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ON SOME INEQUALITIES FOR τ-MEASURABLE OPERATORS

Abstract. This paper deals with the Choi's inequality for measurable operators affiliated with a given von Neumann algebra. Some Young and Cauchy-Schwarz type inequalities for τ-measurable operators are also given.

Key words: *von Neumann algebra, positive operator, noncommutative L_p-space, Young inequality, Cauchy-Schwarz inequality*

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1. Introduction and Preliminaries. Throughout the paper, we denote by $\mathcal{M}$ a semi-finite von Neumann algebra acting on the Hilbert space $\mathcal{H}$, with a normal faithful semi-finite trace τ. We denote the identity in $\mathcal{M}$ by $\mathbf{1}$ and let $\mathcal{P}$ denote the projection lattice of $\mathcal{M}$. We write $p \sim q$ for $p, q \in \mathcal{P}$ if $p = u^*u$ and $q = uu^*$ for some $u \in \mathcal{M}$. A closed densely defined linear operator x in $\mathcal{H}$ with the domain $D(x) \subseteq \mathcal{H}$ is said to be *affiliated with* $\mathcal{M}$ if $u^*xu = x$ for all unitary u that belong to the commutant $\mathcal{M}'$ of $\mathcal{M}$. If x is affiliated with $\mathcal{M}$, then x is said to be τ-*measurable* if for every $\varepsilon > 0$ there exists a projection $e \in \mathcal{M}$ such that $e(\mathcal{H}) \subseteq D(x)$ and $\tau(1-e) < \varepsilon$. The set of all τ-measurable operators will be denoted by $L_0(\mathcal{M},\tau)$, or, simply, $L_0(\mathcal{M})$. The set $L_0(\mathcal{M})$ is a $*$-algebra with sum and product being the respective closures of the algebraic sum and product; see [7]. A closed densely defined linear operator x admits a unique polar decomposition $x = u|x|$, where u is a partial isometry such that $u^*u = (\ker x)^\perp$ and $uu^* = \overline{\mathrm{im} x}$ (with $\mathrm{im} x = x(D(x))$. We call $r(x) = (\ker x)^\perp$ and $l(x) = \overline{\mathrm{im} x}$ the left and right supports of x, respectively. Thus, $l(x) \sim r(x)$. Note that $l(x)$ (resp., $r(x)$) is the least projection e such that $ex = x$ (resp., $xe = x$). If x is self-adjoint, then $r(x) = l(x)$. This common projection is then said to be the support of x and denoted by $s(x)$. For further details, we see [8].

Let $\mathcal{M}_+$ be the positive part of $\mathcal{M}$. Set $S_+(\mathcal{M}) = \{x \in \mathcal{M}_+ : \tau(s(x)) < \infty\}$ and let $S(\mathcal{M})$ be the linear span of $S_+(M)$. Let $0 < p < \infty$, the non-commutative L_p-space $L_p(\mathcal{M}, \tau)$ is the completion of $(S, \|\cdot\|_p)$, where $\|x\|_p = \tau(|x|^p)^{\frac{1}{p}} < \infty$ for each $x \in L_p(\mathcal{M}, \tau)$. In addition, we put $L^\infty(\mathcal{M}, \tau) = \mathcal{M}$ and denote by $\|\cdot\|_p (= \|\cdot\|)$ the usual operator norm. It is well known that $L_p(\mathcal{M}, \tau)$ are Banach spaces under $\|\cdot\|_p$ for $1 \leqslant p < \infty$ and they have a lot of expected properties of classical L_p-spaces. Let x be a τ-measurable operator and $t > 0$. The "t-th singular number (or generalized s-number) of x" is defined by [5]

$$\mu_t(x) = \inf\{\|xe\| : \ e \in P, \tau(1-e) \leqslant t\}.$$

Recall that a linear map Φ is positive if $\Phi(X)$ is positive whenever X is positive. The celebrated Jensen inequality for operators [2] states that if X is a positive operator (self-adjointness is enough), Φ is a positive linear map, and f is an operator monotone on the interval $[0, \infty)$, then

$$\Phi(f(X)) \leqslant f(\Phi(X)). \tag{1}$$

In this paper, we prove the same result for measurable operators affiliated with a given von Neumann algebra. Furthermore, we use the technique of Zhao and Wu [11], via the notion of generalized singular numbers studied by Fack and Kosaki [5], to obtain generalizations of results in [11] for τ-measurable operators case. The obtained inequalities improve known results in [9]. In addition, Audenaert in [1] obtained that if X and Y are two $n \times n$ matrices and $0 \leqslant \nu \leqslant 1$, then for any unitarily invariant norm $\|\cdot\|_u$,

$$\|XY^*\|_u^2 \leqslant \|\nu X^*X + (1-\nu)Y^*Y\|_u \|(1-\nu)X^*X + \nu Y^*Y\|_u. \tag{2}$$

In the next section, we present a τ-measurable version of (2).

