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A NOTE ON A TWO-PARAMETER FAMILY OF OPERATORS $\mathcal{A}^{b,c}$ ON WEIGHTED BERGMAN SPACES

Abstract. In this article, we prove that the two-parameter family of operators $\mathcal{A}^{b,c}$ is bounded on the weighted Bergman spaces $B^p_{\alpha+c-1}$ if $\alpha + 2 < p$ and unbounded if $\alpha + 2 = p$.

Key words: *Generalized Cesáro operator, weighted Bergman space, boundedness*

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1. Introduction. Let $\mathbb{D}$ denote the unit disc in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ its boundary, $H(\mathbb{D})$ the set of all analytic functions on $\mathbb{D}$ and $dm(\cdot) = {}^1\!/\!_\pi\, r\,drd\theta$ the normalized Lebesgue area measure on $\mathbb{D}$. For $0 < p < \infty$, the weighted Bergman space B^p_α for $-1 < \alpha < \infty$ consists of functions $f \in H(\mathbb{D})$, such that

$$\|f\|^p_{B^p_\alpha} = (\alpha+1)\int\limits_{\mathbb{D}} |f(z)|^p(1-|z|^2)^\alpha dm(z) =$$

$$= \frac{\alpha+1}{\pi}\int\limits_0^1 M_p^p(r,f)(1-r^2)^\alpha r dr < \infty,$$

where $M_p^p(r,f) = \int\limits_0^{2\pi} |f(re^{i\theta})|^p d\theta.$

To define the adjoint of the generalized Cesáro operator, we need the Gaussian hypergeometric function. Let (a,n) be the shifted factorial defined by Appel's symbol

$$(a,n) = a(a+1)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \ \ n \in \mathbb{N} = \{1,2,3,\dots\}$$

and $(a, 0) = 1$ for $a \neq 0$. The Gaussain hypergeometric function is defined by the power series expansion

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{z^n}{n!} \quad (|z| < 1),$$

where a, b, c are complex numbers, such that $c \neq -m$, $m = 0, 1, 2, 3, \ldots$, and we assume $c \neq -m$, $m = 0, 1, 2, 3, \ldots$, to avoid zero denominators. Clearly, $F(a, b, c, z)$ belongs to $H(\mathbb{D})$. Many properties of the hypergeometric functions are found in [1]. Asymptotic behavior of the zero-balanced (i. e., the $c = a+b$ case) is well-known. For the non-zero balanced case, improved formulation is obtained in [4, 12], whereas the geometric properties of Gaussian hypergeometric functions are considered, for example, in [9, 10]. The same problems for linear and convolution operator are dealt with in [11].

Let $b, c \in \mathbb{C}$ with $Re\, b > 0$, $Re\, c > 0$. For a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $f(z) \in H(\mathbb{D})$, the two-parameter family of Cesáro averaging operators $\mathcal{P}^{b,c}$ is given by

$$\mathcal{P}^{b,c} f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{A_n^{b+1;c}} \sum_{k=n}^{\infty} b_{n-k} a_k \right) z^n,$$

where

$$A_n^{b;c} = \frac{(b, n)}{(c, n)},$$

and b_k are given by $b_0 = 1$,

$$b_k = \frac{1 + b - c}{c} A_{k-1}^{b+1;c+1} = \frac{1 + b - c}{b} A_k^{b;c}$$

for $k \geqslant 1$. The operators $\mathcal{P}^{b,c}$ were introduced in [2] and have been studied for boundedness on various function spaces, such as H^p, BMOA, B^a [7, 8], on mixed norm spaces [6], as well as on the Dirichlet space [7]. For $b = 1 + \alpha$ and $c = 1$, we obtain the generalized Cesáro operators $\mathcal{P}^{1+\gamma,1} = \mathcal{C}^\gamma$ introduced in [14]. It is known that operators $\mathcal{C}^\gamma$ are bounded on the Hardy space for $0 < p < \infty$, BMOA, and Bloch space [17] and on the Dirichlet space [16].

