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ON THE CONVERGENCE OF THE LEAST SQUARE METHOD IN CASE OF NON-UNIFORM GRIDS

Abstract. Let $f(t)$ be a continuous on $[-1,1]$ function, which values are given at the points of arbitrary non-uniform grid $\Omega_N = \{t_j\}_{j=0}^{N-1}$, where nodes t_j satisfy the only condition $\eta_j \leqslant t_j \leqslant \eta_{j+1}$, $0 \leqslant j \leqslant N-1$, and nodes η_j are such that $-1 = \eta_0 < \eta_1 < \eta_2 < \cdots < \eta_{N-1} < \eta_N = 1$. We investigate approximative properties of the finite Fourier series for $f(t)$ by algebraic polynomials $\hat{P}_{n,N}(t)$, that are orthogonal on $\Omega_N = \{t_j\}_{j=0}^{N-1}$. Lebesgue-type inequalities for the partial Fourier sums by $\hat{P}_{n,N}(t)$ are obtained.

Key words: *random net, non-uniform grid, orthogonal polynomials, Legendre polynomials, least square method, Fourier series, function approximation*

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1. Introduction.

Let $\{\eta_j\}_{j=0}^{N}$ be a system of points, such that

$$-1 = \eta_0 < \eta_1 < \eta_2 < \cdots < \eta_{N-1} < \eta_N = 1. \tag{1}$$

We assume $\Delta\eta_j = \eta_{j+1} - \eta_j$, $0 \leqslant j \leqslant N-1$, $\lambda_N = \max\limits_{0\leqslant j\leqslant N-1} \Delta\eta_j$. Now, we construct a grid Ω_N from the points

$$\eta_j \leqslant t_j \leqslant \eta_{j+1}, \quad j = 0, 1, \ldots, N-1, \tag{2}$$

selected on each segment $[\eta_j, \eta_{j+1}]$. Without loss of generality, we can consider all the nodes $\{t_j\}_{j=0}^{N-1}$ distinct, because if $t_j = t_{j+1}$ for some j, we can leave only one of them and denote the grid by Ω_{N-1}.

Consider the space $l_2(\Omega_N)$ of discrete functions $f : \Omega_N \to R$, where the inner product is given by

$$\langle f, g\rangle = \sum_{j=0}^{N-1} f(t_j)g(t_j)\Delta\eta_j = \lambda_N \sum_{j=0}^{N-1} f(t_j)g(t_j)\rho_j. \tag{3}$$

By $\hat{P}_{n,N}(t)$, $0 \leqslant n \leqslant N-1$, we denote polynomials that form finite orthonormal system with respect to this inner product:

$$\left\langle \hat{P}_{n,N}, \hat{P}_{m,N} \right\rangle = \sum_{j=0}^{N-1} \hat{P}_{n,N}(t_j)\hat{P}_{m,N}(t_j)\Delta\eta_j = \begin{cases} 0, & n \neq m, \\ 1, & n = m. \end{cases} \tag{4}$$

We call polynomials $\hat{P}_{n,N}(t)$, $0 \leqslant n \leqslant N-1$, *the discrete orthonormal Legendre polynomials.*

Since the system $\left\{\hat{P}_{n,N}(t)\right\}_{n=0}^{N-1}$ is complete in $l_2(\Omega_N)$, any function $f \in l_2(\Omega_N)$ can be expanded in a finite Fourier series by this system. Let $\Lambda_{n,N}(f,t)$ be the partial Fourier sum of order n for the function $f = f(t)$ by the system $\left\{\hat{P}_{k,N}\right\}_{k=0}^{N-1}$, in other words

$$\Lambda_{n,N}(f,t) = \sum_{k=0}^{n} \hat{f}_k \hat{P}_{k,N}(t), \quad \text{where} \quad \hat{f}_k = \sum_{j=0}^{N-1} f(t_j)\hat{P}_{k,N}(t_j)\Delta\eta_j.$$

The main goal of this article is to study the approximative properties of $\Lambda_{n,N}(f,t)$ in case when $f(t)$ is continuous on $[-1,1]$ and $t \in [-1,1]$. More precisely, we want to obtain an estimate for the value

$$|R_{n,N}(f,t)| = |f(t) - \Lambda_{n,N}(f,t)|, \quad t \in [-1,1]. \tag{5}$$

Note that the value $|R_{n,N}(f,t)|$ for the discrete Legendre polynomials was studied in [2] for the case of $t_j = \eta_j$ and was studied in [3] for the case of $t_j = \frac{\eta_j+\eta_{j+1}}{2}$. But the results obtained there are valid only when $n = O(\lambda_N^{-1/5})$ and $n = O(\lambda_N^{-2/7})$, respectively, while we managed to get estimates for $n = O(\lambda_N^{-1/3})$ and for a more general case when t_j is arbitrary on the segment $[\eta_j, \eta_{j+1}]$.

To solve this problem, we need some information about discrete Legendre polynomials $\hat{P}_{k,N}(t)$, as well as discrete Jacobi polynomials $\hat{P}_{k,N}^{\alpha,\beta}(t)$, which are a generalization of $\hat{P}_{k,N}(t)$. This information is based on the properties of classical continuous Legendre and Jacobi polynomials.

2. Some information about Jacobi and Legendre polynomials. The Jacobi polynomials can be written using Rodrigues' formula (see, for example, [4]) as follows:

$$P_n^{\alpha,\beta}(t) = \frac{(-1)^n}{2^n n!} \frac{1}{\kappa^{\alpha,\beta}(t)} \frac{d^n}{dt^n}\{\kappa^{\alpha,\beta}(t)\sigma^n(t)\},$$

where α, β are arbitrary real numbers, $\kappa^{\alpha,\beta}(t) = (1-t)^{\alpha}(1+t)^{\beta}$, $\sigma(t) = 1-t^2$. In the case when $\alpha, \beta > -1$, the Jacobi polynomials form an orthogonal system with the weight $\kappa^{\alpha,\beta}(t)$:

$$\int_{-1}^{1} P_n^{\alpha,\beta}(t) P_m^{\alpha,\beta}(t) \kappa^{\alpha,\beta}(t) dt = h_n^{\alpha,\beta} \delta_{nm},$$

where

$$h_n^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)},$$

and, therefore, $h_n^{\alpha,\beta} \asymp n^{-1}$, $n = 1, 2, \ldots$. For the derivative of $P_n^{\alpha,\beta}(t)$, the following equality holds:

$$\left(P_n^{\alpha,\beta}(t)\right)' = \frac{\alpha+\beta+n+1}{2} P_{n-1}^{\alpha+1,\beta+1}(t). \tag{6}$$

