\langle \sum_{i=1}^{n} \Phi_i(A_i)x, x\right\rangle\right)\left\langle \sum_{i=1}^{n} \Phi_i(g(A_i))x, x\right\rangle.$$

The reverse inequality holds when f, g are asynchronous functions.

Proof. We consider only the case of synchronous functions. It follows from (8) that

$$f(t)\,g(t)+f(s)\,g(s)\geqslant f(t)\,g(s)+f(s)\,g(t) \tag{9}$$

for each $t,s\in J$.

Fix $s\in J$. Since J contains the spectra of the A_i for $i=1,\ldots,n$, we may replace t in the inequality (9) by A_i, via functional calculus to get

$$f(A_i)\,g(A_i)+f(s)\,g(s)\,\mathbf{1}_{\mathcal{H}}\geqslant g(s)\,f(A_i)+f(s)\,g(A_i)\,.$$

Applying the positive linear mappings Φ_i and summing on i from 1 to n, this implies

$$\begin{aligned}&\sum_{i=1}^{n}\Phi_i\left(f(A_i)\,g(A_i)\right)+f(s)\,g(s)\,\mathbf{1}_{\mathcal{K}}\geqslant\\&\geqslant g(s)\sum_{i=1}^{n}\Phi_i\left(f(A_i)\right)+f(s)\sum_{i=1}^{n}\Phi_i\left(g(A_i)\right).\end{aligned} \tag{10}$$

The inequality (10) easily implies, for any $x\in\mathcal{K}$ with $\|x\|=1$,

$$\begin{aligned}&\left\langle\sum_{i=1}^{n}\Phi_i\left(f(A_i)\,g(A_i)\right)x,x\right\rangle+f(s)\,g(s)\geqslant\\&\geqslant g(s)\left\langle\sum_{i=1}^{n}\Phi_i\left(f(A_i)\right)x,x\right\rangle+f(s)\left\langle\sum_{i=1}^{n}\Phi_i\left(g(A_i)\right)x,x\right\rangle.\end{aligned} \tag{11}$$

On the other hand, since $\sum\limits_{i=1}^{n}\Phi_i\left(\mathbf{1}_{\mathcal{H}}\right)=\mathbf{1}_{\mathcal{K}}$ and the spectra of operators A_i, $i=1,\ldots,n$, are contained in the interval J, we have $\left\langle\sum\limits_{i=1}^{n}\Phi_i\left(A_i\right)x,x\right\rangle\in J$, where $x\in\mathcal{K}$ with $\|x\|=1$. So, we may replace s by $\left\langle\sum\limits_{i=1}^{n}\Phi_i\left(A_i\right)x,x\right\rangle$ in (11). This yields

$$\left\langle\sum_{i=1}^{n}\Phi_i\left(f(A_i)\,g(A_i)\right)x,x\right\rangle+f\left(\left\langle\sum_{i=1}^{n}\Phi_i\left(A_i\right)x,x\right\rangle\right)g\left(\left\langle\sum_{i=1}^{n}\Phi_i\left(A_i\right)x,x\right\rangle\right)\geqslant$$

$$\geqslant g\Big(\Big\langle \sum_{i=1}^{n} \Phi_i \left(A_i\right)x, x\Big\rangle\Big)\Big\langle \sum_{i=1}^{n} \Phi_i \left(f \left(A_i\right)\right)x, x\Big\rangle +$$
$$+f\Big(\Big\langle \sum_{i=1}^{n} \Phi_i \left(A_i\right)x, x\Big\rangle\Big)\Big\langle \sum_{i=1}^{n} \Phi_i \left(g \left(A_i\right)\right)x, x\Big\rangle.$$