For $b, c \in \mathbb{C}$, such that $Re\, b > 0$, $Re\, c > 0$, let $\mathcal{A}^{b,c}$ be the adjoint operator of $\mathcal{P}^{b,c}$, given by

$$\mathcal{A}^{b,c} f(z) = \sum_{n=0}^{\infty} \left(\sum_{k=n}^{\infty} \frac{b_{k-n} a_k}{A_k^{b+1;c}} \right) z^n, \tag{1}$$

where $f(z)=\sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$ and $A_k^{b;c}$ and b_k are the same as defined for $\mathcal{P}^{b,c}$. These operators were formally introduced in [3] and studied for boundedness on the space of Cauchy transforms.

In the notation of Stempak [14], we find that

$$\mathcal{A}^{1+\gamma,1} f = \mathcal{A}^{\gamma} f.$$

In particular, for $\gamma = 0$

$$\mathcal{A}^{1,1} f = \mathcal{A} f = \sum_{n=0}^{\infty} \left(\sum_{k=n}^{\infty} \frac{1}{k+1} a_k \right) z^n,$$

where $\mathcal{A}^{\gamma}$ is the adjoint operator of the generalized Cesáro operator $\mathcal{C}^{\gamma}$ (see [17]). If $\gamma = 0$, the $\mathcal{A}^{\gamma}$ is simply adjoint of the classical Cesáro operator $\mathcal{C}$ (see [13]). Now we recall a known result that gives an integral representation of the operator $\mathcal{A}^{b,c}$.

Lemma 1. [3] *Let $b, c \in \mathbb{C}$ with $\operatorname{Re} b > 0$, $\operatorname{Re} c > 1$ and function $\varphi_{t,s}(z) = 1 - t - s + st + tz$. Then*

$$\mathcal{A}^{b,c} f(z) = M \int_0^1 \int_0^1 s^{c-2} (1-t)^{b-1} f(\varphi_{t,s}(z)) * F(c,1;1;\varphi_{t,s}(z)) ds dt,$$

where $M = (1 + b - c)(c - 1)$.

Here $*$ denotes the Hadamard product (or convolution) of power series. That is, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are two analytic functions in $|z| < R$, then convolution between f and g is denoted by $f * g$ and is defined by $(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. This series converges for $|z| < R^2$. Moreover,

$$(f * g)(z) = \frac{1}{2\pi i} \int_{|w|=p} f(w) g(z/w) \frac{dw}{w}, \quad |z| < \rho R < R^2.$$

2. Preliminary results. In this section, we recall few preliminary results, which are used to state and prove the main results of this article. The adjoint operator was considered in [13] for the case ($\gamma = 0$) and in [17]. In [13], Siskakis proved the following result.

Theorem 1. *The operator $\mathcal{A}$ is bounded on the weighted Bergman space B_{α}^{p} if and only if $\alpha + 2 < p$.*

Stević proved, in [15], a generalization of Theorem 1 for the operators $\mathcal{A}^\gamma$, when $\gamma \neq 0$:

Theorem 2. *The operator $\mathcal{A}^\gamma$ is bounded on the weighted Bergman space B^p_α if and only if $\alpha + 2 < p$.*

The main aim of this article is to generalize Theorem 2 by finding conditions on the parameters b and c for which the operators $\mathcal{A}^{b,c}$ are bounded on the weighted Bergman spaces.

We will use the following lemma in the sequel.

Lemma 2. [5, p. 65] *For each $1 < \alpha < \infty$ there is a positive constant $C = C(\alpha)$, such that*

$$\int\limits_{-\pi}^{\pi} |1 - re^{i\theta}|^{-\alpha}\, d\theta \leqslant C(1-r)^{-(\alpha-1)},$$

if $0 \leqslant r < 1$.

Henceforth, C, K, and C_1 denote positive constants, whose values are different at different occurrences.

3. Main Results. In this section, we consider the so-called convolution operator and prove its boundedness on the weighted Bergman space $B^p_{\alpha+c-1}$ for $c \geqslant 1$. Also, we state and prove the main result of this paper.

From now onwards, we denote $F(z) = F(1, c; 1; z)$ for all $z \in \mathbb{D}$.

