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# ON THE CONVERGENCE OF THE LEAST SQUARE METHOD IN CASE OF NON-UNIFORM GRIDS

**Abstract.** Let $f(t)$ be a continuous on $[-1,1]$ function, which values are given at the points of arbitrary non-uniform grid $\Omega_N = \{t_j\}_{j=0}^{N-1}$, where nodes $t_j$ satisfy the only condition $\eta_j \leqslant t_j \leqslant \eta_{j+1}$, $0 \leqslant j \leqslant N-1$, and nodes $\eta_j$ are such that $-1 = \eta_0 < \eta_1 < \eta_2 < \cdots < \eta_{N-1} < \eta_N = 1$. We investigate approximative properties of the finite Fourier series for $f(t)$ by algebraic polynomials $\hat{P}_{n,N}(t)$, that are orthogonal on $\Omega_N = \{t_j\}_{j=0}^{N-1}$. Lebesgue-type inequalities for the partial Fourier sums by $\hat{P}_{n,N}(t)$ are obtained.

**Key words:** *random net, non-uniform grid, orthogonal polynomials, Legendre polynomials, least square method, Fourier series, function approximation*

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**1. Introduction.** Let $\{\eta_j\}_{j=0}^{N}$ be a system of points, such that

$$-1 = \eta_0 < \eta_1 < \eta_2 < \cdots < \eta_{N-1} < \eta_N = 1. \tag{1}$$

We assume $\Delta\eta_j = \eta_{j+1} - \eta_j$, $0 \leqslant j \leqslant N-1$, $\lambda_N = \max\limits_{0 \leqslant j \leqslant N-1} \Delta\eta_j$. Now, we construct a grid $\Omega_N$ from the points

$$\eta_j \leqslant t_j \leqslant \eta_{j+1}, \quad j = 0, 1, \ldots, N-1, \tag{2}$$

selected on each segment $[\eta_j, \eta_{j+1}]$. Without loss of generality, we can consider all the nodes $\{t_j\}_{j=0}^{N-1}$ distinct, because if $t_j = t_{j+1}$ for some $j$, we can leave only one of them and denote the grid by $\Omega_{N-1}$.

Consider the space $l_2(\Omega_N)$ of discrete functions $f : \Omega_N \to R$, where the inner product is given by

$$\langle f, g \rangle = \sum_{j=0}^{N-1} f(t_j) g(t_j) \Delta\eta_j = \lambda_N \sum_{j=0}^{N-1} f(t_j) g(t_j) \rho_j. \tag{3}$$

By $\hat{P}_{n,N}(t)$, $0 \leqslant n \leqslant N-1$, we denote polynomials that form finite orthonormal system with respect to this inner product:

$$\left\langle \hat{P}_{n,N}, \hat{P}_{m,N} \right\rangle = \sum_{j=0}^{N-1} \hat{P}_{n,N}(t_j)\hat{P}_{m,N}(t_j)\Delta\eta_j = \begin{cases} 0, & n \neq m, \\ 1, & n = m. \end{cases} \tag{4}$$

We call polynomials $\hat{P}_{n,N}(t)$, $0 \leqslant n \leqslant N-1$, *the discrete orthonormal Legendre polynomials.*

Since the system $\left\{\hat{P}_{n,N}(t)\right\}_{n=0}^{N-1}$ is complete in $l_2(\Omega_N)$, any function $f \in l_2(\Omega_N)$ can be expanded in a finite Fourier series by this system. Let $\Lambda_{n,N}(f,t)$ be the partial Fourier sum of order $n$ for the function $f = f(t)$ by the system $\left\{\hat{P}_{k,N}\right\}_{k=0}^{N-1}$, in other words

$$\Lambda_{n,N}(f,t) = \sum_{k=0}^{n} \hat{f}_k \hat{P}_{k,N}(t), \quad \text{where} \quad \hat{f}_k = \sum_{j=0}^{N-1} f(t_j)\hat{P}_{k,N}(t_j)\Delta\eta_j.$$

The main goal of this article is to study the approximative properties of $\Lambda_{n,N}(f,t)$ in case when $f(t)$ is continuous on $[-1,1]$ and $t \in [-1,1]$. More precisely, we want to obtain an estimate for the value

$$|R_{n,N}(f,t)| = |f(t) - \Lambda_{n,N}(f,t)|, \quad t \in [-1,1]. \tag{5}$$

Note that the value $|R_{n,N}(f,t)|$ for the discrete Legendre polynomials was studied in [2] for the case of $t_j = \eta_j$ and was studied in [3] for the case of $t_j = \frac{\eta_j+\eta_{j+1}}{2}$. But the results obtained there are valid only when $n = O(\lambda_N^{-1/5})$ and $n = O(\lambda_N^{-2/7})$, respectively, while we managed to get estimates for $n = O(\lambda_N^{-1/3})$ and for a more general case when $t_j$ is arbitrary on the segment $[\eta_j, \eta_{j+1}]$.

To solve this problem, we need some information about discrete Legendre polynomials $\hat{P}_{k,N}(t)$, as well as discrete Jacobi polynomials $\hat{P}_{k,N}^{\alpha,\beta}(t)$, which are a generalization of $\hat{P}_{k,N}(t)$. This information is based on the properties of classical continuous Legendre and Jacobi polynomials.

**2. Some information about Jacobi and Legendre polynomials.** The Jacobi polynomials can be written using Rodrigues' formula (see, for example, [4]) as follows:

$$P_n^{\alpha,\beta}(t) = \frac{(-1)^n}{2^n n!}\frac{1}{\kappa^{\alpha,\beta}(t)}\frac{d^n}{dt^n}\{\kappa^{\alpha,\beta}(t)\sigma^n(t)\},$$

where $\alpha, \beta$ are arbitrary real numbers, $\kappa^{\alpha,\beta}(t) = (1-t)^\alpha(1+t)^\beta$, $\sigma(t) = 1-t^2$. In the case when $\alpha, \beta > -1$, the Jacobi polynomials form an orthogonal system with the weight $\kappa^{\alpha,\beta}(t)$:

$$\int_{-1}^{1} P_n^{\alpha,\beta}(t) P_m^{\alpha,\beta}(t) \kappa^{\alpha,\beta}(t) dt = h_n^{\alpha,\beta} \delta_{nm},$$

where

$$h_n^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)},$$

and, therefore, $h_n^{\alpha,\beta} \asymp n^{-1}$, $n = 1, 2, \ldots$. For the derivative of $P_n^{\alpha,\beta}(t)$, the following equality holds:

$$\left(P_n^{\alpha,\beta}(t)\right)' = \frac{\alpha+\beta+n+1}{2} P_{n-1}^{\alpha+1,\beta+1}(t). \tag{6}$$