This completes the proof. □

Remark. *Let $A_1, \dots, A_n \in \mathcal{B}(\mathcal{H})$ be positive operators, and let $\Phi_1, \dots, \Phi_n : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be positive linear mappings such that $\sum_{i=1}^{n} \Phi_i \left(\mathbf{1}_{\mathcal{H}}\right) = \mathbf{1}_{\mathcal{K}}$. Then, for any $p, q > 0$,*

$$\Big\langle \sum_{i=1}^{n} \Phi_i \left(A_i^{p+q}\right)x, x\Big\rangle + \Big\langle \sum_{i=1}^{n} \Phi_i \left(A_i\right)x, x\Big\rangle^{p+q} \geqslant$$
$$\geqslant \Big\langle \sum_{i=1}^{n} \Phi_i \left(A_i\right)x, x\Big\rangle^{q} \Big\langle \sum_{i=1}^{n} \Phi_i \left(A_i^{p}\right)x, x\Big\rangle + \tag{12}$$
$$+ \Big\langle \sum_{i=1}^{n} \Phi_i \left(A_i\right)x, x\Big\rangle^{p} \Big\langle \sum_{i=1}^{n} \Phi_i \left(A_i^{q}\right)x, x\Big\rangle$$

for any unit vector $x \in \mathcal{K}$. If $A_1, \dots, A_n \in \mathcal{B}(\mathcal{H})$ are positive invertible operators, then (12) also holds for $p, q < 0$.

If $A_1 \dots, A_n \in \mathcal{B}(\mathcal{H})$ are positive invertible and either $p > 0$, $q < 0$ or $p < 0$, $q > 0$, then the reverse inequality holds in (12).

Corollary. *Let $f, g : J \to \mathbb{R}$ be synchronous functions, $A_1, \dots, A_n \in \mathcal{B}(\mathcal{H})$ self-adjoint operators with the spectra in J, and let $w_1, \dots, w_n$ be positive scalars such that $\sum_{i=1}^{n} w_i = 1$. Then, for any unit vector $x \in \mathcal{H}$,*

$$\Big\langle \sum_{i=1}^{n} w_i f \left(A_i\right) g \left(A_i\right)x, x\Big\rangle + f\Big(\Big\langle \sum_{i=1}^{n} w_i A_i x, x\Big\rangle\Big) g\Big(\Big\langle \sum_{i=1}^{n} w_i A_i x, x\Big\rangle\Big) \geqslant$$
$$\geqslant g\Big(\Big\langle \sum_{i=1}^{n} w_i A_i x, x\Big\rangle\Big)\Big\langle \sum_{i=1}^{n} w_i f \left(A_i\right)x, x\Big\rangle +$$
$$+f\Big(\Big\langle \sum_{i=1}^{n} w_i A_i x, x\Big\rangle\Big)\Big\langle \sum_{i=1}^{n} w_i g \left(A_i\right)x, x\Big\rangle.$$

The reverse inequality holds when f, g are asynchronous functions.

Proof. Apply Theorem 1 for positive linear mappings $\Phi_i : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ determined by $\Phi_i : T \mapsto w_i T\ (i = 1, \ldots, n)$. □

Remark. *Suppose, in addition to the assumptions in Theorem 1, that f is convex on J; then*

$$g\Big(\Big\langle \sum_{i=1}^n \Phi_i(A_i)x, x\Big\rangle\Big)\Big\langle \sum_{i=1}^n \Phi_i(f(A_i))x, x\Big\rangle +$$
$$+ f\Big(\Big\langle \sum_{i=1}^n \Phi_i(A_i)x, x\Big\rangle\Big)\Big\langle \sum_{i=1}^n \Phi_i(g(A_i))x, x\Big\rangle \leqslant$$
$$\leqslant \Big\langle \sum_{i=1}^n \Phi_i(f(A_i)g(A_i))x, x\Big\rangle + f\Big(\Big\langle \sum_{i=1}^n \Phi_i(A_i)x, x\Big\rangle\Big) g\Big(\Big\langle \sum_{i=1}^n \Phi_i(A_i)x, x\Big\rangle\Big) \leqslant$$
$$\leqslant \Big\langle \sum_{i=1}^n \Phi_i(f(A_i)g(A_i))x, x\Big\rangle + \Big\langle \sum_{i=1}^n \Phi_i(f(A_i))x, x\Big\rangle g\Big(\Big\langle \sum_{i=1}^n \Phi_i(A_i)x, x\Big\rangle\Big)$$