We will also need the following weighted estimate

$$\sqrt{n}\left|P_n^{\alpha,\beta}(t)\right| \leqslant c(\alpha,\beta)\left(\sqrt{1-t}+\frac{1}{n}\right)^{-\alpha-\frac{1}{2}}\left(\sqrt{1+t}+\frac{1}{n}\right)^{-\beta-\frac{1}{2}}, \tag{7}$$

where $-1 \leqslant t \leqslant 1$. An important particular case of Jacobi polynomials with $\alpha = \beta = 0$ is Legendre polynomials $P_n(t)$, orthogonal on $[-1, 1]$ with the unit weight $\rho(t) \equiv 1$. Denote by $\hat{P}_n(t) = \sqrt{\frac{2n+1}{2}} P_n(t)$, $n = 0, 1, 2, \ldots$ the corresponding orthonormal Legendre polynomials. The leading coefficient of polynomial $\hat{P}_n(t)$ can be written as

$$k_n = \frac{(2n)!}{(n!)^2 2^n}\sqrt{\frac{2n+1}{2}}. \tag{8}$$

3. Discrete Jacobi and Legendre polynomials. We will use the integral analogue of the Markov inequality for estimating the derivative of an algebraic polynomial (see [5, 6]), which for $r = 1$ has the following form:

$$\int_{-1}^{1} |q_m'(t)| dt \leqslant c(m) m^2 \int_{-1}^{1} |q_m(t)| dt, \tag{9}$$

where $q_m(t)$ is an arbitrary algebraic polynomial of degree m. For every m, denote by χ_m the minimum of constants $c(m)$ that satisfy inequality (9), i.e.,

$$\chi_m = \sup_{q_m} \frac{\int\limits_{-1}^{1} |q_m'(t)|dt}{m^2 \int\limits_{-1}^{1} |q_m(t)|dt},$$

where the upper bound is taken by polynomials $q_m(t)$ of degree at most m and not equal to zero identically. In work [5] by N. K. Bari, it is shown that $\chi = \sup\limits_{m\geqslant 1} \chi_m < \infty$. Given this fact, we derive from (9):

$$\int\limits_{-1}^{1} |q_m'(t)|dt \leqslant \chi m^2 \int\limits_{-1}^{1} |q_m(t)|dt. \tag{10}$$

Let $\left\{\hat{P}_{n,N}^{\alpha,\beta}(t)\right\}_{n=0}^{N-1}$ be polynomials that form a finite orthonormal system with respect to the inner product

$$\left\langle \hat{P}_{n,N}^{\alpha,\beta}, \hat{P}_{m,N}^{\alpha,\beta} \right\rangle = \sum_{j=0}^{N-1} \hat{P}_{n,N}^{\alpha,\beta}(t_j)\hat{P}_{m,N}^{\alpha,\beta}(t_j)\kappa^{\alpha,\beta}(t_j)\Delta\eta_j = \begin{cases} 0, & n \neq m, \\ 1, & n = m. \end{cases}$$

We call these polynomials *discrete orthonormal Jacobi polynomials.*

In the case when the grid Ω_N consists of equidistant nodes $t_j = -1 + \frac{2j}{N-1}$, the asymptotic properties and weighted estimates for the polynomials orthogonal on Ω_N were first studied in the papers by I. I. Sharapudinov (see [7]). Later, I. I. Sharapudinov [8–10] and A. A. Nurmagomedov [11], [12] studied the asymptotic properties of polynomials that are orthogonal on nonuniform grids of the real axis. In particular, in [12] the author investigated the asymptotic properties of the discrete Jacobi polynomials $\hat{P}_{n,N}^{\alpha,\beta}(t)$ (α and β are integers), orthogonal on nonuniform grid Ω_N with $t_j = \dfrac{\eta_j + \eta_{j+1}}{2}$, $0 \leqslant j \leqslant N-1$.

In our work [13], we investigated asymptotic properties of these polynomials in the general case of random t_j (α, β are still integers). When $n = O(\lambda_N^{-\frac{1}{3}})$ and $n, N \to \infty$ we obtained asymptotic formula

$$\hat{P}_{n,N}^{\alpha,\beta}(t) = \hat{P}_n^{\alpha,\beta}(t) + v_{n,N}^{\alpha,\beta}(t),$$

here $\hat{P}_n^{\alpha,\beta}(t)$ is a normed Jacobi polynomial, and $\upsilon_{n,N}^{\alpha,\beta}(t)$ is the remainder, for which the following estimate is established:

$$\left|\upsilon_{n,N}^{\alpha,\beta}(\cos\theta)\right| \leqslant$$

$$\leqslant c(\alpha,\beta,\gamma)\left(\frac{3-\lambda_N\chi(2n+\alpha+\beta)^2}{1-\lambda_N^2\chi^2(2n+\alpha+\beta)^4}\right)^{\frac{1}{2}}\begin{cases}\theta^{-\alpha-\frac{1}{2}}n^{\frac{3}{2}}\sqrt{\lambda_N}, & \gamma n^{-1}\leqslant\theta\leqslant\frac{\pi}{2},\\ n^{\alpha+2}\sqrt{\lambda_N}, & 0\leqslant\theta\leqslant\gamma n^{-1},\end{cases}$$

where χ is the smallest of the constants in the Markov integral inequality for estimating the derivative of an algebraic polynomial. Here and further in the text, c, $c(\alpha)$, $c(\alpha,\beta)$, $c(\alpha,\beta,\ldots,\gamma)$ are positive constants depending only on the specified parameters, which, generally speaking, may be different in different places. *For the sake of simplicity, these estimates are given for the segment* $[0,1]$*; they apply to* $[-1,0]$ *in the similar way.*

In the article, the indicated asymptotic formula is directly used to study the value $|R_{n,N}(f,t)|$.

4. Auxiliary statements. In this section, we collect some of the statements that will be needed in the future.

Lemma 1. *Let* $f(t)$ *be a function, absolutely continuous on* $[-1,1]$*;* $\{\eta_j\}_{j=0}^{N}$ *and* $\{t_j\}_{j=0}^{N-1}$ *be systems of nodes that satisfy* (1) *and* (2)*, respectively. Then*

$$\int_a^b f(t)dt = \sum_{a\leqslant t_j\leqslant b} f(t_j)\Delta\eta_j + r_N(f)$$

for every segment $[a,b]\subset[-1,1]$*, where*

$$|r_N(f)|\leqslant\lambda_N\int_a^b|f'(t)|dt.$$

Proof of this lemma can be found in [13].

