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ON THE NECESSARY AND SUFFICIENT CONDITIONS FOR THE MEASURABILITY OF A POSITIVE SEQUENCE

Abstract. The work is devoted to finding out the necessary and sufficient conditions for the measurability of a sequence of positive numbers. The concept of logarithmic measurability of a sequence is also introduced. It is assumed that the considered sequences form a sequence of zeros of some entire function of exponential type. Therefore, clarification of this question can be useful in solving the problem of completeness of the system of exponents or exponential monomials in some convex domain. Such characteristics of the sequence as lower and upper densities, minimum and maximum densities, lower and upper logarithmic block densities play an important role.

Key words: *upper density, maximal density, logarithmic block density, zeros of entire function*

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Let $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ be an unbounded nondecreasing (numbers λ_n can be repeated) sequence of positive numbers.

The symbol $B(z,t)$ denotes a closed circle with the center at the point z and radius t. The *upper* and *lower densities* of the sequence Λ are, respectively, the following quantities:

$$\bar{n}(\Lambda) = \varlimsup_{t\to+\infty} \frac{n(t,\Lambda)}{t}, \quad \underline{n}(\Lambda) = \varliminf_{t\to+\infty} \frac{n(t,\Lambda)}{t},$$

where $n(t,\Lambda)$ is the counting function of the sequence Λ, that is, the number of its elements counting the multiplicities located in the circle $B(z,t)$:

$$n(t,\Lambda) = \sum_{|\lambda_n|\leqslant t} 1.$$

If $\bar{n}(\Lambda) = \underline{n}(\Lambda)$, the sequence Λ is called *measurable* and the quantity

$$n(\Lambda) = \lim_{t\to+\infty} \frac{n(t,\Lambda)}{t}$$

exists and is called the *density* of the sequence Λ.

We recall that the *maximal* and *minimal* density of the sequence Λ are, respectively, the quantities

$$\bar{n}_0(\Lambda) = \lim_{\delta\to+0} \varlimsup_{t\to+\infty} \frac{n(t,\Lambda) - n(t(1-\delta),\Lambda)}{\delta t}, \quad \delta \in (0;1).$$

$$\underline{n}_0(\Lambda) = \lim_{\delta\to+0} \varliminf_{t\to+\infty} \frac{n(t,\Lambda) - n(t(1-\delta),\Lambda)}{\delta t}, \quad \delta \in (0;1).$$

According to the Lemma in Section E3, Chapter VI in [3], the limit as $\delta \to 0+$ always exists and the maximal density is well-defined.

If the sequence Λ is a sequence of positive numbers, then the *logarithmic block density*, introduced in [6], is:

$$\bar{L}(\Lambda) = \inf_{a>1} \varlimsup_{t\to+\infty} \frac{\lambda(at) - \lambda(t)}{\ln a}, \quad \lambda(t) = \sum_{\lambda_n \leqslant t} \frac{1}{\lambda_n}.$$

According to Lemma 3.2 of [5], the quantity $\bar{L}(\Lambda)$ can be calculated as follows:

$$\bar{L}(\Lambda) = \lim_{a\to+\infty} \varlimsup_{t\to+\infty} \frac{\lambda(at) - \lambda(t)}{\ln a},$$

that is, the limit as $a \to \infty$ exists. Making the change of variables in the latter identity, we can write

$$\bar{L}(\Lambda) = \lim_{\delta\to 1-0} \varlimsup_{t\to+\infty} \frac{\lambda(t) - \lambda(t(1-\delta))}{-\ln(1-\delta)},$$

where $\delta \in (0;1)$. In what follows, we shall employ exactly this identity while working with the logarithmic density.

For an arbitrary $\delta \in (0;1)$, consider the following quantities

$$L(\Lambda,\delta,t) = \frac{\lambda(t) - \lambda(t(1-\delta))}{-\ln(1-\delta)},$$

$$\bar{L}(\Lambda,\delta) = \varlimsup_{t\to+\infty} L(\Lambda,\delta,t), \qquad \underline{L}(\Lambda,\delta) = \varliminf_{t\to+\infty} L(\Lambda,\delta,t),$$

$$L(\Lambda, \delta) = \lim_{t \to +\infty} L(\Lambda, \delta, t),$$

and, likewise,

$$n_0(\Lambda, \delta, t) = \frac{n(t, \Lambda) - n(t(1-\delta), \Lambda)}{\delta t},$$