for any unit vector $x \in \mathcal{K}$, due to (3). Therefore,

$$f\Big(\Big\langle \sum_{i=1}^n \Phi_i(A_i)x, x\Big\rangle\Big)\Big\langle \sum_{i=1}^n \Phi_i(g(A_i))x, x\Big\rangle \leqslant$$
$$\leqslant \Big\langle \sum_{i=1}^n \Phi_i(f(A_i)g(A_i))x, x\Big\rangle +$$
$$+ g\Big(\Big\langle \sum_{i=1}^n \Phi_i(A_i)x, x\Big\rangle\Big)\Big[f\Big(\Big\langle \sum_{i=1}^n \Phi_i(A_i)x, x\Big\rangle\Big) - \Big\langle \sum_{i=1}^n \Phi_i(f(A_i))x, x\Big\rangle\Big] \leqslant$$
$$\leqslant \Big\langle \sum_{i=1}^n \Phi_i(f(A_i)g(A_i))x, x\Big\rangle.$$

If, in addition, g is convex on J, then

$$f\Big(\Big\langle \sum_{i=1}^n \Phi_i(A_i)x, x\Big\rangle\Big) g\Big(\Big\langle \sum_{i=1}^n \Phi_i(A_i)x, x\Big\rangle\Big) \leqslant$$
$$\leqslant f\Big(\Big\langle \sum_{i=1}^n \Phi_i(A_i)x, x\Big\rangle\Big)\Big\langle \sum_{i=1}^n \Phi_i(g(A_i))x, x\Big\rangle \leqslant \Big\langle \sum_{i=1}^n \Phi_i(f(A_i)g(A_i))x, x\Big\rangle +$$
$$+ g\Big(\Big\langle \sum_{i=1}^n \Phi_i(A_i)x, x\Big\rangle\Big)\Big[f\Big(\Big\langle \sum_{i=1}^n \Phi_i(A_i)x, x\Big\rangle\Big) - \Big\langle \sum_{i=1}^n \Phi_i(f(A_i))x, x\Big\rangle\Big] \leqslant$$

$$\leqslant \left\langle \sum_{i=1}^{n} \Phi_i \left(f\left(A_i\right) g\left(A_i\right)\right)x, x \right\rangle \quad (13)$$

holds for any unit vector $x \in \mathcal{K}$.

As a direct consequence of (13), we have:

Corollary. *Let $A_1, \ldots, A_n \in \mathcal{B}(\mathcal{H})$ be positive operators, and let $\Phi_1, \ldots, \Phi_n : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be positive linear mappings, such that $\sum\limits_{i=1}^{n} \Phi_i(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{K}}$. Then, for any $p, q > 1$,*

$$\begin{aligned} \left\langle \sum_{i=1}^{n} \Phi_i(A_i)x, x \right\rangle^{p+q} &\leqslant \left\langle \sum_{i=1}^{n} \Phi_i(A_i)x, x \right\rangle^{p} \left\langle \sum_{i=1}^{n} \Phi_i(A_i^q)x, x \right\rangle \leqslant \\ &\leqslant \left\langle \sum_{i=1}^{n} \Phi_i\left(A_i^{p+q}\right)x, x \right\rangle + \\ + \left\langle \sum_{i=1}^{n} \Phi_i(A_i)x, x \right\rangle^{q} &\left[\left\langle \sum_{i=1}^{n} \Phi_i(A_i)x, x \right\rangle^{p} - \left\langle \sum_{i=1}^{n} \Phi_i(A_i^p)x, x \right\rangle \right] \leqslant \\ &\leqslant \left\langle \sum_{i=1}^{n} \Phi_i\left(A_i^{p+q}\right)x, x \right\rangle \quad (14) \end{aligned}$$

for any unit vector $x \in \mathcal{K}$.

If $A_1, \ldots, A_n \in \mathcal{B}(\mathcal{H})$ are positive invertible operators, then (14) also holds for $p, q < 0$.