From Lemma 1 the next statement also follows:

Lemma 2. *Let* $\{\eta_j\}_{j=0}^{N}$ *and* $\{t_j\}_{j=0}^{N-1}$ *be systems of nodes that satisfy* (1) *and* (2)*, respectively. Then the following inequality holds for an absolutely continuous on* $[-1,1]$ *monotonous non-negative function* $f(x)$*:*

$$\sum_{a\leqslant t_j\leqslant b} f(t_j)\Delta\eta_j\leqslant\int_a^b f(t)dt+\lambda_N|f(b)-f(a)|.$$

Lemma 3. *For the leading coefficients of the discrete Legendre polynomials, the inequality*

$$\frac{k_{n,\,N}}{k_{n+1,\,N}} \leqslant 1 \tag{11}$$

holds; here $k_{n,\,N}$ and $k_{n+1,\,N}$ are the leading coefficients of the polynomials $\hat{P}_{n,\,N}$ and $\hat{P}_{n+1,\,N}$, respectively.

Proof. Following [14], let us consider the expression

$$\sum_{j=0}^{N-1} \hat{P}_{n+1,\,N}(t_j) t_j \hat{P}_{n,\,N}(t_j) \Delta\eta_j =$$

$$= \frac{k_{n,\,N}}{k_{n+1,\,N}} \sum_{j=0}^{N-1} \hat{P}^2_{n+1,\,N}(t_j) + \sum_{j=0}^{N-1} \hat{Q}_{n,\,N}(t_j) = \frac{k_{n,\,N}}{k_{n+1,\,N}}.$$

On the other hand,

$$\frac{k_{n,\,N}}{k_{n+1,\,N}} \leqslant \sum_{j=0}^{N-1} \left|\hat{P}_{n+1,\,N}(t_j)\right| \left|t_j \hat{P}_{n,\,N}(t_j)\right| \Delta\eta_j \leqslant$$

$$\leqslant \max_{0 \leqslant j \leqslant N-1} \{|t_j|\} \sum_{j=0}^{N-1} \left|\hat{P}_{n+1,\,N}(t_j)\right| \left|\hat{P}_{n,\,N}(t_j)\right| \Delta\eta_j.$$

Applying the Cauchy–Bunyakovsky inequality, we finally get

$$\frac{k_{n,\,N}}{k_{n+1,\,N}} \leqslant \max_{0 \leqslant j \leqslant N-1} \{|t_j|\} \left(\sum_{j=0}^{N-1} \left|\hat{P}_{n+1,\,N}(t_j)\right| \Delta\eta_j\right)^{\frac{1}{2}} \times$$

$$\times \left(\sum_{j=0}^{N-1} \left|\hat{P}_{n,\,N}(t_j)\right| \Delta\eta_j\right)^{\frac{1}{2}} = \max_{0 \leqslant j \leqslant N-1} \{|t_j|\} \leqslant 1.$$

This completes the proof. □

The following lemma establishes the relation between polynomials of degrees n and $n+1$.

Lemma 4. *For $A_n = \sqrt{\dfrac{k_{n,\,N}}{k_{n+1,\,N}}}$ the following equalities hold:*

$$(1-t)\hat{P}^{1,0}_{n,N}(t) = \\ = A_n\left(\hat{P}_{n,N}(t)\sqrt{\frac{\hat{P}_{n+1,N}(1)}{\hat{P}_{n,N}(1)}} - \hat{P}_{n+1,N}(t)\sqrt{\frac{\hat{P}_{n,N}(1)}{\hat{P}_{n+1,N}(1)}}\right), \quad (12)$$

$$(1+t)\hat{P}^{0,1}_{n,N}(t) = \\ = A_n\left(\hat{P}_{n,N}(t)\sqrt{\frac{-\hat{P}_{n+1,N}(-1)}{\hat{P}_{n,N}(-1)}} - \hat{P}_{n+1,N}(t)\sqrt{\frac{-\hat{P}_{n,N}(-1)}{\hat{P}_{n+1,N}(-1)}}\right). \quad (13)$$

Proof. Consider the polynomial $Q_n(t)$, given by the equality

$$(1-t)Q_n(t) = \hat{P}_{n+1,N}(1)\hat{P}_{n,N}(t) - \hat{P}_{n,N}(1)\hat{P}_{n+1,N}(t). \quad (14)$$

From its definition, we have

$$\sum_{j=0}^{N-1} Q_n(t_j)\hat{P}_{k,N}(t_j)(1-t_j)\Delta\eta_j = 0, \quad 0 \leqslant k \leqslant n-1. \quad (15)$$

Let $M_l(t)$ be an arbitrary polynomial of degree $l \leqslant n-1$. Since each polynomial $\hat{P}_{k,N}(t)$ has degree k, it is obvious that $M_l(t)$ can be represented as their linear combination:

$$M_l(t) = \sum_{k=0}^{l} d_k\hat{P}_{k,N}(t).$$

Then, from (15) we get

$$\sum_{j=0}^{N-1} Q_n(t_j)M_l(t_j)(1-t_j)\Delta\eta_j = 0,$$

i. e., polynomials $Q_0(t), \ldots, Q_{N-1}(t)$ form an orthogonal system with the weight $\kappa^{1,0}(t) = 1-t$ on the grid Ω_N. Hence,

$$Q_n(t) = \gamma_n\hat{P}^{1,0}_{n,N}(t), \quad \gamma_n > 0. \quad (16)$$

To find γ_n, taking into account (14), consider the expression

$$H_n = \sum_{j=0}^{N-1} Q_n^2(t_j)(1-t_j)\Delta\eta_j = \tag{17}$$

$$= \hat{P}_{n+1,\,N}(1) \sum_{j=0}^{N-1} \hat{P}_{n,\,N}(t_j) Q_n(t_j)\Delta\eta_j = \hat{P}_{n+1,\,N}(1)\frac{\tilde{k}_n}{\hat{k}_{n,\,N}}, \tag{18}$$

where $\tilde{k}_n$, $\hat{k}_{n,\,N}$ are the leading coefficients of polynomials $Q_n(t)$ and $\hat{P}_{n,\,N}(t)$, respectively.

In addition, notice that $\tilde{k}_n = \hat{P}_{n,\,N}(1)\hat{k}_{n+1,\,N}$ from (14), and, therefore

$$H_n = \hat{P}_{n+1,\,N}(1)\hat{P}_{n,\,N}(1)\frac{\hat{k}_{n+1,\,N}}{\hat{k}_{n,\,N}}. \tag{19}$$

On the other hand, we get, from (16) and (17),

$$H_n = \gamma_n^2 \sum_{j=0}^{N-1} \left(\hat{P}^{1,\,0}_{n,\,N}(t_j)\right)^2 (1-t_j)\Delta\eta_j = \gamma_n^2. \tag{20}$$

Comparing (19) with (20), we derive

$$\gamma_n = \sqrt{\hat{P}_{n+1,\,N}(1)\hat{P}_{n,\,N}(1)\frac{\hat{k}_{n+1,\,N}}{\hat{k}_{n,\,N}}}. \tag{21}$$

Returning to equality (14) and using (16), we have

$$(1-t)\gamma_n \hat{P}^{1,\,0}_{n,\,N}(t) = \hat{P}_{n+1,\,N}(1)\hat{P}_{n,\,N}(t) - \hat{P}_{n,\,N}(1)\hat{P}_{n+1,\,N}(t).$$

This equality, together with (21), gives us (12).