2. Main Theorems. We need the following lemma [3, Theorem 5]:

Lemma 1. *Let $\mathcal{M}$ be a von Neumann algebra on Hilbert space $\mathcal{H}$ and a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an operator monotone with respect to $\mathcal{M}$. Then $f(A) \leqslant f(B)$ for any pair of positive self-adjoint operators A, B affiliated with $\mathcal{M}$, such that $A \leqslant B$.*

We are ready to prove our promised extension of inequality (1).

Theorem 1. *Let Φ be a unital positive linear continuous map, $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an operator monotone function with respect to $\mathcal{M}$*

and $x \in S(\mathcal{M})$. *Then*

$$\Phi(f(x)) \leqslant f(\Phi(x)).$$

Proof. We use the same strategy as in [4, Corollary 3.2]. Put $x_n = x_{\chi([0,n])}$. It is clear that x_n is a increasing sequence of positive operators in $\mathcal{M}$ and converges nearly everywhere to x. Note that x_n commute with x for every n. So, the convergence nearly everywhere of the sequence x_n to x can be considered as in the commutative case. Therefore, for an operator monotone function f with respect to $\mathcal{M}$, and thus continuous on $\mathbb{R}^+$, the sequence $f(x_n)$ converges nearly everywhere to $f(x)$. By Lemma 1, since $x_n \leqslant x_{n+1} \leqslant \ldots \leqslant x$, we also have

$$f(x_n) \leqslant f(x_{n+1}) \leqslant \ldots \leqslant f(x),$$

and, since Φ is a positive linear continuous map,

$$\Phi(f(x_n)) \leqslant \Phi(f(x_{n+1})) \leqslant \ldots \leqslant \Phi(f(x)).$$

Consequently, $\Phi(f(x_n))$ converge nearly everywhere to $\Phi(f(x))$. On the other hand, for every x_n, by inequality (1), we have

$$\Phi(f(x_n)) \leqslant f(\Phi(x_n)) \leqslant f(\Phi(x)).$$

Tending $n \to \infty$, we obtain the desired inequality. □

The following result can be found in [10, Lemma 3.2].

Lemma 2. *Let* $x, y \in S(\mathcal{M})$ *and* $z \in \mathcal{M}$. *Then, for every* $r > 0$,

$$\||x^*zy|^r\|_p^2 \leqslant \||xx^*z|^r\|_p \||zyy^*|^r\|_p.$$

Theorem 2. *Let* $x, y \in S(\mathcal{M})$ *and* $z \in \mathcal{M}$. *Then, for every* $r > 0$,

$$g(s,t) = \left\||x^{1-t}zy^{1+s}|^r\right\|_p \left\||x^{1+t}zy^{1-s}|^r\right\|_p$$

is log-convex on $[-1,1] \times [-1,1]$, *hence is convex, and attains its minimum at* $(0,0)$.

Proof. The function g is continuous and $g(s,t) = g(-s,-t)$ $(s,t \in [0,1])$. Thus, it is enough to show that

$$g(s_1,t_1) \leqslant \frac{1}{2}\{g(s_1+s_2,t_1+t_2) + g(s_1-s_2,t_1-t_2)\}$$

where $s_1 \pm s_2, t_1 \pm t_2 \in [-1,1] \times [-1,1]$.

Applying Lemma 2,

$$\left\| |x^{1-t_1} z y^{1+s_1}|^r \right\|_2 = \left\| |x^{t_2} \left(x^{1-t_1-t_2} z y^{1+s_1-s_2} \right) y^{s_2}|^r \right\|_p \leqslant \\ \leqslant \left\{ \left\| |x^{1-(t_1-t_2)} z y^{1+(s_1-s_2)}|^r \right\|_p \left\| |x^{1-(t_1+t_2)} z y^{1+(s_1+s_2)}|^r \right\|_p \right\}^{\frac{1}{2}} \quad (3)$$

and

$$\left\| |x^{1+t_1} z y^{1-s_1}|^r \right\|_p = \left\| |x^{t_2} \left(x^{1+t_1-t_2} z y^{1-s_1-s_2} \right) y^{s_2}|^r \right\|_p \leqslant \\ \leqslant \left\{ \left\| |x^{1+(t_1+t_2)} z y^{1-(s_1+s_2)}|^r \right\|_p \left\| |x^{1+(t_1-t_2)} z y^{1-(s_1-s_2)}|^r \right\|_p \right\}^{\frac{1}{2}}. \quad (4)$$