The second main result reads as follows:

Theorem 2. *Let $f, g : J \to \mathbb{R}$ be synchronous functions, $A_1, \ldots, A_n \in \mathcal{B}(\mathcal{H})$ self-adjoint operators with the spectra in J, and let $\Phi_1, \ldots, \Phi_n : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be positive linear mappings such that $\sum\limits_{i=1}^{n} \Phi_i(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{K}}$. Then, for any unit vector $x \in \mathcal{K}$,*

$$\begin{aligned} &\left\langle f\left(\sum_{i=1}^{n} \Phi_i(A_i)\right) g\left(\sum_{i=1}^{n} \Phi_i(A_i)\right)x, x \right\rangle + \\ &\quad + f\left(\left\langle \sum_{i=1}^{n} \Phi_i(A_i)x, x \right\rangle\right) g\left(\left\langle \sum_{i=1}^{n} \Phi_i(A_i)x, x \right\rangle\right) \geqslant \end{aligned}$$

$$\geqslant g\Big(\Big\langle \sum_{i=1}^{n} \Phi_i\,(A_i)x, x\Big\rangle\Big)\Big\langle f\Big(\sum_{i=1}^{n} \Phi_i\,(A_i)\Big)x, x\Big\rangle + \\ + f\Big(\Big\langle \sum_{i=1}^{n} \Phi_i\,(A_i)x, x\Big\rangle\Big)\Big\langle g\Big(\sum_{i=1}^{n} \Phi_i\,(A_i)\Big)x, x\Big\rangle.$$

The reverse inequality holds when f, g are asynchronous functions.

Proof. Fix $s \in J$. Since J contains the spectra of the A_i for $i = 1, \ldots, n$ and $\sum_{i=1}^{n} \Phi_i\,(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{K}}$, the spectra of $\sum_{i=1}^{n} \Phi_i\,(A_i)$ are also contained in J. Then, we may replace t in the inequality (9) by $\sum_{i=1}^{n} \Phi_i\,(A_i)$, via functional calculus to get

$$f\Big(\sum_{i=1}^{n} \Phi_i\,(A_i)\Big)g\Big(\sum_{i=1}^{n} \Phi_i\,(A_i)\Big) + f(s)g(s)\mathbf{1}_{\mathcal{K}} \geqslant \\ \geqslant g(s)f\Big(\sum_{i=1}^{n} \Phi_i(A_i)\Big) + f(s)g\Big(\sum_{i=1}^{n} \Phi_i\,(A_i)\Big).$$

This inequality implies, for any $x \in \mathcal{K}$ with $\|x\| = 1$,

$$\Big\langle f\Big(\sum_{i=1}^{n} \Phi_i\,(A_i)\Big)g\Big(\sum_{i=1}^{n} \Phi_i\,(A_i)\Big)x, x\Big\rangle + f(s)g(s) \geqslant \\ \geqslant g\,(s)\,\Big\langle f\Big(\sum_{i=1}^{n} \Phi_i\,(A_i)\Big)x, x\Big\rangle + f\,(s)\,\Big\langle g\Big(\sum_{i=1}^{n} \Phi_i\,(A_i)\Big)x, x\Big\rangle. \tag{15}$$

Substituting s with $\Big\langle \sum_{i=1}^{n} \Phi_i\,(A_i)x, x\Big\rangle$ in (15), gives the desired result. □

Remark. *Suppose, in addition to the assumptions in Theorem 2, that f is convex on J; then*

$$f\Big(\Big\langle \sum_{i=1}^{n} \Phi_i\,(A_i)x, x\Big\rangle\Big)\Big\langle g\Big(\sum_{i=1}^{n} \Phi_i\,(A_i)\Big)x, x\Big\rangle \leqslant \\ \leqslant \Big\langle f\Big(\sum_{i=1}^{n} \Phi_i\,(A_i)\Big)g\Big(\sum_{i=1}^{n} \Phi_i\,(A_i)\Big)x, x\Big\rangle + g\Big(\Big\langle \sum_{i=1}^{n} \Phi_i\,(A_i)x, x\Big\rangle\Big)\times \\ \times \Big[f\Big(\Big\langle \sum_{i=1}^{n} \Phi_i\,(A_i)x, x\Big\rangle\Big) - \Big\langle f\Big(\sum_{i=1}^{n} \Phi_i\,(A_i)\Big)x, x\Big\rangle\Big] \leqslant$$