Similarly, we derive equality (13). $\square$

Next, let us agree on the following notation:

$$K_{n,\,N}(x,y) = \sum_{k=0}^{n} \hat{P}_{k,\,N}(x)\hat{P}_{k,\,N}(y). \tag{22}$$

Then, using the Christoffel–Darboux formula and Lemma 4, we can also prove the following assertion.

Lemma 5. *The following equality holds:*

$$K_{n,N}(x,y) =$$
$$= C_{n,N}\left[\frac{1-x}{y-x}\hat{P}^{1,0}_{n,N}(x)\hat{P}_{n,N}(y) - \frac{1-y}{y-x}\hat{P}^{1,0}_{n,N}(y)\hat{P}_{n,N}(x)\right], \quad (23)$$

where

$$C_{n,N} = \sqrt{\frac{\hat{k}_{n,N}\hat{P}_{n+1,N}(1)}{\hat{k}_{n+1,N}\hat{P}_{n,N}(1)}}.$$

The weighted estimate for the discrete Legendre polynomials, obtained by the author in [13], takes the following form:

Theorem A. *Let us put $4\lambda_N\chi n^2 < 1$; then there is a constant $a > 0$, such that*

$$\left|\hat{P}_{n,N}(t)\right| \leqslant c(a)\left(1 + B\sqrt{n^3\lambda_N}\right)\left(\sqrt{1-t^2} + \frac{1}{n}\right)^{-\frac{1}{2}} \leqslant$$

$$\leqslant c(a)\left(1 + B\sqrt{n^3\lambda_N}\right)\begin{cases} n^{\frac{1}{2}}, & -1 \leqslant t \leqslant -1 + an^{-2}, \\ (1+t)^{-\frac{1}{4}}, & -1 + an^{-2} \leqslant t \leqslant 0, \\ (1-t)^{-\frac{1}{4}}, & 0 \leqslant t \leqslant 1 - an^{-2}, \\ n^{\frac{1}{2}}, & 1 - an^{-2} \leqslant t \leqslant 1, \end{cases} \quad (24)$$

where

$$B = \left(\frac{3 - 4\lambda_N\chi n^2}{1 - 16\lambda_N^2\chi^2 n^4}\right)^{\frac{1}{2}}.$$

5. Approximative properties of the Fourier sums by $\hat{P}^{\alpha,\beta}_{n,N}(t)$. Suppose we are given the values of some continuous on $[-1,1]$ function $f(t)$ at the points of the grid Ω_N. Our main goal is to estimate the value

$$|R_{n,N}(f,t)| = |f(t) - \Lambda_{n,N}(f,t)|, \quad t \in [-1,1].$$

Denote by $\mathcal{P}_n$ the Hilbert space of all polynomials of degree n and by

$$E_n(f) = \min_{p_n\in\mathcal{P}_n}\max_{t\in[-1,1]}|f(t) - p_n(t)|$$

the best approximation for the function $f(t)$ by polynomials of degree at most n.

It is easy to show that $\Lambda_{n,N}(p_n, t) = p_n(t)$ for any polynomial $p_n \in \mathcal{P}_n$. Hence, using the Lebesgue-type inequality, we get

$$|R_{n,N}(f,t)| = |f(t) - \Lambda_{n,N}(f,t)| \leqslant E_n(f)\left[1 + L_{n,N}(t)\right], \tag{25}$$

where

$$L_{n,N}(t) = \sum_{j=0}^{N-1} |K_{n,N}(t_j, t)|\, \Delta\eta_j \tag{26}$$

is the Lebesgue function for $\left\{\hat{P}_{n,N}\right\}_{n=0}^{N-1}$ and $K_{n,N}(t_j, t)$ is the kernel from (22).

Thus, it is necessary to study the Lebesgue function $L_{n,N}(t)$.

Theorem 1. *There exists a real number $\gamma > 0$, such that for $2 \leqslant n \leqslant \gamma\lambda_N^{-1/3}$ and $0 < \varepsilon < 1$ the following estimates hold:*

$$\max_{-1\leqslant t\leqslant 1} L_{n,N}(t) \leqslant c(\gamma) n^{\frac{1}{2}},$$

$$L_{n,N}(t) \leqslant c(\gamma)\ln n, \quad -1+\varepsilon \leqslant t \leqslant 1-\varepsilon.$$

Proof. We consider only the case $t \in [0,1]$, because for $t \in [-1,0]$ the proof is quite similar.

Let us start with $t \in [0, 1-4n^{-2}]$. We divide the sum on the right-hand side of (26) according to the following scheme:

$$L_{n,N}(t) = \left[\sum_{-1\leqslant t_j\leqslant -\frac{1}{2}} + \sum_{-\frac{1}{2}\leqslant t_j\leqslant q_1} + \sum_{q_1\leqslant t_j\leqslant q_2} + \sum_{q_2\leqslant t_j\leqslant 1}\right] |K_{n,N}(t_j,t)|\,\Delta\eta_j$$

$$= A_1 + A_2 + A_3 + A_4,$$

where $q_1 = t - \frac{\sqrt{1-t^2}}{n}$, $q_2 = t + \frac{\sqrt{1-t^2}}{n}$.

1. To estimate A_1, we use the Christoffel–Darboux formula and Lemma 3, as well as the fact that $|t - t_j| \geqslant \frac{1}{2}$ for $t \in [0, 1-4n^{-2}]$ and $t_j \in [-1, -\frac{1}{2}]$. We have

$$A_1 \leqslant 2 \sum_{-1\leqslant t_j\leqslant -\frac{1}{2}} \left(\left|\hat{P}_{n+1,N}(t)\hat{P}_{n,N}(t_j)\right| + \left|\hat{P}_{n,N}(t)\hat{P}_{n+1,N}(t_j)\right|\right)\Delta\eta_j.$$

Again, we divide the sum into two parts: denote by A_{11} the sum over $-1 \leqslant t_j \leqslant -1+4n^{-2}$, and by A_{12} that over $-1+4n^{-2} \leqslant t_j \leqslant -\frac{1}{2}$.