$$\bar{n}_0(\Lambda, \delta) = \varlimsup_{t \to +\infty} n_0(\Lambda, \delta, t), \quad \underline{n}_0(\Lambda, \delta) = \varliminf_{t \to +\infty} n_0(\Lambda, \delta, t),$$

$$n_0(\Lambda, \delta) = \lim_{t \to +\infty} n_0(\Lambda, \delta, t).$$

It is clear that

$$\bar{n}_0(\Lambda) = \lim_{\delta \to +0} \bar{n}_0(\Lambda, \delta), \;\; \underline{n}_0(\Lambda) = \lim_{\delta \to +0} \underline{n}_0(\Lambda, \delta),$$

$$\bar{L}(\Lambda) = \lim_{\delta \to 1-0} \bar{L}(\Lambda, \delta).$$

Later in this paper, the quantity $\bar{L}(\Lambda)$ will be called the *upper* logarithmic block density of the sequence Λ in order to distinguish it from the *lower* logarithmic block density, which we define as

$$\underline{L}(\Lambda) = \sup_{\delta \in (0;1)} \underline{L}(\Lambda, \delta).$$

Lemma 1. *The function $\underline{L}(\Lambda, \delta)$ has a limit at $\delta \to 1-0$, and the following equality holds:*

$$\underline{L}(\Lambda) = \lim_{\delta \to 1-0} \underline{L}(\Lambda, \delta).$$

Proof. We reason as in the proof of Lemma 3.2 of [5]. Consider, for $u > 0$, the function

$$\varphi(u) = \varliminf_{t \to +\infty} \big(\lambda(e^u r) - \lambda(r)\big).$$

Since $e^{u_2} r \to \infty$ at $r \to \infty$ for any fixed $u_2 > 0$, it follows from the properties of the lower limit that

$$\varphi(u_1 + u_2) = \varliminf_{t \to +\infty} \big(\lambda(e^{u_1+u_2} r) - \lambda(r)\big) =$$

$$= \varliminf_{t \to +\infty} \big(\lambda(e^{u_1+u_2} r) - \lambda(e^{u_2} r) + \lambda(e^{u_2} r) - \lambda(r)\big) \geqslant$$

$$\geqslant \varliminf_{t \to +\infty} \big(\lambda(e^{u_1+u_2} r) - \lambda(e^{u_2} r)\big) + \varliminf_{t \to +\infty} \big(\lambda(e^{u_2} r) - \lambda(e^{u} r)\big) = \varphi(u_1) + \varphi(u_2).$$

Let $v>1$ be fixed, u be large enough; then there is such a positive integer N that $u=Nv+w$, where $0\leqslant w<r$. In this case

$$\frac{\varphi(u)}{u}\geqslant\frac{N\varphi(v)+\varphi(w)}{Nv+w}=\frac{\varphi(v)}{v}\cdot\frac{1}{1+w/Nv}+\frac{\varphi(w)}{Nv+w}.$$

Since $N\to+\infty$ at $u\to+\infty$, at any $v>1$ we have

$$\mathop{\underline{\lim}}_{t\to+\infty}\frac{\varphi(u)}{u}\geqslant\frac{\varphi(v)}{v}.$$

It follows from this inequality that

$$\mathop{\underline{\lim}}_{u\to+\infty}\frac{\varphi(u)}{u}\geqslant\mathop{\overline{\lim}}_{u\to+\infty}\frac{\varphi(u)}{u},\quad\mathop{\underline{\lim}}_{u\to+\infty}\frac{\varphi(u)}{u}\geqslant\sup_{u>1}\frac{\varphi(u)}{u}.$$

Since inequalities with the opposite signs are obviously true,

$$\lim_{u\to+\infty}\frac{\varphi(u)}{u}=\sup_{u>1}\frac{\varphi(u)}{u}.$$

Making the change of variables $u=-\ln(1-\delta)$, $r=t(1-\delta)$, $\delta\in(0;1)$, we obtain

$$\lim_{\delta\to1-0}\mathop{\overline{\lim}}_{t\to+\infty}\frac{\lambda(t)-\lambda(t(1-\delta))}{-\ln(1-\delta)}=\lim_{\delta\to1-0}\underline{L}(\Lambda,\delta)=\sup_{\delta\in(0;1)}\underline{L}(\Lambda,\delta)=\underline{L}(\Lambda).$$

The lemma is proved. □

Exploring relationships between different densities is of the most interest in the case when the sequence Λ is a sequence of zeros of some entire function of exponential type. According to Lindelöf's theorem, the sequence Λ must have a finite upper density. Therefore, all the reasonings given below can be carried out assuming that $\bar{n}(\Lambda)<\infty$.