$$\leqslant \left\langle f\left(\sum_{i=1}^{n}\Phi_i\left(A_i\right)\right)g\left(\sum_{i=1}^{n}\Phi_i\left(A_i\right)\right)x,x\right\rangle$$

for any unit vector $x\in\mathcal{K}$, due to (7). If, in addition, g is convex on J,

$$\begin{aligned}
f\left(\left\langle \sum_{i=1}^{n}\Phi_i\left(A_i\right)x,x\right\rangle\right)g\left(\left\langle \sum_{i=1}^{n}\Phi_i\left(A_i\right)x,x\right\rangle\right)\leqslant&\\
\leqslant f\left(\left\langle \sum_{i=1}^{n}\Phi_i\left(A_i\right)x,x\right\rangle\right)\left\langle g\left(\sum_{i=1}^{n}\Phi_i\left(A_i\right)\right)x,x\right\rangle\leqslant&\\
\leqslant \left\langle f\left(\sum_{i=1}^{n}\Phi_i\left(A_i\right)\right)g\left(\sum_{i=1}^{n}\Phi_i\left(A_i\right)\right)x,x\right\rangle + g\left(\left\langle \sum_{i=1}^{n}\Phi_i\left(A_i\right)x,x\right\rangle\right)\times&\\
\times\left[f\left(\left\langle \sum_{i=1}^{n}\Phi_i\left(A_i\right)x,x\right\rangle\right)-\left\langle f\left(\sum_{i=1}^{n}\Phi_i\left(A_i\right)\right)x,x\right\rangle\right]\leqslant&\\
\leqslant \left\langle f\left(\sum_{i=1}^{n}\Phi_i\left(A_i\right)\right)g\left(\sum_{i=1}^{n}\Phi_i\left(A_i\right)\right)x,x\right\rangle&
\end{aligned}$$

holds for any unit vector $x\in\mathcal{K}$.

3. Refinement via the Cartesian Decomposition. We start this section by proving the following theorem, that can be considered as a refinement of [4, Theorem 1].

Theorem 3. *Let $\Phi:\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{K})$ be a unital positive linear map, $A\in\mathcal{B}(\mathcal{H})$ be a positive operator with the Cartesian decomposition $A=B+iC$, and let f be a non-negative function on $[0,\infty)$, such that $g(t)=f(\sqrt{t})$ is convex. Then, for any unit vector $x\in\mathcal{K}$,*

$$f\left(\left\langle\Phi\left(A\right)x,x\right\rangle\right)\leqslant\begin{cases} f\left(\left(\left\langle\Phi(B)^2x,x\right\rangle+\left\langle\Phi\left(C\right)x,x\right\rangle^2\right)^{\frac{1}{2}}\right)\\ f\left(\left(\left\langle\Phi\left(B\right)x,x\right\rangle^2+\left\langle\Phi(C)^2x,x\right\rangle\right)^{\frac{1}{2}}\right)\end{cases}\leqslant\left\langle\Phi\left(f(A)\right)x,x\right\rangle.$$

Proof. First of all, note that

$$\left\langle\Phi\left(A\right)x,x\right\rangle^2=\left\langle\Phi\left(B\right)x,x\right\rangle^2+\left\langle\Phi\left(C\right)x,x\right\rangle^2,\ \left(x\in\mathcal{K};\ \|x\|=1\right) \quad (16)$$

and

$$\Phi(A)^2=\Phi(B)^2+\Phi(C)^2. \quad (17)$$

Since g is non-negative and convex on $[0,\infty)$, it follows that g is increasing. Now,