Using the weighted estimate (24) and Lemma 2, we obtain

$$A_{11} \leqslant cn^{-1}, \qquad A_{12} \leqslant c\,(1-t)^{-\frac{1}{4}}\,, \tag{27}$$

which means that

$$A_1 \leqslant c\,(1-t)^{-\frac{1}{4}}\,. \tag{28}$$

2. We proceed to the estimation of A_2, for which we again use the Christoffel–Darboux formula and the transformation (23) from Lemma 5. We have

$$A_2 \leqslant C_{n,N}\Big[\sum_{-\frac{1}{2}\leqslant t_j\leqslant q_1} \frac{1-t_j}{t-t_j}\left|\hat{P}^{1,0}_{n,N}(t_j)\hat{P}_{n,N}(t)\right|\Delta\eta_j +$$

$$+\sum_{-\frac{1}{2}\leqslant t_j\leqslant q_1} \frac{1-t}{t-t_j}\left|\hat{P}^{1,0}_{n,N}(t)\hat{P}_{n,N}(t_j)\right|\Delta\eta_j\Big] = C_{n,N}\left[A_{21}+A_{22}\right].$$

Consider, firstly, A_{21}:

$$A_{21} \leqslant c(1-t)^{-\frac{1}{4}} \sum_{-\frac{1}{2}\leqslant t_j\leqslant q_1} \frac{1-t_j}{t-t_j}\left|\hat{P}^{1,0}_{n,N}(t_j)\right|\Delta\eta_j = c(1-t)^{-\frac{1}{4}}A_{211},$$

where

$$A_{211} = \Big[\sum_{-\frac{1}{2}\leqslant t_j\leqslant 0} + \sum_{0\leqslant t_j\leqslant q_1}\Big]\frac{1-t_j}{t-t_j}\left|\hat{P}^{1,0}_{n,N}(t_j)\right|\Delta\eta_j \leqslant$$

$$\leqslant c\Big[\sum_{-\frac{1}{2}\leqslant t_j\leqslant 0}\frac{1-t_j}{t-t_j}\Delta\eta_j + \sum_{0\leqslant t_j\leqslant q_1}\frac{(1-t_j)^{\frac{1}{4}}}{t-t_j}\Delta\eta_j\Big] \leqslant$$

$$\leqslant c \sum_{-\frac{1}{2}\leqslant t_j\leqslant q_1}\frac{(1-t_j)^{\frac{1}{4}}}{t-t_j}\Delta\eta_j.$$

Due to the obvious inequality $(1-t_j)^{\frac{1}{4}} \leqslant (1-t)^{\frac{1}{4}} + (t-t_j)^{\frac{1}{4}}$, we can rewrite

$$A_{21} \leqslant c\,(1-t)^{-\frac{1}{4}} \sum_{-\frac{1}{2}\leqslant t_j\leqslant q_1}\left[\frac{(1-t)^{\frac{1}{4}}}{t-t_j} + \frac{(t-t_j)^{\frac{1}{4}}}{t-t_j}\right]\Delta\eta_j =$$

$$= c\Big[\sum_{-\frac{1}{2}\leqslant t_j\leqslant q_1}\frac{\Delta\eta_j}{t-t_j} + (1-t)^{-\frac{1}{4}}\sum_{-\frac{1}{2}\leqslant t_j\leqslant q_1}\frac{\Delta\eta_j}{(t-t_j)^{\frac{3}{4}}}\Big] =$$

$$= c\left[A_{21}^{(1)} + (1-t)^{-\frac{1}{4}} A_{21}^{(2)}\right]. \tag{29}$$

Using Lemma 2, the theorem condition of $\lambda_N n^3 \leqslant \gamma^3$ and the fact that $(1-t^2)^{-\frac{1}{2}} \leqslant n$ at $t \in [0, 1-4n^{-2}]$, we get the estimates

$$A_{21}^{(1)} \leqslant c \ln n, \qquad A_{21}^{(2)} \leqslant c,$$

wherefrom

$$A_{21} \leqslant c\left(\ln n + (1-t)^{-\frac{1}{4}}\right). \tag{30}$$

We proceed to studying A_{22}. Applying the weighted estimate for $\hat{P}^{1,0}_{n,N}(t)$, we derive

$$A_{22} \leqslant c(1-t)^{\frac{1}{4}} \sum_{-\frac{1}{2} \leqslant t_j \leqslant q_1} \frac{\left|\hat{P}_{n,N}(t_j)\right|}{t-t_j} \Delta\eta_j.$$

Before applying the weighted estimate for $\hat{P}_{n,N}(t_j)$, note that $(1+t_j)^{-\frac{1}{4}} \leqslant$ $\leqslant 3^{\frac{1}{4}}(1-t_j)^{-\frac{1}{4}}$ and $(1-t_j)^{-\frac{1}{4}} \leqslant (1-t)^{-\frac{1}{4}}$ for $-\frac{1}{2} \leqslant t_j \leqslant 0$. Then

$$A_{22} \leqslant c(1-t)^{\frac{1}{4}} \sum_{-\frac{1}{2} \leqslant t_j \leqslant q_1} \frac{(1-t_j)^{-\frac{1}{4}}}{t-t_j} \Delta\eta_j \leqslant$$

$$\leqslant c \sum_{-\frac{1}{2} \leqslant t_j \leqslant q_1} \frac{\Delta\eta_j}{t-t_j} \leqslant c(a) \ln n. \tag{31}$$

Substituting (30)–(31) in (29), we finally get

$$A_2 \leqslant C_{n,N}\left[A_{21} + A_{22}\right] \leqslant c\left((1-t)^{-\frac{1}{4}} + \ln n\right). \tag{32}$$

3. To estimate A_3, we do not apply any transformations, but substitute the weighted estimates directly in (26):

$$A_3 \leqslant c\frac{(n+1)}{(1-t)^{\frac{1}{4}}} \sum_{q_1 \leqslant t_j \leqslant q_2} \frac{\Delta\eta_j}{(1-t_j)^{\frac{1}{4}}} \leqslant$$

$$\leqslant c\frac{(n+1)}{(1-t)^{\frac{1}{4}}} \frac{q_2-q_1}{(1-q_2)^{\frac{1}{4}}} \leqslant c(a). \tag{33}$$

4. The study of the last part of the sum is similar to the study of A_2:

$$A_4 \leqslant C_{n,N}\Big[\sum_{q_2\leqslant t_j\leqslant 1}\frac{1-t_j}{t_j-t}\left|\hat{P}^{1,0}_{n,N}(t_j)\hat{P}_{n,N}(t)\right|\Delta\eta_j+$$

$$+\sum_{q_2\leqslant t_j\leqslant 1}\frac{1-t}{t_j-t}\left|\hat{P}^{1,0}_{n,N}(t)\hat{P}_{n,N}(t_j)\right|\Delta\eta_j\Big] = C_{n,N}\left[A_{41}+A_{42}\right]. \qquad (34)$$