Lemma 3. *If $p \in [1, \infty), \alpha > -1, c \geqslant 1$, $f \in B^p_\alpha$, then $f * F \in B^p_{\alpha+(c-1)p}$.*

Proof. Let $f \in B^p_\alpha$. Then

$$\frac{\alpha+1}{\pi} \int\limits_0^1 M_p^p(f, r)(1-r^2)^\alpha r dr < \infty. \tag{2}$$

Using the definition of convolution and the fact that $F(1, c; 1;\ z) = (1-z)^{-c}$, for $0 < r < \rho < 1$, we have

$$M_p^p(f * F, r) = \frac{1}{2\pi} \int\limits_0^{2\pi} |(f * F)(\rho e^{i\theta})|^p d\theta =$$

$$= \frac{1}{2\pi} \int\limits_0^{2\pi} \Big| \frac{1}{2\pi} \int\limits_0^{2\pi} F(\rho e^{it}) f\Big(\frac{r}{\rho} e^{i(\theta-t)}\Big) dt \Big|^p d\theta =$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\Big|\frac{1}{2\pi}\int_0^{2\pi}(1-\rho e^{it})^{-c} f\Big(\frac{r}{\rho}e^{i(\theta-t)}\Big)dt\Big|^p d\theta.$$

Applying Minkowski's integral inequality and Lemma 2 above, we have

$$M_p(f*F,r) \leqslant \frac{1}{2\pi}\int_0^{2\pi}\Big(\frac{1}{2\pi}\int_0^{2\pi}\Big|(1-\rho e^{it})^{-c} f\Big(\frac{r}{\rho}e^{i(\theta-t)}\Big)\Big|^p d\theta\Big)^{\frac{1}{p}} dt \leqslant$$

$$\leqslant \frac{1}{2\pi}\int_0^{2\pi}\Big(\frac{1}{2\pi}\int_0^{2\pi}\Big|1-\rho e^{it}\Big|^{-cp}\Big|f\Big(\frac{r}{\rho}e^{i(\theta-t)}\Big)\Big|^p d\theta\Big)^{\frac{1}{p}} dt =$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\big|1-\rho e^{it}\big|^{-c} dt\Big(\frac{1}{2\pi}\int_0^{2\pi}\Big|f\Big(\frac{r}{\rho}e^{i(\theta-t)}\Big)\Big|^p d\theta\Big)^{\frac{1}{p}} =$$

$$= \frac{1}{2\pi}C(1-\rho)^{-c+1}M_p\Big(f,\frac{r}{\rho}\Big).$$

From the above inequality, we find

$$\int_0^1 (1-\rho)^{(c-1)p}M_p^p(f*F,r)(1-r)^\alpha dr \leqslant K\int_0^1 M_p^p\Big(f,\frac{r}{\rho}\Big)(1-r)^\alpha dr,$$

where $K = C/2\pi$.

Now, taking $r=\rho^2$, we have

$$\int_0^1 M_p^p(f*F,\rho^2)(1-\rho^2)^{\alpha+(c-1)p}d\rho^2 \leqslant$$

$$\leqslant K(1+\rho)^{(c-1)p}\int_0^1 M_p^p\,(f,\rho)\,(1-\rho^2)^\alpha d\rho^2.$$

A simple calculation shows:

$$\int_0^1 M_p^p(f*F,\rho^2)(1-\rho^4)^{\alpha+(c-1)q}\rho^2 d\rho^2 \leqslant K\int_0^1 M_p^p\,(f,\rho)\,(1-\rho^2)^\alpha \rho^2 d\rho^2.$$

The last inequality and (2) give

$$\int_0^1 M_p^p(f*F,\rho^2)(1-\rho^4)^{\alpha+(c-1)q}\rho^2 d\rho^2 < \infty.$$

This completes the proof. □

Now we give an estimate on the norm of the convolution operator $I_\phi(f)$ on B^p_α.

Theorem 3. *Let $p \in (0,\infty), \alpha > -1, c \geqslant 1, \phi : \mathbb{D} \to \mathbb{D}$ be a non-constant analytic function. Then the operator $I_\phi(f) = (f * F)(\phi)$, where $F = F(1,c;1,z)$ on $B^p_{\alpha+c-1}(\mathbb{D})$, satisfies the following inequality:*

$$\|I_\phi(f)\|_{B^p_{\alpha+c-1}} \leqslant C\left(\frac{\|\phi\|_\infty + |\phi(0)|}{\|\phi\|_\infty - |\phi(0)|}\right)^{\frac{\alpha+2}{p}} \|f\|_{B^p_\alpha},$$

$C \equiv \left(\frac{D}{(2\pi)^2}\right)$ *if* $\alpha \geqslant 0$ *and* $C = \left(\frac{D}{(2\pi)^2}\right)(\|\phi\|_\infty + |\phi(0)|)^{\frac{\alpha}{p}}(\|\phi\|_\infty + 3|\phi(0)|)^{-\frac{\alpha}{p}}$.