We will also need the following weighted estimate

$$\sqrt{n}\left|P_n^{\alpha,\beta}(t)\right| \leqslant c(\alpha,\beta)\left(\sqrt{1-t}+\frac{1}{n}\right)^{-\alpha-\frac{1}{2}}\left(\sqrt{1+t}+\frac{1}{n}\right)^{-\beta-\frac{1}{2}}, \tag{7}$$

where $-1 \leqslant t \leqslant 1$. An important particular case of Jacobi polynomials with $\alpha = \beta = 0$ is Legendre polynomials $P_n(t)$, orthogonal on $[-1, 1]$ with the unit weight $\rho(t) \equiv 1$. Denote by $\hat{P}_n(t) = \sqrt{\frac{2n+1}{2}} P_n(t)$, $n = 0, 1, 2, \ldots$ the corresponding orthonormal Legendre polynomials. The leading coefficient of polynomial $\hat{P}_n(t)$ can be written as

$$k_n = \frac{(2n)!}{(n!)^2 2^n}\sqrt{\frac{2n+1}{2}}. \tag{8}$$

**3. Discrete Jacobi and Legendre polynomials.** We will use the integral analogue of the Markov inequality for estimating the derivative of an algebraic polynomial (see [5, 6]), which for $r = 1$ has the following form:

$$\int_{-1}^{1} |q_m'(t)| dt \leqslant c(m) m^2 \int_{-1}^{1} |q_m(t)| dt, \tag{9}$$

where $q_m(t)$ is an arbitrary algebraic polynomial of degree $m$. For every $m$, denote by $\chi_m$ the minimum of constants $c(m)$ that satisfy inequality (9), i.e.,

$$\chi_m = \sup_{q_m} \frac{\int_{-1}^{1} |q_m'(t)|dt}{m^2 \int_{-1}^{1} |q_m(t)|dt},$$

where the upper bound is taken by polynomials $q_m(t)$ of degree at most $m$ and not equal to zero identically. In work [5] by N. K. Bari, it is shown that $\chi = \sup_{m\geqslant 1} \chi_m < \infty$. Given this fact, we derive from (9):

$$\int_{-1}^{1} |q_m'(t)|dt \leqslant \chi m^2 \int_{-1}^{1} |q_m(t)|dt. \tag{10}$$

Let $\{\hat{P}_{n,N}^{\alpha,\beta}(t)\}_{n=0}^{N-1}$ be polynomials that form a finite orthonormal system with respect to the inner product

$$\langle \hat{P}_{n,N}^{\alpha,\beta}, \hat{P}_{m,N}^{\alpha,\beta} \rangle = \sum_{j=0}^{N-1} \hat{P}_{n,N}^{\alpha,\beta}(t_j)\hat{P}_{m,N}^{\alpha,\beta}(t_j)\kappa^{\alpha,\beta}(t_j)\Delta\eta_j = \begin{cases} 0, & n \neq m, \\ 1, & n = m. \end{cases}$$

We call these polynomials *discrete orthonormal Jacobi polynomials.*

In the case when the grid $\Omega_N$ consists of equidistant nodes $t_j = -1 + \frac{2j}{N-1}$, the asymptotic properties and weighted estimates for the polynomials orthogonal on $\Omega_N$ were first studied in the papers by I. I. Sharapudinov (see [7]). Later, I. I. Sharapudinov [8–10] and A. A. Nurmagomedov [11], [12] studied the asymptotic properties of polynomials that are orthogonal on nonuniform grids of the real axis. In particular, in [12] the author investigated the asymptotic properties of the discrete Jacobi polynomials $\hat{P}_{n,N}^{\alpha,\beta}(t)$ ($\alpha$ and $\beta$ are integers), orthogonal on nonuniform grid $\Omega_N$ with $t_j = \dfrac{\eta_j + \eta_{j+1}}{2}$, $0 \leqslant j \leqslant N-1$.

In our work [13], we investigated asymptotic properties of these polynomials in the general case of random $t_j$ ($\alpha$, $\beta$ are still integers). When $n = O(\lambda_N^{-\frac{1}{3}})$ and $n, N \to \infty$ we obtained asymptotic formula

$$\hat{P}_{n,N}^{\alpha,\beta}(t) = \hat{P}_{n}^{\alpha,\beta}(t) + v_{n,N}^{\alpha,\beta}(t),$$

here $\hat{P}_n^{\alpha,\beta}(t)$ is a normed Jacobi polynomial, and $\upsilon_{n,N}^{\alpha,\beta}(t)$ is the remainder, for which the following estimate is established:

$$\left|\upsilon_{n,N}^{\alpha,\beta}(\cos\theta)\right| \leqslant$$

$$\leqslant c(\alpha,\beta,\gamma)\left(\frac{3-\lambda_N\chi(2n+\alpha+\beta)^2}{1-\lambda_N^2\chi^2(2n+\alpha+\beta)^4}\right)^{\frac{1}{2}}\begin{cases}\theta^{-\alpha-\frac{1}{2}}n^{\frac{3}{2}}\sqrt{\lambda_N}, & \gamma n^{-1}\leqslant\theta\leqslant\dfrac{\pi}{2},\\ n^{\alpha+2}\sqrt{\lambda_N}, & 0\leqslant\theta\leqslant\gamma n^{-1},\end{cases}$$

where $\chi$ is the smallest of the constants in the Markov integral inequality for estimating the derivative of an algebraic polynomial. Here and further in the text, $c$, $c(\alpha)$, $c(\alpha,\beta)$, $c(\alpha,\beta,\ldots,\gamma)$ are positive constants depending only on the specified parameters, which, generally speaking, may be different in different places. *For the sake of simplicity, these estimates are given for the segment* $[0,1]$*; they apply to* $[-1,0]$ *in the similar way.*

In the article, the indicated asymptotic formula is directly used to study the value $|R_{n,N}(f,t)|$.

**4. Auxiliary statements.** In this section, we collect some of the statements that will be needed in the future.

**Lemma 1**. *Let $f(t)$ be a function, absolutely continuous on $[-1,1]$; $\{\eta_j\}_{j=0}^{N}$ and $\{t_j\}_{j=0}^{N-1}$ be systems of nodes that satisfy (1) and (2), respectively. Then*

$$\int_a^b f(t)dt=\sum_{a\leqslant t_j\leqslant b} f(t_j)\Delta\eta_j+r_N(f)$$

*for every segment $[a,b]\subset[-1,1]$, where*

$$|r_N(f)|\leqslant\lambda_N\int_a^b|f'(t)|dt.$$

Proof of this lemma can be found in [13].