Let $\delta\in(0;1)$; then, according to Lemma 1 from [4] and Lemma 2.1 from [1], the series of inequalities is valid:

$$\underline{n}_0(\Lambda)\leqslant\underline{n}(\Lambda,\delta)\leqslant\underline{n}(\Lambda)\leqslant\bar{L}(\Lambda)\leqslant\bar{n}(\Lambda)\leqslant\bar{n}_0(\Lambda,\delta)\leqslant\bar{n}_0(\Lambda)\qquad(1)$$

If the sequence Λ is measurable, then all inequalities in this chain pass into equalities.

It is easy to see that for any sequence Λ the inequality holds:

$$\underline{L}(\Lambda)\leqslant\bar{L}(\Lambda).$$

Lemma 2. *The following inequality holds:*

$$\underline{n}(\Lambda) \leqslant \underline{L}(\Lambda).$$

Proof. If $\underline{n}(\Lambda) = 0$, then the statement is trivial. Consider the case $\underline{n}(\Lambda) > 0$. Since $\bar{n}(\Lambda) < \infty$, then, by (1), we have $\underline{n}(\Lambda) < \infty$. Hence, for any sufficiently small $\varepsilon > 0$ there is r_ε such that for $r \geqslant r_\varepsilon$ the inequalities hold:

$$n(r, \Lambda) \geqslant (\underline{n}(\Lambda) - \varepsilon)r,$$
$$n(r, \Lambda) \leqslant (\bar{n}(\Lambda) + \varepsilon)r.$$

Let $\delta \in (0; 1)$ and $t(1 - \delta) \geqslant r_\varepsilon$. We have:

$$\sum_{t(1-\delta)<\lambda_n\leqslant t} \frac{1}{\lambda_n} = \int_{t(1-\delta)}^{t} \frac{dn(r, \Lambda)}{r} = \frac{n(t, \Lambda)}{t} - \frac{n(t(1-\delta), \Lambda)}{t(1-\delta)} +$$
$$+ \int_{t(1-\delta)}^{t} \frac{n(r, \Lambda)dr}{r^2} > \frac{n(t, \Lambda)}{t} - \frac{n(t(1-\delta), \Lambda)}{t(1-\delta)} + (\underline{n}(\Lambda) - \varepsilon) \int_{t(1-\delta)}^{t} \frac{dr}{r} \geqslant$$
$$\geqslant -\frac{n(t(1-\delta), \Lambda)}{t(1-\delta)} + (\underline{n}(\Lambda) - \varepsilon) \int_{t(1-\delta)}^{t} \frac{dr}{r} >$$
$$> -(\bar{n}(\Lambda) + \varepsilon) + (\underline{n}(\Lambda) - \varepsilon) \ln\left(\frac{1}{1-\delta}\right).$$

Hence, based on Lemma 1 and letting $\delta \to 1 - 0$, we obtain the desired result. The lemma is proved. □

Lemma 3. *The equality holds:*

$$\underline{n}_0(\Lambda) = \lim_{\delta\to 0+} \underline{L}(\Lambda, \delta).$$

Proof. Let $\delta \in (0; 1)$. Then

$$n(\Lambda, \delta, t) = \frac{n(t, \Lambda) - n(t(1-\delta), \Lambda)}{\delta t} \leqslant \frac{1}{\delta} \sum_{t(1-\delta)<\lambda_n\leqslant t} \frac{1}{\lambda_n} =$$
$$= \frac{-\ln(1-\delta)}{\delta} \bar{L}(\Lambda, \delta, t).$$

From here we obtain

$$\underline{n}_0(\Lambda,\delta) \leqslant \frac{-\ln(1-\delta)}{\delta}\underline{L}(\Lambda,\delta). \tag{2}$$

Likewise,

$$L(\Lambda,\delta,t) = \frac{1}{-\ln(1-\delta)}\sum_{t(1-\delta)<\lambda_n\leqslant t}\frac{m_n}{\lambda_n} \leqslant$$

$$\leqslant \frac{1}{-\ln(1-\delta)}\frac{n(t,\Lambda)-n(t(1-\delta),\Lambda)}{(1-\delta)t} \leqslant \frac{\delta}{(\delta-1)\ln(1-\delta)}\bar{n}_0(\Lambda,\delta,t).$$