Proof. We will use the method of Siskakis [13]. Let $a = |\phi(0)|$ and $b = \|\phi\|_\infty$ and fix $0 < r < 1$. By the well known consequence of the Schwarz-pick lemma on the map $\phi_1 = b^{-1}\phi$, we have $|\phi(z)| \leqslant \frac{(ba+b^2r)}{(b+ar)}$, for $|z| \leqslant r$. Since $a \leqslant b$, we have $\frac{(a+br)}{(b+ar)} \leqslant \frac{((b-a)r+2a)}{(b+a)}$ for all $0 < r < 1$, so $|\phi(z)| \leqslant bR \leqslant R$, where $R = R(r) = \frac{((b-a)r+2a)}{(b+a)}$ for all $0 < r < 1$. If $f * F \in B^p_{\alpha+c-1}(\mathbb{D})$, let $|(u * F)(z)|$ be the harmonic extension $|(f * F)(Re^{i\theta})|^p$ on $|z| \leqslant bR$; $|(u * F)(z)|$ is continuous on $|z| \leqslant bR$ and majorizes $|(f * F)(z)|^p$ there, so

$$|(f*F)(\phi(z))|^p \leqslant |(u*F)(\phi(z))| \text{ for } |z| \leqslant r.$$

It follows that

$$M_p^p((f*F)\phi, r) = \int_0^{2\pi} |(f*F)(\phi(re^{i\theta}))|^p d\theta \leqslant \int_0^{2\pi} |(u*F)(\phi(re^{i\theta}))| d\theta. \quad (3)$$

Now, for $0 < \rho < 1$,

$$(u*F)(\rho\phi(0)) = \frac{1}{2\pi}\int_0^{2\pi} F(\rho e^{i\theta})u(\phi(0)e^{-i\theta})d\theta,$$

$$|(u * F)(\rho\phi(0))| = \left|\frac{1}{2\pi}\int_0^{2\pi} F(\rho e^{i\theta})u(\phi(0)e^{-i\theta})d\theta\right| \leqslant$$
$$\leqslant \frac{1}{2\pi}\int_0^{2\pi} |F(\rho e^{i\theta})||u(\phi(0)e^{-i\theta})|d\theta =$$
$$= \frac{1}{2\pi}\int_0^{2\pi} |1-\rho e^{i\theta}|^{-c} u(\phi(0)e^{-i\theta})d\theta. \tag{4}$$

Finally, by Harnack's inequality and the Mean Value Theorem, we have

$$u(\phi(0)e^{-i\theta}) \leqslant \frac{bR+a}{bR-a}u(0) = \frac{bR+a}{bR-a}\frac{1}{2\pi}\int_0^{2\pi} |f(bRe^{i\theta})|^p d\theta. \tag{5}$$

From (3), (4), and (5) and using Lemma 2, we obtain

$$M_p^p((f*F)\phi, r) \leqslant \frac{1}{(2\pi)^2}\left(\frac{bR+a}{bR-a}\right)\int_0^{2\pi} |1-\rho e^{i\theta}|^{-c}\Big[\int_0^{2\pi} |f(dRe^{it})|^p dt\Big]d\theta \leqslant$$
$$\leqslant \frac{D(1-\rho)^{-c+1}}{(2\pi)^2}\left(\frac{bR+a}{bR-a}\right)\int_0^{2\pi} |f(bRe^{i\theta})|^p d\theta =$$
$$= \frac{D(1-\rho)^{-c+1}}{(2\pi)^2}\left(\frac{bR+a}{bR-a}\right) M_p^p(f,R). \tag{6}$$

Now multiply both sides of (6) by $(1-r^2)^\alpha r$ and integrate with respect to r from 0 to 1 to get

$$\int_0^1 M_p^p((f*F)\phi, r)(1-\rho)^{c-1}(1-r^2)^\alpha r dr \leqslant$$
$$\leqslant \frac{D}{(2\pi)^2}\int_0^1 \left(\frac{bR+a}{bR-a}\right) M_p^p(f,R)(1-r^2)^\alpha r dr.$$