From Lemma 1 the next statement also follows:

**Lemma 2**. *Let $\{\eta_j\}_{j=0}^{N}$ and $\{t_j\}_{j=0}^{N-1}$ be systems of nodes that satisfy (1) and (2), respectively. Then the following inequality holds for an absolutely continuous on $[-1,1]$ monotonous non-negative function $f(x)$:*

$$\sum_{a\leqslant t_j\leqslant b} f(t_j)\Delta\eta_j\leqslant\int_a^b f(t)dt+\lambda_N|f(b)-f(a)|.$$

**Lemma 3**. *For the leading coefficients of the discrete Legendre polynomials, the inequality*

$$\frac{k_{n,\,N}}{k_{n+1,\,N}} \leqslant 1 \tag{11}$$

*holds; here $k_{n,\,N}$ and $k_{n+1,\,N}$ are the leading coefficients of the polynomials $\hat{P}_{n,\,N}$ and $\hat{P}_{n+1,\,N}$, respectively.*

**Proof.** Following [14], let us consider the expression

$$\sum_{j=0}^{N-1} \hat{P}_{n+1,\,N}(t_j) t_j \hat{P}_{n,\,N}(t_j) \Delta\eta_j =$$

$$= \frac{k_{n,\,N}}{k_{n+1,\,N}} \sum_{j=0}^{N-1} \hat{P}^2_{n+1,\,N}(t_j) + \sum_{j=0}^{N-1} \hat{Q}_{n,\,N}(t_j) = \frac{k_{n,\,N}}{k_{n+1,\,N}}.$$

On the other hand,

$$\frac{k_{n,\,N}}{k_{n+1,\,N}} \leqslant \sum_{j=0}^{N-1} \left|\hat{P}_{n+1,\,N}(t_j)\right| \left|t_j \hat{P}_{n,\,N}(t_j)\right| \Delta\eta_j \leqslant$$

$$\leqslant \max_{0\leqslant j\leqslant N-1} \{|t_j|\} \sum_{j=0}^{N-1} \left|\hat{P}_{n+1,\,N}(t_j)\right| \left|\hat{P}_{n,\,N}(t_j)\right| \Delta\eta_j.$$

Applying the Cauchy–Bunyakovsky inequality, we finally get

$$\frac{k_{n,\,N}}{k_{n+1,\,N}} \leqslant \max_{0\leqslant j\leqslant N-1} \{|t_j|\} \left(\sum_{j=0}^{N-1} \left|\hat{P}_{n+1,\,N}(t_j)\right| \Delta\eta_j\right)^{\frac{1}{2}} \times$$

$$\times \left(\sum_{j=0}^{N-1} \left|\hat{P}_{n,\,N}(t_j)\right| \Delta\eta_j\right)^{\frac{1}{2}} = \max_{0\leqslant j\leqslant N-1} \{|t_j|\} \leqslant 1.$$

This completes the proof. □

The following lemma establishes the relation between polynomials of degrees $n$ and $n+1$.

**Lemma 4**. *For $A_n = \sqrt{\dfrac{k_{n,\,N}}{k_{n+1,\,N}}}$ the following equalities hold:*

$$(1-t)\hat{P}^{1,\,0}_{n,\,N}(t) = \\ = A_n\left(\hat{P}_{n,\,N}(t)\sqrt{\frac{\hat{P}_{n+1,\,N}(1)}{\hat{P}_{n,\,N}(1)}} - \hat{P}_{n+1,\,N}(t)\sqrt{\frac{\hat{P}_{n,\,N}(1)}{\hat{P}_{n+1,\,N}(1)}}\right), \quad (12)$$

$$(1+t)\hat{P}^{0,\,1}_{n,\,N}(t) = \\ = A_n\left(\hat{P}_{n,\,N}(t)\sqrt{\frac{-\hat{P}_{n+1,\,N}(-1)}{\hat{P}_{n,\,N}(-1)}} - \hat{P}_{n+1,\,N}(t)\sqrt{\frac{-\hat{P}_{n,\,N}(-1)}{\hat{P}_{n+1,\,N}(-1)}}\right). \quad (13)$$

**Proof.** Consider the polynomial $Q_n(t)$, given by the equality

$$(1-t)Q_n(t) = \hat{P}_{n+1,\,N}(1)\hat{P}_{n,\,N}(t) - \hat{P}_{n,\,N}(1)\hat{P}_{n+1,\,N}(t). \quad (14)$$

From its definition, we have

$$\sum_{j=0}^{N-1} Q_n(t_j)\hat{P}_{k,\,N}(t_j)(1-t_j)\Delta\eta_j = 0, \quad 0 \leqslant k \leqslant n-1. \quad (15)$$

Let $M_l(t)$ be an arbitrary polynomial of degree $l \leqslant n-1$. Since each polynomial $\hat{P}_{k,\,N}(t)$ has degree $k$, it is obvious that $M_l(t)$ can be represented as their linear combination:

$$M_l(t) = \sum_{k=0}^{l} d_k\hat{P}_{k,\,N}(t).$$

Then, from (15) we get

$$\sum_{j=0}^{N-1} Q_n(t_j)M_l(t_j)(1-t_j)\Delta\eta_j = 0,$$

i. e., polynomials $Q_0(t), \dots, Q_{N-1}(t)$ form an orthogonal system with the weight $\kappa^{1,\,0}(t) = 1-t$ on the grid $\Omega_N$. Hence,

$$Q_n(t) = \gamma_n\hat{P}^{1,\,0}_{n,\,N}(t), \quad \gamma_n > 0. \quad (16)$$

To find $\gamma_n$, taking into account (14), consider the expression

$$H_n = \sum_{j=0}^{N-1} Q_n^2(t_j)(1-t_j)\Delta\eta_j = \tag{17}$$

$$= \hat{P}_{n+1,\,N}(1) \sum_{j=0}^{N-1} \hat{P}_{n,\,N}(t_j) Q_n(t_j)\Delta\eta_j = \hat{P}_{n+1,\,N}(1)\frac{\tilde{k}_n}{\hat{k}_{n,\,N}}, \tag{18}$$

where $\tilde{k}_n$, $\hat{k}_{n,\,N}$ are the leading coefficients of polynomials $Q_n(t)$ and $\hat{P}_{n,\,N}(t)$, respectively.

In addition, notice that $\tilde{k}_n = \hat{P}_{n,\,N}(1)\hat{k}_{n+1,\,N}$ from (14), and, therefore

$$H_n = \hat{P}_{n+1,\,N}(1)\hat{P}_{n,\,N}(1)\frac{\hat{k}_{n+1,\,N}}{\hat{k}_{n,\,N}}. \tag{19}$$

On the other hand, we get, from (16) and (17),

$$H_n = \gamma_n^2 \sum_{j=0}^{N-1} \left(\hat{P}^{1,\,0}_{n,\,N}(t_j)\right)^2 (1-t_j)\Delta\eta_j = \gamma_n^2. \tag{20}$$

Comparing (19) with (20), we derive

$$\gamma_n = \sqrt{\hat{P}_{n+1,\,N}(1)\hat{P}_{n,\,N}(1)\frac{\hat{k}_{n+1,\,N}}{\hat{k}_{n,\,N}}}. \tag{21}$$

Returning to equality (14) and using (16), we have

$$(1-t)\gamma_n \hat{P}^{1,\,0}_{n,\,N}(t) = \hat{P}_{n+1,\,N}(1)\hat{P}_{n,\,N}(t) - \hat{P}_{n,\,N}(1)\hat{P}_{n+1,\,N}(t).$$

This equality, together with (21), gives us (12).