Therefore,

$$\bar{L}(\Lambda,\delta) \leqslant \frac{\delta}{(\delta-1)\ln(1-\delta)}\bar{n}_0(\Lambda,\delta). \tag{3}$$

Moving in (2) and (3) to the limit at $\delta \to 0+$, we obtain the lemma statement. □

We will say that the positive sequence Λ is logarithmically measurable if $\underline{L}(\Lambda) = \bar{L}(\Lambda)$.

The following statement is true, which is the main result of this work.

Theorem 1. *Let $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ be a unbounded nondecreasing sequence of positive numbers. Then the following statements are equivalent:*

1) *the sequence Λ is measurable;*
2) *the quantity $L(\Lambda,\delta)$ exists and does not depend on $\delta \in (0;1)$;*
3) *the quantity $n_0(\Lambda,\delta)$ exists and does not depend on $\delta \in (0;1)$.*

Proof. The implication 1) $\Longrightarrow$ 2) follows from Lemma 2 and the chain of inequalities (1).

If the quantity $L(\Lambda,\delta)$ exists and does not depend on $\delta \in (0;1)$, then it follows from Lemma 3 and [4, Theorem 1] that $\bar{n}_0(\Lambda) = \underline{n}_0(\Lambda)$. This gives us an implication 2) $\Longrightarrow$ 3). Finally, if the quantity $n_0(\Lambda,\delta)$ exists and does not depend on δ, then we obtain that the maximal density is equal to the minimal density, and then from the chain of inequalities (1) the measurability of the sequence Λ follows. This gives an implication 3) $\Longrightarrow$ 1). The proof is complete. □

Remark 1. *We give an example of a sequence that shows that logarithmic measurability not entails measurability. Consider a sequence of natural numbers $\lambda_n \in \mathbb{N}$ satisfying the condition*

$$\frac{\lambda_{n+1}}{\lambda_n} \to \infty, \quad n \to \infty. \tag{4}$$

Suppose that the multiplicity of each point λ_n is $\mu_n = \lambda_n$. Let $\delta \in (0;1)$ be fixed. Then, by virtue of (4), there is such a number N_0 that for any $n \geqslant N_0$ the half-interval of the form $(\lambda_n(1-\delta), \lambda_n]$ contains only the point λ_n. It follows that at any $t > 0$ the interval of the form $(t(1-\delta), t]$ contains no more than one point of sequence Λ_n. We have

$$\frac{\lambda(t) - \lambda(t(1-\delta))}{-\ln(1-\delta)} = \frac{1}{-\ln(1-\delta)} \sum_{t(1-\delta)<\lambda_n\leqslant t} \frac{\lambda_n}{\lambda_n} \leqslant \frac{1}{-\ln(1-\delta)}.$$

From here we get that $\underline{L}(\Lambda) = \bar{L}(\Lambda) = 0$. Now calculate the upper and lower densities. Since for any $n \geqslant 1$ and for a sufficiently small $\varepsilon > 0$, inequalities

$$\frac{n(\lambda_n - \varepsilon, \Lambda)}{\lambda_n - \varepsilon} \leqslant \frac{n(\lambda_n, \Lambda)}{\lambda_n}, \quad \frac{n(\lambda_n + \varepsilon, \Lambda)}{\lambda_n + \varepsilon} \leqslant \frac{n(\lambda_n, \Lambda)}{\lambda_n},$$

are true; so,

$$\bar{n}(\Lambda) = \varlimsup_{n\to\infty} \frac{n(\lambda_n, \Lambda)}{\lambda_n}, \quad \underline{n}(\Lambda) = \varliminf_{n\to\infty} \frac{n(\lambda_n - \varepsilon, \Lambda)}{\lambda_n - \varepsilon} = \varliminf_{n\to\infty} \frac{n(\lambda_n - \varepsilon, \Lambda)}{\lambda_n}.$$

We have

$$\bar{n}(\Lambda) = \varlimsup_{n\to\infty} \frac{n(\lambda_n, \Lambda)}{\lambda_n} = \varlimsup_{n\to\infty} \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_n}{\lambda_n} =$$
$$= 1 + \varlimsup_{n\to\infty} \left(\frac{\lambda_1}{\lambda_n} + \frac{\lambda_2}{\lambda_n} + \cdots + \frac{\lambda_{n-1}}{\lambda_n} \right).$$