In turn, A_{41} can also be represented in the form of several sums

$$A_{41} \leqslant c(1-t)^{-\frac{1}{4}}\times$$

$$\times\Big[\sum_{q_2\leqslant t_j\leqslant\frac{1+t}{2}}+\sum_{\frac{1+t}{2}\leqslant t_j\leqslant 1-n^{-2}}+\sum_{1-n^{-2}\leqslant t_j\leqslant 1}\Big]\frac{1-t_j}{t_j-t}\left|\hat{P}^{1,0}_{n,N}(t_j)\right|\Delta\eta_j =$$

$$= c(1-t)^{-\frac{1}{4}}\left[A^{(1)}_{41}+A^{(2)}_{41}+A^{(3)}_{41}\right]. \qquad (35)$$

Using Lemma 2 and the weighted estimates, we can obtain the following estimates for these terms:

$$A^{(1)}_{41} \leqslant c\left[(1-t)^{\frac{1}{4}}A^{(1)}_{411}+A^{(1)}_{412}\right] = c\left[(1-t)^{\frac{1}{4}}\ln n+1\right], \qquad (36)$$

$$A^{(2)}_{41} \leqslant cn^{-\frac{1}{2}}, \qquad A^{(3)}_{41} \leqslant cn^{-\frac{1}{2}}. \qquad (37)$$

Returning to inequality (35) and using (36)–(37), we derive

$$A_{41} \leqslant c\left[\ln n+(1-t)^{-\frac{1}{4}}\left(n^{-\frac{1}{2}}+1\right)\right] \leqslant c\left[(1-t)^{-\frac{1}{4}}+\ln n\right]. \qquad (38)$$

Similarly, the second term from (34) is estimated as

$$A_{42} = c(1-t)^{\frac{1}{4}}\times$$

$$\times\Big[\sum_{q_2\leqslant t_j\leqslant\frac{1+t}{2}}+\sum_{\frac{1+t}{2}\leqslant t_j\leqslant 1-n^{-2}}+\sum_{1-n^{-2}\leqslant t_j\leqslant 1}\Big]\frac{\left|\hat{P}_{n,N}(t_j)\right|}{t_j-t}\Delta\eta_j =$$

$$= c(1-t)^{\frac{1}{4}}\left[A^{(1)}_{42}+A^{(2)}_{42}+A^{(3)}_{42}\right]. \qquad (39)$$

Applying Lemma 2 and a series of transformations, we obtain estimates for these parts:

$$A^{(1)}_{42} \leqslant cn^{\frac{1}{2}}, \quad A^{(2)}_{42} \leqslant cn^{\frac{1}{2}}, \quad A^{(3)}_{42} \leqslant cn^{\frac{1}{2}}.$$

Then

$$A_4 \leqslant c\left[A_{41} + A_{42}\right] \leqslant c\left[(1-t)^{\frac{1}{4}} + \ln n\right]. \tag{40}$$

Finally, all the estimates (28), (32), (33) and (40) in total allow us to display for $t \in [0, 1 - 4n^{-2}]$

$$L_{n,N}(t) = A_1 + A_2 + A_3 + A_4 \leqslant c\left[(1-t)^{\frac{1}{4}} + \ln n\right]. \tag{41}$$

Now we consider the behavior of the Lebesgue function $L_{n,N}(t)$ for $t \in [1 - 4n^{-2}, 1]$. Let us represent the Lebesgue function in the following form:

$$L_{n,N}(t) = \left[\sum_{-1 \leqslant t_j \leqslant -\frac{1}{2}} + \sum_{-\frac{1}{2} \leqslant t_j \leqslant q_1} + \sum_{q_1 \leqslant t_j \leqslant 1}\right] |K_{n,N}(t_j, t)|\, \Delta\eta_j =$$

$$= I_1 + I_2 + I_3. \tag{42}$$

1. The estimation of I_1 is similar to the estimation of A_1:

$$I_1 \leqslant 2\left[\sum_{-1 \leqslant t_j \leqslant -1+4n^{-2}} + \sum_{-1+4n^{-2} \leqslant t_j \leqslant -\frac{1}{2}}\right] \left(\left|\hat{P}_{n+1,N}(t)\hat{P}_{n,N}(t_j)\right| + \right.$$

$$\left. + \left|\hat{P}_{n,N}(t)\hat{P}_{n+1,N}(t_j)\right|\right) \Delta\eta_j = 2(I_{11} + I_{12}).$$

Using the weighted estimates for the discrete Legendre polynomials, we get for I_{11} and I_{12} the following inequalities:

$$I_{11} \leqslant cn^{-1}, \quad I_{12} \leqslant cn^{\frac{1}{2}}.$$

Therefore,

$$I_1 \leqslant cn^{\frac{1}{2}}. \tag{43}$$

2. To estimate I_2, we use Lemma 5 and the Christoffel – Darboux formula again:

$$I_2 \leqslant C_{n,N}\left[\sum_{-\frac{1}{2} \leqslant t_j \leqslant 1 - \frac{8}{n^2}} \frac{1 - t_j}{t - t_j} \left|\hat{P}^{1,0}_{n,N}(t_j)\hat{P}_{n,N}(t)\right| \Delta\eta_j + \right.$$

$$\left. + \sum_{-\frac{1}{2} \leqslant t_j \leqslant 1 - \frac{8}{n^2}} \frac{1 - t}{t - t_j} \left|\hat{P}^{1,0}_{n,N}(t)\hat{P}_{n,N}(t_j)\right| \Delta\eta_j\right] = C_{n,N}\left[I_{21} + I_{22}\right].$$

Consider, firstly, I_{21}:

$$I_{21} \leqslant c\,n^{\frac{1}{2}}\Bigg[\sum_{-\frac{1}{2}\leqslant t_j\leqslant 0} + \sum_{0\leqslant t_j\leqslant 1-\frac{8}{n^2}}\Bigg]\frac{1-t_j}{t-t_j}\left|\hat{P}^{1,0}_{n,N}(t_j)\right|\Delta\eta_j \leqslant$$

$$\leqslant c \sum_{-\frac{1}{2}\leqslant t_j\leqslant 1-\frac{8}{n^2}} \frac{(1-t_j)^{\frac{1}{4}}}{t-t_j}\Delta\eta_j.$$

Due to the obvious inequality $(1-t_j)^{\frac{1}{4}} \leqslant (1-t)^{\frac{1}{4}} + (t-t_j)^{\frac{1}{4}}$, we can rewrite

$$I_{21} \leqslant cn^{\frac{1}{2}}\Bigg[(1-t)^{\frac{1}{4}} \sum_{-\frac{1}{2}\leqslant t_j\leqslant 1-\frac{8}{n^2}} \frac{\Delta\eta_j}{t-t_j} + \sum_{-\frac{1}{2}\leqslant t_j\leqslant 1-\frac{8}{n^2}} \frac{\Delta\eta_j}{(t-t_j)^{\frac{3}{4}}}\Bigg] =$$

$$= c\,n^{\frac{1}{2}}\left[(1-t)^{\frac{1}{4}} I_{21}^{(1)} + I_{21}^{(2)}\right]. \tag{44}$$