Similarly, we derive equality (13). □

Next, let us agree on the following notation:

$$K_{n,\,N}(x,y) = \sum_{k=0}^{n} \hat{P}_{k,\,N}(x)\hat{P}_{k,\,N}(y). \tag{22}$$

Then, using the Christoffel–Darboux formula and Lemma 4, we can also prove the following assertion.

**Lemma 5**. *The following equality holds:*

$$K_{n,N}(x,y)=$$
$$=C_{n,N}\left[\frac{1-x}{y-x}\hat{P}^{1,0}_{n,N}(x)\hat{P}_{n,N}(y)-\frac{1-y}{y-x}\hat{P}^{1,0}_{n,N}(y)\hat{P}_{n,N}(x)\right], \quad (23)$$

*where*

$$C_{n,N}=\sqrt{\frac{\hat{k}_{n,N}\hat{P}_{n+1,N}(1)}{\hat{k}_{n+1,N}\hat{P}_{n,N}(1)}}.$$

The weighted estimate for the discrete Legendre polynomials, obtained by the author in [13], takes the following form:

**Theorem A.** *Let us put $4\lambda_N\chi n^2<1$; then there is a constant $a>0$, such that*

$$\left|\hat{P}_{n,N}(t)\right|\leqslant c(a)\left(1+B\sqrt{n^3\lambda_N}\right)\left(\sqrt{1-t^2}+\frac{1}{n}\right)^{-\frac{1}{2}}\leqslant$$

$$\leqslant c(a)\left(1+B\sqrt{n^3\lambda_N}\right)\begin{cases} n^{\frac{1}{2}}, & -1\leqslant t\leqslant -1+an^{-2},\\ (1+t)^{-\frac{1}{4}}, & -1+an^{-2}\leqslant t\leqslant 0,\\ (1-t)^{-\frac{1}{4}}, & 0\leqslant t\leqslant 1-an^{-2},\\ n^{\frac{1}{2}}, & 1-an^{-2}\leqslant t\leqslant 1,\end{cases} \quad (24)$$

*where*

$$B=\left(\frac{3-4\lambda_N\chi n^2}{1-16\lambda_N^2\chi^2n^4}\right)^{\frac{1}{2}}.$$

**5. Approximative properties of the Fourier sums by $\hat{P}^{\alpha,\beta}_{n,N}(t)$.** Suppose we are given the values of some continuous on $[-1,1]$ function $f(t)$ at the points of the grid $\Omega_N$. Our main goal is to estimate the value

$$|R_{n,N}(f,t)|=|f(t)-\Lambda_{n,N}(f,t)|\,,\quad t\in[-1,1].$$

Denote by $\mathcal{P}_n$ the Hilbert space of all polynomials of degree $n$ and by

$$E_n(f)=\min_{p_n\in\mathcal{P}_n}\max_{t\in[-1,1]}|f(t)-p_n(t)|$$

the best approximation for the function $f(t)$ by polynomials of degree at most $n$.

It is easy to show that $\Lambda_{n,N}(p_n,t)=p_n(t)$ for any polynomial $p_n\in\mathcal{P}_n$. Hence, using the Lebesgue-type inequality, we get

$$|R_{n,N}(f,t)|=|f(t)-\Lambda_{n,N}(f,t)|\leqslant E_n(f)\left[1+L_{n,N}(t)\right], \tag{25}$$

where

$$L_{n,N}(t)=\sum_{j=0}^{N-1}|K_{n,N}(t_j,t)|\,\Delta\eta_j \tag{26}$$

is the Lebesgue function for $\left\{\hat{P}_{n,N}\right\}_{n=0}^{N-1}$ and $K_{n,N}(t_j,t)$ is the kernel from (22).

Thus, it is necessary to study the Lebesgue function $L_{n,N}(t)$.

**Theorem 1.** *There exists a real number $\gamma>0$, such that for $2\leqslant n\leqslant$ $\leqslant\gamma\lambda_N^{-1/3}$ and $0<\varepsilon<1$ the following estimates hold:*

$$\max_{-1\leqslant t\leqslant 1}L_{n,N}(t)\leqslant c(\gamma)n^{\frac{1}{2}},$$

$$L_{n,N}(t)\leqslant c(\gamma)\ln n,\quad -1+\varepsilon\leqslant t\leqslant 1-\varepsilon.$$

**Proof.** We consider only the case $t\in[0,1]$, because for $t\in[-1,0]$ the proof is quite similar.

Let us start with $t\in[0,1-4n^{-2}]$. We divide the sum on the right-hand side of (26) according to the following scheme:

$$L_{n,N}(t)=\left[\sum_{-1\leqslant t_j\leqslant-\frac{1}{2}}+\sum_{-\frac{1}{2}\leqslant t_j\leqslant q_1}+\sum_{q_1\leqslant t_j\leqslant q_2}+\sum_{q_2\leqslant t_j\leqslant 1}\right]|K_{n,N}(t_j,t)|\,\Delta\eta_j$$

$$=A_1+A_2+A_3+A_4,$$

where $q_1=t-\frac{\sqrt{1-t^2}}{n}$, $q_2=t+\frac{\sqrt{1-t^2}}{n}$.

**1.** To estimate $A_1$, we use the Christoffel–Darboux formula and Lemma 3, as well as the fact that $|t-t_j|\geqslant\frac{1}{2}$ for $t\in[0,1-4n^{-2}]$ and $t_j\in[-1,-\frac{1}{2}]$. We have

$$A_1\leqslant 2\sum_{-1\leqslant t_j\leqslant-\frac{1}{2}}\left(\left|\hat{P}_{n+1,N}(t)\hat{P}_{n,N}(t_j)\right|+\left|\hat{P}_{n,N}(t)\hat{P}_{n+1,N}(t_j)\right|\right)\Delta\eta_j.$$

Again, we divide the sum into two parts: denote by $A_{11}$ the sum over $-1\leqslant t_j\leqslant-1+4n^{-2}$, and by $A_{12}$ that over $-1+4n^{-2}\leqslant t_j\leqslant-\frac{1}{2}$.