It follows that $\bar{n}(\Lambda) \geqslant 1$. On the other hand, by virtue of (4), for any number $M > 1$ there exists a number n_0 such that for all $n \geqslant n_0$ the inequality

$$\frac{\lambda_{n+1}}{\lambda_n} > M \quad or \quad \frac{\lambda_{n-1}}{\lambda_n} < 1/M, \quad n \geqslant n_0 \tag{5}$$

is satisfied. Since the dropping of a finite number of sequence points does not affect the value of its densities, it can be assumed, without loss of generality, that (5) is fulfilled for all $n \geqslant 1$. Then

$$0 \leqslant \frac{\lambda_1}{\lambda_n} + \frac{\lambda_2}{\lambda_n} + \cdots + \frac{\lambda_{n-1}}{\lambda_n} =$$
$$= \frac{\lambda_1}{\lambda_2}\frac{\lambda_2}{\lambda_3} \cdots \frac{\lambda_{n-1}}{\lambda_n} + \frac{\lambda_2}{\lambda_3} \cdots \frac{\lambda_{n-1}}{\lambda_n} + \cdots + \frac{\lambda_{n-1}}{\lambda_n} <$$

$$< \frac{1}{M^{n-1}} + \frac{1}{M^{n-2}} + \cdots + \frac{1}{M} =$$
$$= \frac{1}{M-1}\left(1 - \frac{1}{M^{n-1}}\right) \to 0, \quad M \to +\infty.$$

From here we get $\bar{n}(\Lambda) \leqslant 1$. Therefore, $\bar{n}(\Lambda) = 1$. At the same time,

$$\underline{n}(\Lambda) = \varliminf_{n\to\infty} \frac{n(\lambda_n - \varepsilon, \Lambda)}{\lambda_n} = \varliminf_{n\to\infty} \left(\frac{\lambda_1}{\lambda_n} + \frac{\lambda_2}{\lambda_n} + \cdots + \frac{\lambda_{n-1}}{\lambda_n}\right) = 0.$$

Hence, $\underline{n}(\Lambda) < \bar{n}(\Lambda)$. This means that the Λ sequence is not measurable, though $\underline{L}(\Lambda) = \bar{L}(\Lambda)$.

Remark 2. *A natural question arises: can this result be extended to complex sequences? It turned out that the answer is no. For such sequences the upper logarithmic block density is defined as*

$$\bar{L}(\Lambda) = \inf_{a>1} \varlimsup_{t\to+\infty} \frac{l(at, t, \Lambda)}{\ln a},$$

where

$$l(t_1, t_2, \Lambda) = \max\{l^-(t_1, t_2, \Lambda), l^+(t_1, t_2, \Lambda)\},$$
$$l^-(t_1, t_2, \Lambda) = \sum_{t_1 < |\lambda_n| \leqslant t_2, \operatorname{Re}\lambda_n < 0} -\operatorname{Re}\frac{1}{\lambda_n},$$
$$l^+(t_1, t_2, \Lambda) = \sum_{t_1 < |\lambda_n| \leqslant t_2, \operatorname{Re}\lambda_n > 0} \operatorname{Re}\frac{1}{\lambda_n},$$

(see [2], [7], [8]). This is explained by the fact that the chain of inequalities similar to the chain (1) and the complex version of Lemma 2 are necessary for the validity of the theorem statement in the complex case. However, Lemma 2 is no longer valid in the case of a complex sequence. For example, for the sequence $\Lambda = \{i\sqrt{n}\}_{n=1}^{\infty}$ we obtain the following equations:

$$\underline{n}(\Lambda) = \varliminf_{n\to\infty} = \frac{n}{|\lambda_n|} = \lim_{n\to\infty} \frac{n}{\sqrt{n}} = \infty, \quad \bar{L}(\Lambda) = 0.$$

This is a counterexample to Lemma 2 in the complex case. Note that in [9, Theorem 3, Theorem 6] there are more general statements where it is shown that the upper density of a positive sequence of points can be arbitrarily large, while the logarithmic block densities in all directions are *zero. From [9, Theorem 6] it can be seen that the upper density can be replaced by the lower density arbitrarily large.*

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