Using Lemma 1, we obtain for these new parts the estimates

$$I_{21}^{(1)} \leqslant c\ln n, \qquad I_{21}^{(2)} \leqslant c,$$

and finally

$$I_{21} \leqslant cn^{\frac{1}{2}}\left(n^{-\frac{1}{2}}\ln n + 1\right) \leqslant cn^{\frac{1}{2}}. \tag{45}$$

Let us start with I_{22}. Using the weighted estimates and the fact that $(1+t_j)^{-\frac{1}{4}} \leqslant 3^{\frac{1}{4}}(1-t_j)^{-\frac{1}{4}} \leqslant cn^{\frac{1}{2}}$ for $-\frac{1}{2} \leqslant t_j \leqslant 0$, we derive

$$I_{22} \leqslant c\,n^{-\frac{1}{2}} \sum_{-\frac{1}{2}\leqslant t_j\leqslant 1-\frac{8}{n^2}} \frac{(1-t_j)^{-\frac{1}{4}}}{t-t_j}\Delta\eta_j \leqslant$$

$$\leqslant c\,n^{-\frac{1}{2}}n^{\frac{1}{2}} \sum_{-\frac{1}{2}\leqslant t_j\leqslant 1-\frac{8}{n^2}} \frac{\Delta\eta_j}{t-t_j} \leqslant c\ln n. \tag{46}$$

Combining (45) and (46), we finally get

$$I_2 \leqslant C_{n,N}\left[I_{21} + I_{22}\right] \leqslant c\left(n^{\frac{1}{2}} + \ln n\right) \leqslant c\,n^{\frac{1}{2}}. \tag{47}$$

3. For the last part, we just use (24):

$$I_3 \leqslant \sum_{1-n^{-2}\leqslant t_j\leqslant 1}\sum_{k=0}^{n}\left|\hat{P}_{k,N}(t)\hat{P}_{k,N}(t_j)\right|\Delta\eta_j \leqslant$$

$$\leqslant c \sum_{1-n^{-2}\leqslant t_j\leqslant 1} \sum_{k=0}^{n} k^{\frac{1}{2}} k^{\frac{1}{2}} \Delta\eta_j \leqslant cn^2 \sum_{1-n^{-2}\leqslant t_j\leqslant 1} \Delta\eta_j \leqslant c(a). \tag{48}$$

Combining estimates (43), (47) and (48), we finally get

$$L_{n,N}(t) = I_1 + I_2 + I_3 \leqslant c\left(n^{\frac{1}{2}} + \ln n\right), \quad t \in [1-4n^{-2}, 1]. \tag{49}$$

Note that for $t \in [1-4n^{-2}, 1]$ the expressions $(1-t^2)^{\frac{1}{4}}$ and $n^{\frac{1}{2}}$ are of the same order. Hence, from (41) and (49) we deduce the assertion of the theorem. □

Returning to (25), we also get the following statement from Theorem 1:

Theorem 2. *The estimate*

$$|R_{n,N}(t)| \leqslant c(\gamma)\, E_n(f)\left[\ln n + \left(\sqrt{1-t^2} + \frac{1}{n}\right)^{-\frac{1}{2}}\right]$$

holds for the remainder $R_{n,N}(t)$, where $2 \leqslant n \leqslant \gamma\lambda_N^{-1/3}$, $\gamma > 0$, and $E_n(f)$ is the best approximation for the function $f(t)$ by polynomials of degree at most n.

6. Some applications. Once again, let $f(t)$ be a continuous on $[-1,1]$ function, which is measured at the nodes of some arbitrary grid $\Omega_N = \{t_j\}_{j=0}^{N-1}$, satisfying (1)–(2). We denote these measurements by $y_j = f(t_j) + \xi_j$, $0 \leqslant j \leqslant N-1$. Here ξ_j are observation errors, which are independent random variables satisfying the following conditions:

$$E[\xi_j] = 0, \quad E[\xi_i\xi_j] = \sigma_j^2\delta_{ij}, \quad 0 \leqslant j \leqslant N-1, \tag{50}$$

where $E[X]$ is the expected value of a random variable X. It is required to approximately restore $f(t)$ at the point $t \in [-1,1]$ using discrete information $\{y_j\}_{j=0}^{N-1}$. To solve this problem, we introduce an algebraic polynomial $S_{n,N}(t)$ that minimizes the sum

$$J(a_0,\ldots,a_n) = \sum_{j=0}^{N-1} \left(y_j - p_n(t_j)\right)^2 \rho_j$$

on the set of all polynomials $p_n(t) = a_0 + a_1 t + \ldots + a_n t^n$ of degree $n \leqslant N-1$, where ρ_j are positive weight factors.

The question is, how precise $S_{n,N}(t)$ approximates the original function $f(t)$ at $t \in [-1, 1]$, i. e., it is required to estimate the value $(f(t) - S_{n,\,N}(t))^2$. Since this value depends on random errors $\xi_0, \ldots, \xi_{N-1}$, a more accurate formulation of the problem is to estimate its average value

$$J_{n,\,N}(f, t) = E\left[(f(t) - S_{n,\,N}(t))^2\right]. \tag{51}$$

In [1] this problem was studied for the uniform grid $t_j = -1 + \frac{2j}{N-1}$, $\rho_j = 1$, $0 \leqslant j \leqslant N-1$. In this article, we consider a more general case when the nodes t_j form a non-uniform grid $\Omega_N = \{t_j\}_{j=0}^{N-1} \subset [-1, 1]$, and weights ρ_j satisfy certain natural conditions.

More precisely, the values of σ_j, appearing in (50), and corresponding weights ρ_j are defined for a given real σ using equalities

$$\sigma_j^2 = \sigma^2 \frac{\lambda_N}{\Delta\eta_j}, \quad \rho_j = (\sigma/\sigma_j)^2 = \frac{\Delta\eta_j}{\lambda_N}. \tag{52}$$

It is well-known (see [15]) that polynomials $S_{n,\,N}(t)$ minimizing the value (51), can be represented as

$$S_{n,\,N}(t) = \sum_{k=0}^{n} \hat{y}_k \hat{P}_{k,\,N}(t), \quad \text{where} \quad \hat{y}_k = \sum_{j=0}^{N-1} y_j \hat{P}_{k,\,N}(t_j)\Delta\eta_j.$$

Let $\Lambda_{n,\,N}(f, t)$ be the partial Fourier sum of order n for the original (noiseless) function $f = f(t)$ by the system $\left\{\hat{P}_{k,\,N}\right\}_{k=0}^{N-1}$, i. e.,

$$\Lambda_{n,\,N}(f, t) = \sum_{k=0}^{n} \hat{f}_k \hat{P}_{k,\,N}(t), \quad \text{where} \quad \hat{f}_k = \sum_{j=0}^{N-1} f(t_j) \hat{P}_{k,\,N}(t_j)\Delta\eta_j.$$