Using the weighted estimate (24) and Lemma 2, we obtain

$$A_{11} \leqslant cn^{-1}, \qquad A_{12} \leqslant c\,(1-t)^{-\frac{1}{4}}\,, \tag{27}$$

which means that

$$A_1 \leqslant c\,(1-t)^{-\frac{1}{4}}\,. \tag{28}$$

**2.** We proceed to the estimation of $A_2$, for which we again use the Christoffel–Darboux formula and the transformation (23) from Lemma 5. We have

$$A_2 \leqslant C_{n,N}\Big[\sum_{-\frac{1}{2}\leqslant t_j\leqslant q_1} \frac{1-t_j}{t-t_j}\left|\hat{P}^{1,0}_{n,N}(t_j)\hat{P}_{n,N}(t)\right|\Delta\eta_j +$$

$$+\sum_{-\frac{1}{2}\leqslant t_j\leqslant q_1} \frac{1-t}{t-t_j}\left|\hat{P}^{1,0}_{n,N}(t)\hat{P}_{n,N}(t_j)\right|\Delta\eta_j\Big] = C_{n,N}\left[A_{21}+A_{22}\right].$$

Consider, firstly, $A_{21}$:

$$A_{21} \leqslant c(1-t)^{-\frac{1}{4}}\sum_{-\frac{1}{2}\leqslant t_j\leqslant q_1} \frac{1-t_j}{t-t_j}\left|\hat{P}^{1,0}_{n,N}(t_j)\right|\Delta\eta_j = c(1-t)^{-\frac{1}{4}}A_{211},$$

where

$$A_{211} = \Big[\sum_{-\frac{1}{2}\leqslant t_j\leqslant 0} + \sum_{0\leqslant t_j\leqslant q_1}\Big]\frac{1-t_j}{t-t_j}\left|\hat{P}^{1,0}_{n,N}(t_j)\right|\Delta\eta_j \leqslant$$

$$\leqslant c\Big[\sum_{-\frac{1}{2}\leqslant t_j\leqslant 0}\frac{1-t_j}{t-t_j}\Delta\eta_j + \sum_{0\leqslant t_j\leqslant q_1}\frac{(1-t_j)^{\frac{1}{4}}}{t-t_j}\Delta\eta_j\Big] \leqslant$$

$$\leqslant c\sum_{-\frac{1}{2}\leqslant t_j\leqslant q_1}\frac{(1-t_j)^{\frac{1}{4}}}{t-t_j}\Delta\eta_j.$$

Due to the obvious inequality $(1-t_j)^{\frac{1}{4}} \leqslant (1-t)^{\frac{1}{4}} + (t-t_j)^{\frac{1}{4}}$, we can rewrite

$$A_{21} \leqslant c\,(1-t)^{-\frac{1}{4}}\sum_{-\frac{1}{2}\leqslant t_j\leqslant q_1}\left[\frac{(1-t)^{\frac{1}{4}}}{t-t_j} + \frac{(t-t_j)^{\frac{1}{4}}}{t-t_j}\right]\Delta\eta_j =$$

$$= c\Big[\sum_{-\frac{1}{2}\leqslant t_j\leqslant q_1}\frac{\Delta\eta_j}{t-t_j} + (1-t)^{-\frac{1}{4}}\sum_{-\frac{1}{2}\leqslant t_j\leqslant q_1}\frac{\Delta\eta_j}{(t-t_j)^{\frac{3}{4}}}\Big] =$$

$$= c\left[A_{21}^{(1)} + (1-t)^{-\frac{1}{4}}A_{21}^{(2)}\right]. \tag{29}$$

Using Lemma 2, the theorem condition of $\lambda_N n^3 \leqslant \gamma^3$ and the fact that $(1-t^2)^{-\frac{1}{2}} \leqslant n$ at $t \in [0, 1-4n^{-2}]$, we get the estimates

$$A_{21}^{(1)} \leqslant c \ln n, \qquad A_{21}^{(2)} \leqslant c,$$

wherefrom

$$A_{21} \leqslant c\left(\ln n + (1-t)^{-\frac{1}{4}}\right). \tag{30}$$

We proceed to studying $A_{22}$. Applying the weighted estimate for $\hat{P}_{n,N}^{1,0}(t)$, we derive

$$A_{22} \leqslant c(1-t)^{\frac{1}{4}} \sum_{-\frac{1}{2}\leqslant t_j \leqslant q_1} \frac{\left|\hat{P}_{n,N}(t_j)\right|}{t-t_j}\Delta\eta_j.$$

Before applying the weighted estimate for $\hat{P}_{n,N}(t_j)$, note that $(1+t_j)^{-\frac{1}{4}} \leqslant$ $\leqslant 3^{\frac{1}{4}}(1-t_j)^{-\frac{1}{4}}$ and $(1-t_j)^{-\frac{1}{4}} \leqslant (1-t)^{-\frac{1}{4}}$ for $-\frac{1}{2} \leqslant t_j \leqslant 0$. Then

$$A_{22} \leqslant c(1-t)^{\frac{1}{4}} \sum_{-\frac{1}{2}\leqslant t_j \leqslant q_1} \frac{(1-t_j)^{-\frac{1}{4}}}{t-t_j}\Delta\eta_j \leqslant$$

$$\leqslant c \sum_{-\frac{1}{2}\leqslant t_j \leqslant q_1} \frac{\Delta\eta_j}{t-t_j} \leqslant c(a)\ln n. \tag{31}$$

Substituting (30)–(31) in (29), we finally get

$$A_2 \leqslant C_{n,N}\left[A_{21} + A_{22}\right] \leqslant c\left((1-t)^{-\frac{1}{4}} + \ln n\right). \tag{32}$$

**3.** To estimate $A_3$, we do not apply any transformations, but substitute the weighted estimates directly in (26):

$$A_3 \leqslant c\frac{(n+1)}{(1-t)^{\frac{1}{4}}} \sum_{q_1\leqslant t_j \leqslant q_2} \frac{\Delta\eta_j}{(1-t_j)^{\frac{1}{4}}} \leqslant$$

$$\leqslant c\frac{(n+1)}{(1-t)^{\frac{1}{4}}} \frac{q_2-q_1}{(1-q_2)^{\frac{1}{4}}} \leqslant c(a). \tag{33}$$

**4.** The study of the last part of the sum is similar to the study of $A_2$:

$$A_4 \leqslant C_{n,N}\Big[\sum_{q_2\leqslant t_j\leqslant 1}\frac{1-t_j}{t_j-t}\left|\hat{P}^{1,0}_{n,N}(t_j)\hat{P}_{n,N}(t)\right|\Delta\eta_j+$$

$$+\sum_{q_2\leqslant t_j\leqslant 1}\frac{1-t}{t_j-t}\left|\hat{P}^{1,0}_{n,N}(t)\hat{P}_{n,N}(t_j)\right|\Delta\eta_j\Big] = C_{n,N}\left[A_{41}+A_{42}\right]. \qquad (34)$$