Taking $\rho = r^2$, we get

$$\int_0^1 M_p^p((f*F)\phi,r)(1-r^2)^{\alpha+c-1}rdr \leqslant$$
$$\leqslant \frac{D}{(2\pi)^2}\int_0^1 \left(\frac{bR+a}{bR-a}\right) M_p^p(f,R)(1-r^2)^{\alpha}rdr.$$

Proceeding as in [13], we get

$$\int_0^1 M_p^p((f*F)\phi,r)(1-r^2)^{\alpha+c-1}rdr \leqslant$$
$$\leqslant \frac{D}{(2\pi)^2}\left(\frac{b+a}{b+3a}\right)^{\alpha}\left(\frac{b+a}{b-a}\right)^{\alpha+2}\int_0^1 (1-u^2)^{\alpha}M_p^p(f,u)udu$$

for $-1<\alpha<0$.

$$\|I_\phi(f)\|^p_{A^p_{\alpha+c-1}} \leqslant \frac{D}{(2\pi)^2}\left(\frac{b+a}{b+3a}\right)^{\alpha}\left(\frac{b+a}{b-a}\right)^{\alpha+2}\|f\|^p_{A^p_\alpha}.$$

Hence, the conclusion follows. □

The following results, regarding the boundedness of the composition operator on the Weighted Bergman space, was proved in [13], which is a particular case of $c=1$ of our result, as given in Theorem 3.

Corollary. *Let $\phi:\mathbb{D}\to\mathbb{D}$ be a non-constant analytic function. Then the operator $T_\phi(f)=fo\phi$ on B^p_α satisfies the following inequality:*

$$\|T_\phi\| \leqslant C\left(\frac{\|\phi\|_\infty+|\phi(0)|}{\|\phi\|_\infty-|\phi(0)|}\right)^{\frac{\alpha+2}{p}},$$

where $C=1$ if $\alpha\geqslant 0$ and $C=(\|\phi\|_\infty+|\phi(0)|)^{\frac{\alpha}{p}}(\|\phi\|_\infty+3|\phi(0)|)^{-\frac{\alpha}{p}}$, $-1<\alpha<0$.

Now we state and prove the main result of this article.

Theorem 4. *Let $b,c\in\mathbb{C}$ with $Re(b)>0$ and $c\geqslant 1$. Then the operator $\mathcal{A}^{b,c}$ is bounded on the weighted Bergman space $B^p_{\alpha+c-1}$ if $\alpha+2<p$ and unbounded for $\alpha+2=p$.*

Proof. Case (i) Let $\alpha+2<p$.
Here $p>1$, because $\alpha>-1$. Applying Minkowski's inequality twice and taking $\phi=\phi_{t,s}$ in Theorem 3, we obtain

$$\|\mathcal{A}^{b,c}(f_m)\|_{B^p_{\alpha+c-1}}=$$

$$=K\Big[\int\limits_U\Big|\int\limits_0^1\int\limits_0^1((f*F)(\phi_{t,s}(z)))(1-t)^{b-1}s^{c-2}dsdt\Big|^p(1-|z|^2)^{\alpha+c-1}dm(z)\Big]^{\frac{1}{p}}\leqslant$$

$$\leqslant K\int\limits_0^1\Big(\int\limits_U\Big|\int\limits_0^1(f*F)(\phi_{t,s}(z))(1-t)^{b-1}dt\Big|^p(1-|z|^2)^{\alpha+c-1}dm(z)\Big)^{\frac{1}{p}}s^{c-2}ds\leqslant$$

$$\leqslant K\int\limits_0^1\int\limits_0^1\Big(\int\limits_U\Big|(f*F)(\phi_{t,s}(z))\Big|^p(1-|z|^2)^{\alpha+c-1}dm(z)\Big)^{\frac{1}{p}}(1-t)^{b-1}dts^{c-2}ds=$$

$$=K\int\limits_0^1\int\limits_0^1\|I_\phi(f)\|_{B^p_{\alpha+c-1}}(1-t)^{b-1}dts^{c-2}ds\leqslant$$

$$\leqslant KC_1\|f\|_{B^p_\alpha}\int\limits_0^1\int\limits_0^1\left(\frac{2-2s-t+ts}{t}\right)^{\frac{\alpha+2}{p}}(1-t)^{b-1}dts^{c-2}ds\leqslant$$

$$\leqslant KC_1\|f\|_{B^p_\alpha}2^{\frac{\alpha+2}{p}}\frac{1}{c-1}\int\limits_0^1\frac{1}{t^{\frac{\alpha+2}{p}}}(1-t)^{b-1}dt.$$

Here $K=(1+b-c)(c-1)(\alpha+1)$. The above integral is convergent for $\frac{\alpha+2}{p}<1$. This completes the proof.