From (50) it follows that $\Lambda_{n,\,N}(f, t) = E\left[S_{n,\,N}(t)\right]$. In fact,

$$E\left[S_{n,\,N}(t)\right] = E\Big[\sum_{k=0}^{n} \hat{y}_k \hat{P}_{k,\,N}(t)\Big] = \sum_{k=0}^{n} E\left[\hat{y}_k\right] \hat{P}_{k,\,N}(t),$$

where

$$E\left[\hat{y}_k\right] = E\Big[\sum_{j=0}^{N-1} (f(t_j) + \xi_j) \hat{P}_{k,\,N}(t_j)\Delta\eta_j\Big] =$$

$$= \sum_{j=0}^{N-1} E\left[f(t_j) + \xi_j\right] \hat{P}_{k,N}(t_j)\Delta\eta_j = \hat{f}_k.$$

In addition, it can be shown that

$$J_{n,N}(f,t) = (f(t) - \Lambda_{n,N}(f,t))^2 + \sigma^2\lambda_N \sum_{k=0}^{n} \left(\hat{P}_{k,N}(t)\right)^2 =$$

$$= R^2_{n,N}(f,t) + D_{n,N}(t). \tag{53}$$

To do this, consider the expression

$$\Delta_{n,N}(f,t) = S_{n,N}(f,t) - \Lambda_{n,N}(f,t) = \sum_{k=0}^{n} \left(\hat{y}_k - \hat{f}_k\right) \hat{P}_{k,N}(t) =$$

$$= \sum_{k=0}^{n} \hat{\xi}_k \hat{P}_{k,N}(t) = \sum_{k=0}^{n} \sum_{j=0}^{N-1} \xi_j \hat{P}_{k,N}(t)\hat{P}_{k,N}(t_j)\Delta\eta_j.$$

We will need the expected value of this value:

$$E\left[\Delta_{n,N}(f,t)\right] = \sum_{k=0}^{n} E\left[\hat{\xi}_k\right] \hat{P}_{k,N}(t) =$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{N-1} E\left[\xi_j\right] \hat{P}_{k,N}(t)\hat{P}_{k,N}(t_j)\Delta\eta_j = 0, \tag{54}$$

and its square:

$$E\left[\Delta^2_{n,N}(f,t)\right] = E\left[\left(\sum_{k=0}^{n} \hat{\xi}_k \hat{P}_{k,N}(t)\right)\left(\sum_{l=0}^{n} \hat{\xi}_l \hat{P}_{l,N}(t)\right)\right] =$$

$$= \sum_{k=0}^{n} \sum_{l=0}^{n} E\left[\hat{\xi}_k\hat{\xi}_l\right] \hat{P}_{k,N}(t)\hat{P}_{l,N}(t).$$

Due to (50) and (4), we know that

$$E\left[\hat{\xi}_k\hat{\xi}_l\right] = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} E\left[\xi_i\xi_j\right] \hat{P}_{k,N}(t_i)\hat{P}_{l,N}(t_j)\Delta\eta_i\Delta\eta_j =$$

$$= \sum_{j=0}^{N-1} \left(\sigma_j^2 \Delta\eta_j\right) \hat{P}_{k,N}(t_j) \hat{P}_{l,N}(t_j) \Delta\eta_j = \sigma^2 \lambda_N \delta_{kl}, \tag{55}$$

wherefrom

$$E\left[\Delta^2_{n,N}(f,t)\right] = \sigma^2 \lambda_N \sum_{k=0}^{n} \left(\hat{P}_{k,N}(t)\right)^2.$$

Then, taking into account (54) and (55), we have

$$J_{n,N}(f,t) = E\left[\left(f(t) - S_{n,N}(f,t)\right)^2\right] =$$

$$= E\left[\left(f(t) - \Lambda_{n,N}(f,t) + \Delta_{n,N}(f,t)\right)^2\right] = \left(f(t) - \Lambda_{n,N}(f,t)\right)^2 +$$
$$+2\left(f(t) - \Lambda_{n,N}(f,t)\right) E\left[\Delta_{n,N}(f,t)\right] + E\left[\Delta^2_{n,N}(f,t)\right] =$$
$$= \left(f(t) - \Lambda_{n,N}(f,t)\right)^2 + \sigma^2 \lambda_N \sum_{k=0}^{n} \left(\hat{P}_{k,N}(t)\right)^2 = R^2_{n,N}(f,t) + D_{n,N}(t).$$

Thus, the original objective of estimating the deviation of partial sums by discrete Legendre polynomials $\hat{P}_{n,N}(t)$ from the desired function $f(t)$ comes to estimating these two values: $R^2_{n,N}(f,t)$ and $D_{n,N}(t)$.

The estimate for $R_{n,N}(f,t)$ is given in Theorem 2. Let us consider the value $D_{n,N}(t) = \sigma^2 \lambda_N \sum\limits_{k=0}^{n} \left(\hat{P}_{k,N}(t)\right)^2$. Using weighted estimates (24) obtained in Theorem A, we have

$$D_{n,N}(t) \leqslant \sigma^2 \lambda_N \sum_{k=0}^{n} \left(c(a)\left(1 + B\sqrt{k^3 \lambda_N}\right) k^{\frac{1}{2}}\right)^2 \leqslant$$

$$\leqslant c(a)\sigma^2 (n^2 \lambda_N) \left(1 + B\sqrt{n^3 \lambda_N}\right)^2 \leqslant c(a)\sigma^2 (n^{\frac{5}{2}} \lambda_N)^2.$$

So, the value $D_{n,N}(t)$ tends to zero when $n = O(\lambda_N^{-1/3})$.

Finally, we conclude with the following statement.

Theorem 3. *Let $f(t)$ be a continuous on $[-1,1]$ function given by its measurements $y_j = f(t_j) + \xi_j$, $j = 0, 1, \ldots, N-1$, in the nodes of the grid Ω_N, which satisfy (1)–(2), where ξ_j are independent random mistakes of observation that satisfy (50)–(52). Then, for $2 \leqslant n \leqslant \gamma \lambda_N^{-1/3}$, $\gamma > 0$, the following estimate holds:*

$$J_{n,N}(f,t) \leqslant c(a,\gamma,\sigma)\left(E_n(f)\left[\ln n + \left(\sqrt{1-t^2} + \frac{1}{n}\right)^{-\frac{1}{2}}\right] + \lambda_N^{\frac{1}{3}}\right),$$

where $E_n(f)$ is the best approximation for the function $f(t)$ by polynomials of degree at most n.

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