In turn, $A_{41}$ can also be represented in the form of several sums

$$A_{41} \leqslant c(1-t)^{-\frac{1}{4}}\times$$

$$\times\Big[\sum_{q_2\leqslant t_j\leqslant \frac{1+t}{2}}+\sum_{\frac{1+t}{2}\leqslant t_j\leqslant 1-n^{-2}}+\sum_{1-n^{-2}\leqslant t_j\leqslant 1}\Big]\frac{1-t_j}{t_j-t}\left|\hat{P}^{1,0}_{n,N}(t_j)\right|\Delta\eta_j =$$

$$= c(1-t)^{-\frac{1}{4}}\left[A^{(1)}_{41}+A^{(2)}_{41}+A^{(3)}_{41}\right]. \qquad (35)$$

Using Lemma 2 and the weighted estimates, we can obtain the following estimates for these terms:

$$A^{(1)}_{41} \leqslant c\left[(1-t)^{\frac{1}{4}}A^{(1)}_{411}+A^{(1)}_{412}\right] = c\left[(1-t)^{\frac{1}{4}}\ln n+1\right], \qquad (36)$$

$$A^{(2)}_{41} \leqslant cn^{-\frac{1}{2}}, \qquad A^{(3)}_{41} \leqslant cn^{-\frac{1}{2}}. \qquad (37)$$

Returning to inequality (35) and using (36)–(37), we derive

$$A_{41} \leqslant c\left[\ln n+(1-t)^{-\frac{1}{4}}\left(n^{-\frac{1}{2}}+1\right)\right] \leqslant c\left[(1-t)^{-\frac{1}{4}}+\ln n\right]. \qquad (38)$$

Similarly, the second term from (34) is estimated as

$$A_{42} = c(1-t)^{\frac{1}{4}}\times$$

$$\times\Big[\sum_{q_2\leqslant t_j\leqslant \frac{1+t}{2}}+\sum_{\frac{1+t}{2}\leqslant t_j\leqslant 1-n^{-2}}+\sum_{1-n^{-2}\leqslant t_j\leqslant 1}\Big]\frac{\left|\hat{P}_{n,N}(t_j)\right|}{t_j-t}\Delta\eta_j =$$

$$= c(1-t)^{\frac{1}{4}}\left[A^{(1)}_{42}+A^{(2)}_{42}+A^{(3)}_{42}\right]. \qquad (39)$$

Applying Lemma 2 and a series of transformations, we obtain estimates for these parts:

$$A^{(1)}_{42} \leqslant cn^{\frac{1}{2}}, \quad A^{(2)}_{42} \leqslant cn^{\frac{1}{2}}, \quad A^{(3)}_{42} \leqslant cn^{\frac{1}{2}}.$$

Then
$$A_4 \leqslant c\,[A_{41} + A_{42}] \leqslant c\left[(1-t)^{\frac{1}{4}} + \ln n\right]. \tag{40}$$

Finally, all the estimates (28), (32), (33) and (40) in total allow us to display for $t \in [0, 1-4n^{-2}]$
$$L_{n,\,N}(t) = A_1 + A_2 + A_3 + A_4 \leqslant c\left[(1-t)^{\frac{1}{4}} + \ln n\right]. \tag{41}$$

Now we consider the behavior of the Lebesgue function $L_{n,\,N}(t)$ for $t \in [1-4n^{-2}, 1]$. Let us represent the Lebesgue function in the following form:
$$L_{n,\,N}(t) = \Bigg[\sum_{-1\leqslant t_j\leqslant -\frac{1}{2}} + \sum_{-\frac{1}{2}\leqslant t_j\leqslant q_1} + \sum_{q_1\leqslant t_j\leqslant 1}\Bigg] |K_{n,\,N}(t_j, t)|\,\Delta\eta_j =$$
$$= I_1 + I_2 + I_3. \tag{42}$$

**1.** The estimation of $I_1$ is similar to the estimation of $A_1$:
$$I_1 \leqslant 2\Bigg[\sum_{-1\leqslant t_j\leqslant -1+4n^{-2}} + \sum_{-1+4n^{-2}\leqslant t_j\leqslant -\frac{1}{2}}\Bigg]\left(\left|\hat{P}_{n+1,\,N}(t)\hat{P}_{n,\,N}(t_j)\right| + \right.$$
$$\left. + \left|\hat{P}_{n,\,N}(t)\hat{P}_{n+1,\,N}(t_j)\right|\right)\Delta\eta_j = 2(I_{11} + I_{12}).$$

Using the weighted estimates for the discrete Legendre polynomials, we get for $I_{11}$ and $I_{12}$ the following inequalities:
$$I_{11} \leqslant cn^{-1}, \quad I_{12} \leqslant cn^{\frac{1}{2}}.$$

Therefore,
$$I_1 \leqslant cn^{\frac{1}{2}}. \tag{43}$$

**2.** To estimate $I_2$, we use Lemma 5 and the Christoffel – Darboux formula again:
$$I_2 \leqslant C_{n,\,N}\Bigg[\sum_{-\frac{1}{2}\leqslant t_j\leqslant 1-\frac{8}{n^2}} \frac{1-t_j}{t-t_j}\left|\hat{P}^{1,0}_{n,N}(t_j)\hat{P}_{n,N}(t)\right|\Delta\eta_j +$$
$$+ \sum_{-\frac{1}{2}\leqslant t_j\leqslant 1-\frac{8}{n^2}} \frac{1-t}{t-t_j}\left|\hat{P}^{1,\,0}_{n,\,N}(t)\hat{P}_{n,\,N}(t_j)\right|\Delta\eta_j\Bigg] = C_{n,\,N}\,[I_{21} + I_{22}].$$

Consider, firstly, $I_{21}$:

$$I_{21} \leqslant c\,n^{\frac{1}{2}}\Big[\sum_{-\frac{1}{2}\leqslant t_j\leqslant 0} + \sum_{0\leqslant t_j\leqslant 1-\frac{8}{n^2}}\Big]\frac{1-t_j}{t-t_j}\left|\hat{P}^{1,0}_{n,N}(t_j)\right|\Delta\eta_j \leqslant$$

$$\leqslant c \sum_{-\frac{1}{2}\leqslant t_j\leqslant 1-\frac{8}{n^2}} \frac{(1-t_j)^{\frac{1}{4}}}{t-t_j}\Delta\eta_j.$$

Due to the obvious inequality $(1-t_j)^{\frac{1}{4}} \leqslant (1-t)^{\frac{1}{4}} + (t-t_j)^{\frac{1}{4}}$, we can rewrite

$$I_{21} \leqslant cn^{\frac{1}{2}}\left[(1-t)^{\frac{1}{4}} \sum_{-\frac{1}{2}\leqslant t_j\leqslant 1-\frac{8}{n^2}} \frac{\Delta\eta_j}{t-t_j} + \sum_{-\frac{1}{2}\leqslant t_j\leqslant 1-\frac{8}{n^2}} \frac{\Delta\eta_j}{(t-t_j)^{\frac{3}{4}}}\right] =$$

$$= c\,n^{\frac{1}{2}}\left[(1-t)^{\frac{1}{4}} I^{(1)}_{21} + I^{(2)}_{21}\right]. \tag{44}$$