Case (ii) Let $\alpha+2>p$.
Suppose $f_1(z)=\frac{1}{1-z}$. Using Lemma 2, we obtain

$$\int\limits_U\frac{1}{|1-z|^p}(1-|z|^2)^\alpha dm(z)=\frac{1}{\pi}\int\limits_0^1(1-r^2)^\alpha\int\limits_{-pi}^{\pi}|1-re^{i\theta}|^{-p}d\theta rdr\leqslant$$

$$\leqslant\frac{2^\alpha C}{\pi}\int\limits_0^1(1-r)^{\alpha+1-p}dr,$$

which is finite for $\alpha+2>p$. Hence, $f_1\in A^p_{\alpha+c-1}$.

(a) For $c > 1$, we have:

$$\mathcal{A}^{b,c}(f_1(z)) = K\int_0^1\int_0^1 (f_1 * F)(1-t-s+ts+tz)(1-t)^{b-1}s^{c-2}dsdt =$$

$$= K\int_0^1\int_0^1 F(1{,}c;1;1-t-s+ts+tz)(1-t)^{b-1}s^{c-2}dsdt =$$

$$= K\int_0^1\int_0^1 \frac{(1-t)^{b-1}s^{c-2}}{(t+s-ts-tz)^c}dsdt,$$

where $K = (1+b-c)(c-1)$. Now we find

$$\mathcal{A}^{b,c}(f_1(0)) = K\int_0^1\int_0^1 \frac{(1-t)^{b-1}s^{c-2}}{(t+s-ts)^c}dsdt =$$

$$= K\int_0^1\int_0^1 (1-t)^{b-1}t^{-c}s^{c-2}\sum_{n=0}^{\infty}\frac{(c)_n}{n!}s^n\frac{(t-1)^n}{t^n}dsdt =$$

$$= K\sum_{n=0}^{\infty}\frac{(c)_n}{n!}(-1)^n\int_0^1 (1-t)^{n+b-1}t^{-(n+c)}dt\int_0^1 s^{n+c-2}ds =$$

$$= K\sum_{n=0}^{\infty}\frac{(c)_n}{n!}(-1)^n\int_0^1 \frac{(1-t)^{n+b-1}}{t^{(n+c)}}dt\int_0^1 s^{n+c-2}ds.$$

Since $n+c > 1$, the first integral diverges. Hence, $\mathcal{A}^{b,c}(f_1(z))$ is unbounded.

(b) For $c = 1$, from (1) we have

$$\mathcal{A}^{b,1}(f(z)) = \sum_{n=0}^{\infty}\Big(\sum_{k=n}^{\infty}\frac{b_{k-n}a_k}{A_k^{b+1;1}}\Big)z^n.$$

For $f_1(z) = \frac{1}{1-z}$, we find

$$\mathcal{A}^{b,1}(f_1(z)) = \sum_{n=0}^{\infty}\Big(\sum_{k=n}^{\infty}\frac{b_{k-n}}{A_k^{b+1;1}}\Big)z^n = \sum_{k=0}^{\infty}\frac{b_k}{A_k^{b+1;1}} + \sum_{n=1}^{\infty}\Big(\sum_{k=n}^{\infty}\frac{b_{k-n}}{A_k^{b+1;1}}\Big)z^n.$$

Hence, we have

$$\mathcal{A}^{b,1}(f_1(0)) = \sum_{k=0}^{\infty} \frac{b_k}{A_k^{b+1;1}} = \sum_{k=0}^{\infty} \frac{(1+b-1)A_k^{b;1}}{bA_k^{b+1;1}} = b\sum_{k=0}^{\infty} \frac{b+k-1}{b+k},$$

which is divergent. □

Remark. *It is still an open question, whether the operator $\mathcal{A}^{b,c}$ is bounded on the weighted Bergman space $B^p_{\alpha+c-1}$ for $\alpha + 2 > p$.*

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