$$\begin{aligned}
&g\left(\langle\Phi\left(A\right)x,x\rangle^2\right)=g\left(\langle\Phi\left(B\right)x,x\rangle^2+\langle\Phi\left(C\right)x,x\rangle^2\right)\leqslant(\text{by }(16))\\
&\leqslant g\left(\left\langle\Phi(B)^2x,x\right\rangle+\langle\Phi\left(C\right)x,x\rangle^2\right)\leqslant(\text{by the Cauchy-Schwarz inequality})\\
&\leqslant g\left(\left\langle\Phi(B)^2+\Phi(C)^2x,x\right\rangle\right)=(\text{by }(5))\\
&=g\left(\left\langle\Phi(A)^2x,x\right\rangle\right)\leqslant(\text{by }(17))\\
&\leqslant g\left(\left\langle\Phi\left(A^2\right)x,x\right\rangle\right)\leqslant(\text{since }t^2\text{ is operator convex})\\
&\leqslant\left\langle\Phi\left(g\left(A^2\right)\right)x,x\right\rangle\quad(\text{by }(3))
\end{aligned}$$

for any unit vector $x\in\mathcal{K}$. Consequently,

$$\begin{aligned}
f\left(\langle\Phi\left(A\right)x,x\rangle\right)&=g\left(\langle\Phi\left(A\right)x,x\rangle^2\right)\leqslant\\
&\leqslant g\left(\left\langle\Phi(B)^2x,x\right\rangle+\langle\Phi\left(C\right)x,x\rangle^2\right)=\\
&=f\left(\left(\left\langle\Phi(B)^2x,x\right\rangle+\langle\Phi\left(C\right)x,x\rangle^2\right)^{\frac{1}{2}}\right)\leqslant\\
&\leqslant\left\langle\Phi\left(g\left(A^2\right)\right)x,x\right\rangle=\\
&=\langle\Phi\left(f\left(A\right)\right)x,x\rangle\,.
\end{aligned}$$

The other case can be obtained similarly. □

By setting $f(t)=t^p\ (t\geqslant 0, p\geqslant 2)$ in Theorem 3, we find that:

Corollary. *Let $\Phi:\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{K})$ be a unital positive linear map, $A\in\mathcal{B}(\mathcal{H})$ be a positive operator with the Cartesian decomposition $A=B+iC$. Then, for any $p\geqslant 2$,*

$$\langle\Phi\left(A\right)x,x\rangle^p\leqslant\begin{cases}\left(\left\langle\Phi(B)^2x,x\right\rangle+\langle\Phi\left(C\right)x,x\rangle^2\right)^{\frac{p}{2}}\\ \left(\langle\Phi\left(B\right)x,x\rangle^2+\left\langle\Phi(C)^2x,x\right\rangle\right)^{\frac{p}{2}}\end{cases}\leqslant\langle\Phi\left(A^p\right)x,x\rangle$$

for any unit vector $x\in\mathcal{K}$.

Taking $\Phi(T)=T$ in Theorem 3, we get:

Corollary. *Let $A\in\mathcal{B}(\mathcal{H})$ be a positive operator with the Cartesian decomposition $A=B+iC$, and let f be a non-negative function on*

$[0,\infty)$, such that $g(t) = f\left(\sqrt{t}\right)$ is convex. Then, for any unit vector $x \in \mathcal{H}$,

$$f\left(\langle Ax, x\rangle\right) \leqslant \begin{cases} f\left(\left(\left\langle B^2x, x\right\rangle + \langle Cx, x\rangle^2\right)^{\frac{1}{2}}\right) \\ f\left(\left(\langle Bx, x\rangle^2 + \left\langle C^2x, x\right\rangle\right)^{\frac{1}{2}}\right) \end{cases} \leqslant \langle f(A)x, x\rangle .$$

In particular, for any $p \geqslant 2$,

$$\langle Ax, x\rangle^p \leqslant \begin{cases} \left(\left\langle B^2x, x\right\rangle + \langle Cx, x\rangle^2\right)^{\frac{p}{2}} \\ \left(\langle Bx, x\rangle^2 + \left\langle C^2x, x\right\rangle\right)^{\frac{p}{2}} \end{cases} \leqslant \langle A^px, x\rangle$$

holds for any unit vector $x \in \mathcal{H}$.

Acknowledgment. The authors would like to express their hearty thanks to the referees for their valuable comments.

This work was financially supported by Safadasht Branch, Islamic Azad University, Tehran, Iran.

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Received May 28, 2019.
In revised form, July 18, 2019.
Accepted July 21, 2019.
Published online August 14, 2019.

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