Using Lemma 1, we obtain for these new parts the estimates

$$I^{(1)}_{21} \leqslant c\ln n, \qquad I^{(2)}_{21} \leqslant c,$$

and finally

$$I_{21} \leqslant cn^{\frac{1}{2}}\left(n^{-\frac{1}{2}}\ln n + 1\right) \leqslant cn^{\frac{1}{2}}. \tag{45}$$

Let us start with $I_{22}$. Using the weighted estimates and the fact that $(1+t_j)^{-\frac{1}{4}} \leqslant 3^{\frac{1}{4}}(1-t_j)^{-\frac{1}{4}} \leqslant cn^{\frac{1}{2}}$ for $-\frac{1}{2} \leqslant t_j \leqslant 0$, we derive

$$I_{22} \leqslant c\,n^{-\frac{1}{2}} \sum_{-\frac{1}{2}\leqslant t_j\leqslant 1-\frac{8}{n^2}} \frac{(1-t_j)^{-\frac{1}{4}}}{t-t_j}\Delta\eta_j \leqslant$$

$$\leqslant c\,n^{-\frac{1}{2}}n^{\frac{1}{2}} \sum_{-\frac{1}{2}\leqslant t_j\leqslant 1-\frac{8}{n^2}} \frac{\Delta\eta_j}{t-t_j} \leqslant c\ln n. \tag{46}$$

Combining (45) and (46), we finally get

$$I_2 \leqslant C_{n,\,N}\left[I_{21} + I_{22}\right] \leqslant c\left(n^{\frac{1}{2}} + \ln n\right) \leqslant c\,n^{\frac{1}{2}}. \tag{47}$$

**3.** For the last part, we just use (24):

$$I_3 \leqslant \sum_{1-n^{-2}\leqslant t_j\leqslant 1} \sum_{k=0}^{n} \left|\hat{P}_{k,\,N}(t)\hat{P}_{k,\,N}(t_j)\right|\Delta\eta_j \leqslant$$

$$\leqslant c \sum_{1-n^{-2}\leqslant t_j\leqslant 1} \sum_{k=0}^{n} k^{\frac{1}{2}} k^{\frac{1}{2}} \Delta\eta_j \leqslant cn^2 \sum_{1-n^{-2}\leqslant t_j\leqslant 1} \Delta\eta_j \leqslant c(a). \tag{48}$$

Combining estimates (43), (47) and (48), we finally get

$$L_{n,N}(t) = I_1 + I_2 + I_3 \leqslant c\left(n^{\frac{1}{2}} + \ln n\right), \quad t \in [1-4n^{-2}, 1]. \tag{49}$$

Note that for $t \in [1-4n^{-2}, 1]$ the expressions $(1-t^2)^{\frac{1}{4}}$ and $n^{\frac{1}{2}}$ are of the same order. Hence, from (41) and (49) we deduce the assertion of the theorem. □

Returning to (25), we also get the following statement from Theorem 1:

**Theorem 2.** *The estimate*

$$|R_{n,N}(t)| \leqslant c(\gamma)\, E_n(f)\left[\ln n + \left(\sqrt{1-t^2} + \frac{1}{n}\right)^{-\frac{1}{2}}\right]$$

*holds for the remainder $R_{n,N}(t)$, where $2 \leqslant n \leqslant \gamma\lambda_N^{-1/3}$, $\gamma > 0$, and $E_n(f)$ is the best approximation for the function $f(t)$ by polynomials of degree at most $n$.*

**6. Some applications.** Once again, let $f(t)$ be a continuous on $[-1,1]$ function, which is measured at the nodes of some arbitrary grid $\Omega_N = \{t_j\}_{j=0}^{N-1}$, satisfying (1)–(2). We denote these measurements by $y_j = f(t_j) + \xi_j$, $0 \leqslant j \leqslant N-1$. Here $\xi_j$ are observation errors, which are independent random variables satisfying the following conditions:

$$E[\xi_j] = 0, \quad E[\xi_i \xi_j] = \sigma_j^2 \delta_{ij}, \quad 0 \leqslant j \leqslant N-1, \tag{50}$$

where $E[X]$ is the expected value of a random variable $X$. It is required to approximately restore $f(t)$ at the point $t \in [-1,1]$ using discrete information $\{y_j\}_{j=0}^{N-1}$. To solve this problem, we introduce an algebraic polynomial $S_{n,N}(t)$ that minimizes the sum

$$J(a_0, \ldots, a_n) = \sum_{j=0}^{N-1} \left(y_j - p_n(t_j)\right)^2 \rho_j$$

on the set of all polynomials $p_n(t) = a_0 + a_1 t + \ldots + a_n t^n$ of degree $n \leqslant N-1$, where $\rho_j$ are positive weight factors.

The question is, how precise $S_{n,N}(t)$ approximates the original function $f(t)$ at $t \in [-1,1]$, i. e., it is required to estimate the value $(f(t) - S_{n,N}(t))^2$. Since this value depends on random errors $\xi_0, \dots, \xi_{N-1}$, a more accurate formulation of the problem is to estimate its average value

$$J_{n,N}(f,t) = E\left[(f(t) - S_{n,N}(t))^2\right]. \tag{51}$$

In [1] this problem was studied for the uniform grid $t_j = -1 + \frac{2j}{N-1}$, $\rho_j = 1$, $0 \leqslant j \leqslant N-1$. In this article, we consider a more general case when the nodes $t_j$ form a non-uniform grid $\Omega_N = \{t_j\}_{j=0}^{N-1} \subset [-1,1]$, and weights $\rho_j$ satisfy certain natural conditions.

More precisely, the values of $\sigma_j$, appearing in (50), and corresponding weights $\rho_j$ are defined for a given real $\sigma$ using equalities

$$\sigma_j^2 = \sigma^2 \frac{\lambda_N}{\Delta\eta_j}, \quad \rho_j = (\sigma/\sigma_j)^2 = \frac{\Delta\eta_j}{\lambda_N}. \tag{52}$$

It is well-known (see [15]) that polynomials $S_{n,N}(t)$ minimizing the value (51), can be represented as

$$S_{n,N}(t) = \sum_{k=0}^{n} \hat{y}_k \hat{P}_{k,N}(t), \quad \text{where} \quad \hat{y}_k = \sum_{j=0}^{N-1} y_j \hat{P}_{k,N}(t_j)\Delta\eta_j.$$

Let $\Lambda_{n,N}(f,t)$ be the partial Fourier sum of order $n$ for the original (noiseless) function $f = f(t)$ by the system $\left\{\hat{P}_{k,N}\right\}_{k=0}^{N-1}$, i. e.,

$$\Lambda_{n,N}(f,t) = \sum_{k=0}^{n} \hat{f}_k \hat{P}_{k,N}(t), \quad \text{where} \quad \hat{f}_k = \sum_{j=0}^{N-1} f(t_j)\hat{P}_{k,N}(t_j)\Delta\eta_j.$$