Applying (3), (4), and the arithmetic-geometric mean inequality, we get

$$g(s_1, t_1) = \left\| |x^{1-t_1} z y^{1+s_1}|^r \right\|_2 \left\| |x^{1+t_1} z y^{1-s_1}|^r \right\|_2 \leqslant \\ \leqslant \left\{ g(s_1+s_2, t_1+t_2)\, g(s_1-s_2, t_1-t_2) \right\}^{\frac{1}{2}} \leqslant \\ \leqslant \frac{1}{2} \left[g(s_1+s_2, t_1+t_2) + g(s_1-s_2, t_1-t_2) \right].$$

The proof is completed. □

Using this observation, we give the following corollary.

Corollary. *Let $x, y \in S(\mathcal{M})$ and $z \in \mathcal{M}$. Then, for every $r > 0$,*

$$\left\| |x^{\frac{1}{2}} z y^{\frac{1}{2}}|^r \right\|_p^2 \leqslant \left\| |x^t z y^{1-s}|^r \right\|_p \left\| |x^{1-t} z y^s|^r \right\|_p \leqslant \\ \leqslant \max \left\{ \left\| |xz|^r \right\|_p \left\| |zy|^r \right\|_p, \left\| |xzy|^r \right\|_p \left\| |z|^r \right\|_p \right\}$$

where $0 \leqslant s, t \leqslant 1$.

Proof. If we replace s, t, x, y by $2s-1, 2t-1, x^{\frac{1}{2}}, y^{\frac{1}{2}}$, respectively, in Theorem 2, we see that the function $g(s,t) = \left\| |x^t z y^{1-s}|^r \right\|_p \left\| |x^{1-t} z y^s|^r \right\|_p$ is jointly convex on $[0,1] \times [0,1]$ and attains its minimum at $\left(\frac{1}{2}, \frac{1}{2}\right)$. Hence,

$$\left\| |x^{\frac{1}{2}} z y^{\frac{1}{2}}|^r \right\|_p^2 \leqslant \left\| |x^t z y^{1-s}|^r \right\|_p \left\| |x^{1-t} z y^s t|^r \right\|_p.$$

In addition, since the function g is continuous and convex on $[0,1] \times [0,1]$, it follows that g attains its maximum at the vertices of the square. Moreover, due to the symmetry, there are two possibilities for the maximum. □

The Corollary can be regarded as an extension of [10, Corollary 3.4].

In the following result, we present a τ-measurable version of the main result in [1]. We emphasize that the method of proof is completely different from the present proof in [6, Theorem 3.6].

Theorem 3. *Let x, y be two τ-measurable positive operators. Then*

$$\tau(xy)^2 \leqslant \tau(\nu\, x^2 + (1-\nu)y^2)\; \tau((1-\nu)x^2 + \nu\, y^2), \tag{5}$$

for $0 \leqslant \nu \leqslant 1$.

Proof. Note that the function $f(\nu) = \tau\left(x^\nu y^{1-\nu}\right)$ is log-convex. Consequently, the function

$$g(\nu) = f(\nu)f(1-\nu)$$

is log-convex. Since g is symmetric with respect to $\nu = \frac{1}{2}$, it follows that $f(1/2) \leqslant f(\nu)$. This means

$$\tau\left(x^{\frac{1}{2}}y^{\frac{1}{2}}\right) \leqslant \tau\left(x^\nu y^{1-\nu}\right)\; \tau\left(x^{1-\nu}y^{\nu}\right), \; -\infty < \nu < \infty.$$

Now, using Theorem 4, for $0 \leqslant \nu \leqslant 1$, we infer

$$\tau\left(x^{\frac{1}{2}}y^{\frac{1}{2}}\right) \leqslant \tau(\nu\, x + (1-\nu)y)\; \tau((1-\nu)x + \nu\, y).$$

Replacing x and y by their squares, we get the desired inequality. $\square$

Note that inequality (5) interpolates between the arithmetic-geometric mean inequality and Cauchy-Schwarz inequality for τ-measurable operators. That is, for $\nu = 0$ we obtain the Cauchy-Schwarz type inequality

$$\tau(xy)^2 \leqslant \tau(x^2)\tau(y^2),$$

while we obtain the arithmetic-geometric mean inequality

$$\tau(xy) \leqslant \frac{1}{2}\tau(x^2 + y^2)$$

for $\nu = \frac{1}{2}$.