From (50) it follows that $\Lambda_{n,N}(f,t) = E\left[S_{n,N}(t)\right]$. In fact,

$$E\left[S_{n,N}(t)\right] = E\Big[\sum_{k=0}^{n} \hat{y}_k \hat{P}_{k,N}(t)\Big] = \sum_{k=0}^{n} E\left[\hat{y}_k\right] \hat{P}_{k,N}(t),$$

where

$$E\left[\hat{y}_k\right] = E\Big[\sum_{j=0}^{N-1} (f(t_j) + \xi_j)\hat{P}_{k,N}(t_j)\Delta\eta_j\Big] =$$

$$= \sum_{j=0}^{N-1} E\left[f(t_j) + \xi_j\right] \hat{P}_{k,\,N}(t_j) \Delta\eta_j = \hat{f}_k.$$

In addition, it can be shown that

$$J_{n,\,N}(f,t) = \left(f(t) - \Lambda_{n,\,N}(f,t)\right)^2 + \sigma^2 \lambda_N \sum_{k=0}^{n} \left(\hat{P}_{k,\,N}(t)\right)^2 =$$

$$= R^2_{n,\,N}(f,t) + D_{n,\,N}(t). \tag{53}$$

To do this, consider the expression

$$\Delta_{n,\,N}(f,t) = S_{n,\,N}(f,t) - \Lambda_{n,\,N}(f,t) = \sum_{k=0}^{n} \left(\hat{y}_k - \hat{f}_k\right) \hat{P}_{k,\,N}(t) =$$

$$= \sum_{k=0}^{n} \hat{\xi}_k \hat{P}_{k,\,N}(t) = \sum_{k=0}^{n} \sum_{j=0}^{N-1} \xi_j \hat{P}_{k,\,N}(t) \hat{P}_{k,\,N}(t_j) \Delta\eta_j.$$

We will need the expected value of this value:

$$E\left[\Delta_{n,\,N}(f,t)\right] = \sum_{k=0}^{n} E\left[\hat{\xi}_k\right] \hat{P}_{k,\,N}(t) =$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{N-1} E\left[\xi_j\right] \hat{P}_{k,\,N}(t) \hat{P}_{k,\,N}(t_j) \Delta\eta_j = 0, \tag{54}$$

and its square:

$$E\left[\Delta^2_{n,\,N}(f,t)\right] = E\left[\left(\sum_{k=0}^{n} \hat{\xi}_k \hat{P}_{k,\,N}(t)\right)\left(\sum_{l=0}^{n} \hat{\xi}_l \hat{P}_{l,\,N}(t)\right)\right] =$$

$$= \sum_{k=0}^{n} \sum_{l=0}^{n} E\left[\hat{\xi}_k \hat{\xi}_l\right] \hat{P}_{k,\,N}(t) \hat{P}_{l,\,N}(t).$$

Due to (50) and (4), we know that

$$E\left[\hat{\xi}_k \hat{\xi}_l\right] = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} E\left[\xi_i \xi_j\right] \hat{P}_{k,\,N}(t_i) \hat{P}_{l,\,N}(t_j) \Delta\eta_i \Delta\eta_j =$$

$$= \sum_{j=0}^{N-1} \left(\sigma_j^2 \Delta\eta_j\right) \hat{P}_{k,\,N}(t_j) \hat{P}_{l,\,N}(t_j) \Delta\eta_j = \sigma^2 \lambda_N \delta_{kl}, \tag{55}$$

wherefrom

$$E\left[\Delta_{n,\,N}^2(f,t)\right] = \sigma^2 \lambda_N \sum_{k=0}^{n} \left(\hat{P}_{k,\,N}(t)\right)^2.$$

Then, taking into account (54) and (55), we have

$$J_{n,\,N}(f,t) = E\left[(f(t) - S_{n,\,N}(f,t))^2\right] =$$

$$= E\left[(f(t) - \Lambda_{n,\,N}(f,t) + \Delta_{n,\,N}(f,t))^2\right] = (f(t) - \Lambda_{n,\,N}(f,t))^2 +$$

$$+2\,(f(t) - \Lambda_{n,\,N}(f,t))\, E\left[\Delta_{n,\,N}(f,t)\right] + E\left[\Delta_{n,\,N}^2(f,t)\right] =$$

$$= (f(t) - \Lambda_{n,\,N}(f,t))^2 + \sigma^2 \lambda_N \sum_{k=0}^{n} \left(\hat{P}_{k,\,N}(t)\right)^2 = R_{n,\,N}^2(f,t) + D_{n,\,N}(t).$$

Thus, the original objective of estimating the deviation of partial sums by discrete Legendre polynomials $\hat{P}_{n,\,N}(t)$ from the desired function $f(t)$ comes to estimating these two values: $R_{n,\,N}^2(f,t)$ and $D_{n,\,N}(t)$.

The estimate for $R_{n,\,N}(f,t)$ is given in Theorem 2. Let us consider the value $D_{n,\,N}(t) = \sigma^2 \lambda_N \sum\limits_{k=0}^{n} \left(\hat{P}_{k,\,N}(t)\right)^2$. Using weighted estimates (24) obtained in Theorem A, we have

$$D_{n,\,N}(t) \leqslant \sigma^2 \lambda_N \sum_{k=0}^{n} \left(c(a)\left(1 + B\sqrt{k^3 \lambda_N}\right) k^{\frac{1}{2}}\right)^2 \leqslant$$

$$\leqslant c(a)\sigma^2 (n^2 \lambda_N) \left(1 + B\sqrt{n^3 \lambda_N}\right)^2 \leqslant c(a)\sigma^2 (n^{\frac{5}{2}} \lambda_N)^2.$$

So, the value $D_{n,\,N}(t)$ tends to zero when $n = O(\lambda_N^{-1/3})$.

Finally, we conclude with the following statement.

**Theorem 3**. *Let $f(t)$ be a continuous on $[-1,1]$ function given by its measurements $y_j = f(t_j) + \xi_j$, $j = 0, 1, \ldots, N-1$, in the nodes of the grid $\Omega_N$, which satisfy (1)–(2), where $\xi_j$ are independent random mistakes of observation that satisfy (50)–(52). Then, for $2 \leqslant n \leqslant \gamma \lambda_N^{-1/3}$, $\gamma > 0$, the following estimate holds:*

$$J_{n,\,N}(f,t) \leqslant c(a,\gamma,\sigma) \left( E_n(f) \left[ \ln n + \left(\sqrt{1-t^2} + \frac{1}{n}\right)^{-\frac{1}{2}} \right] + \lambda_N^{\frac{1}{3}} \right),$$

*where $E_n(f)$ is the best approximation for the function $f(t)$ by polynomials of degree at most $n$.*

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