Recently, Shao in [?, Theorem 3.1] obtained a refinement of the Young inequality

$$\tau\left(x^\nu y^{1-\nu}\right) + r_0\Big(\tau(x)^{\frac{1}{2}} - \tau(y)^{\frac{1}{2}}\Big)^2 \leqslant \tau\left(\nu x + (1-\nu)\, y\right) \tag{6}$$

where $x, y \in L_1(\mathcal{M})$ are positive operators, and $r_0 = \min\{\nu, 1-\nu\}$ with $\nu \in (0,1)$. We close this paper by improving (6).

Theorem 4. *Let $x, y \in L_1(\mathcal{M})$ be positive operators and $\nu \in (0,1)$.*

1) *If* $0 < \nu \leqslant \frac{1}{2}$, *then*

$$r_0\Big((\tau(xy))^{\frac{1}{4}} - (\tau(x))^{\frac{1}{2}}\Big)^2 + \nu\Big((\tau(x))^{\frac{1}{2}} - (\tau(y))^{\frac{1}{2}}\Big)^2 + \tau\left(x^{1-\nu}y^{\nu}\right) \leqslant \\ \leqslant \tau\left((1-\nu)x + \nu y\right). \tag{7}$$

2) *If* $\frac{1}{2} < \nu < 1$, *then*

$$r_0\Big((\tau(xy))^{\frac{1}{4}} - (\tau(x))^{\frac{1}{2}}\Big)^2 + (1-\nu)\left((\tau(x))^{\frac{1}{2}} - (\tau(y))^{\frac{1}{2}}\right)^2 + \\ + \tau\left(x^{1-\nu}y^{\nu}\right) \leqslant \tau\left((1-\nu)x + \nu y\right). \tag{8}$$

Proof. We prove only (7) as (8) goes similarly. By [11, Lemma 1], we have

$$(1-\nu)\mu_t(x) + \nu\mu_t(y) \geqslant \\ \geqslant r_0\Big(\mu_t(xy)^{\frac{1}{4}} - \mu_t(x)^{\frac{1}{2}}\Big)^2 + \nu\Big(\mu_t(x)^{\frac{1}{2}} - \mu_t(y)^{\frac{1}{2}}\Big)^2 + \mu_t(x)^{1-\nu}\mu_t(y)^{\nu},$$

where $r = \min\{\nu, 1-\nu\}$ and $r_0 = \min\{2r, 1-2r\}$.

Hence

$$\tau((1-\nu)x + \nu y) = (1-\nu)\tau(x) + \nu\tau(y) = \\ = \int_0^\infty \Big[(1-\nu)\mu_t(x) + \nu\mu_t(y)\Big]dt \geqslant \\ \geqslant r_0\Big(\int_0^\infty \Big[\mu_t(xy)^{\frac{1}{2}} + \mu_t(x) - 2(\tau(xy))^{\frac{1}{4}}(\tau(x))^{\frac{1}{2}}\Big]dt\Big) + \\ + \nu\Big(\int_0^\infty \Big[\mu_t(x) + \mu_t(y) - 2(\tau(x))^{\frac{1}{2}}(\tau(y))^{\frac{1}{2}}\Big]dt\Big) + \int_0^\infty \mu_t(x)^{1-\nu}\mu_t(y)^{\nu}dt \geqslant \\ \geqslant r_0\Bigg(\tau(x) + \int_0^\infty \mu_t\big((xy)^{\frac{1}{2}}\big)dt - \\ - 2\Big(\int_0^\infty \Big((\tau(xy))^{\frac{1}{4}}\Big)^2 dt\Big)^{\frac{1}{2}}\Big(\int_0^\infty \Big((\tau(x))^{\frac{1}{2}}\Big)^2 dt\Big)^{\frac{1}{2}}\Bigg) +$$

$$+\nu\Bigg(\tau(x)+\tau(y)-2\Big(\int\limits_0^\infty\Big(\big(\tau(x)\big)^{\frac{1}{2}}\Big)^{\frac{1}{2}}dt\Big)^2\Big(\int\limits_0^\infty\Big(\big(\tau(y)\big)^{\frac{1}{2}}\Big)^2dt\Big)^{\frac{1}{2}}\Bigg)+$$

$$+\int\limits_0^\infty \mu_t\left(x^{1-\nu}y^\nu\right)dt=$$

$$=r_0\Big(\big(\tau(xy)\big)^{\frac{1}{4}}-\big(\tau(x)\big)^2\Big)^2+\nu\Big(\big(\tau(x)\big)^{\frac{1}{2}}-\big(\tau(y)\big)^{\frac{1}{2}}\Big)^2+\tau\left(x^{1-\nu}y^\nu\right). \quad (9)$$

The proof is completed. □

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