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DOUBLE COSINE-SINE SERIES AND NIKOL'SKII CLASSES IN UNIFORM METRIC

Abstract. In the this paper, we give neccessary and sufficient conditions for a function even with respect to the first argument but odd with respect to the second one to belong to the Nikol'skii classes defined by a mixed modulus of smoothness of a mixed derivative (both have arbitrary integer orders). These conditions involve the growth of partial sum of Fourier cosine-sine coefficients with power weights or the rate of decreasing to zero of these coefficients. A similar problem for generalized "small" Nikol'skii classes is also treated.

Key words: *double cosine-sine series, mixed modulus of smoothness, Nikol'skii classes*

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1. Introduction. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of real numbers, such that $\sum\limits_{k=1}^{\infty}|a_k|<\infty$. Then the functions

$$f(x)=\sum_{k=1}^{\infty}a_k\cos kx \tag{1}$$

and

$$g(x)=\sum_{k=1}^{\infty}a_k\sin kx \tag{2}$$

are continuous and 2π-periodic (i. e., $f,g\in C_{2\pi}$) and (1) is the Fourier series of f, correspondingly, (2) is the Fourier series of g. Lorentz [13] established that the condition $\sum\limits_{k=n}^{\infty}|a_k|=O(n^{-\alpha})$, $n\in\mathbb{N}$, for $0<\alpha<1$

implies $f \in Lip(\alpha)$ or $g \in Lip(\alpha)$. For $a_k \geqslant 0$, Boas [2] proved the following

Theorem 1. *(i) Let* $a_k \geqslant 0$, $k \in \mathbb{N}$, $0 < \alpha < 1$ *and* $\{a_k\}_{k=1}^{\infty}$ *be the sequence of Fourier sine or cosine coefficients of* φ. *Then* $\varphi \in Lip(\alpha)$ *if and only if* $\sum\limits_{k=n}^{\infty} a_k = O(n^{-\alpha})$, $n \in \mathbb{N}$, *or, equivalently,* $\sum\limits_{k=1}^{n} k a_k = O(n^{1-\alpha})$, $n \in \mathbb{N}$.

(ii) If $a_k \geqslant 0$, $k \in \mathbb{N}$, $\{a_k\}_{k=1}^{\infty}$ *is the sequence of Fourier sine coefficients of* g, *then* $g \in Lip(1)$ *if and only if* $\sum\limits_{k=1}^{\infty} k a_k < \infty$.

Similar result to (i) was earlier obtained by Rubinstein [19] (see also Theorem A in [26]).

Nemeth [15] established several generalizations of Theorem A and gave a sharp version of Theorem 3 from [2]. Dyachenko [6] studied trigonometric series with coeffcients of fractional order monotonicity, and obtained conditions for sums of such series to belong Lipschitz classes.

If ω is increasing and continuous on $[0; 2\pi]$, $\omega(0) = 0$, then $\omega \in \Phi$. A function $\omega \in \Phi$ belongs to the Bary class B, if $\sum\limits_{k=n}^{\infty} k^{-1}\omega(k^{-1}) = O(\omega(n^{-1}))$, $n \in \mathbb{N}$; respectively, it belongs to the Bary-Stechkin class B_α, $\alpha > 0$, if $\sum\limits_{k=1}^{n} k^{\alpha-1}\omega(k^{-1}) = O(n^\alpha \omega(n^{-1}))$, $n \in \mathbb{N}$ (see [1]).

In the paper [3] by Butzer et al, several properties of fractional modulus of smoothness $\omega_\beta(f, \delta)$, $\beta > 0$, and its applications to the approximation theory were studied. Tikhonov [21], [22] proved a generalization of the Boas results in the case of fractional modulus of smoothness. In [23], the same author obtained the Boas type results for the Nikol'skii spaces $W^\alpha H^\omega_\beta$ of functions. Let us note, that the previous results of the Boas type connected with Nikol'skii classes belong to Chan [4] and Nemeth [16].

For multiple complex Fourier series one can note the papers by Móricz and Fülöp: [10] and [14]. Their results were generalized by the author of this paper in [26]. For the double cosine-sine series, Tevzadze [20] proved the following

Theorem 2. *Let* $m, n \in \mathbb{N}$, $a_{ik} \geqslant 0$ *for all* $i, k \in \mathbb{N}$, $\sum\limits_{i,k=1}^{\infty} a_{ik} < \infty$, $h(x, y) = \sum\limits_{i,k=1}^{\infty} a_{ik} \cos ix \sin ky$ *and* $\omega(t, \tau)$ *be an increasing in each variable* *function on* $[0,1]^2$*, such that* $\omega(0,0)=0$ *and*

$$\int_t^1\int_\tau^1 \omega(u,v)u^{-m-1}v^{-n-1}\,du\,dv \leqslant C_1\omega(t,\tau)t^{-m}\tau^{-n}, \quad t,\tau\in(0,1], \tag{3}$$

$$\int_0^t\int_\tau^1 \omega(u,v)u^{-1}v^{-n-1}\,du\,dv \leqslant C_2\omega(t,\tau)\tau^{-n}, \quad t,\tau\in(0,1], \tag{4}$$

$$\int_t^1\int_0^\tau \omega(u,v)u^{-m-1}v^{-1}\,du\,dv \leqslant C_3\omega(t,\tau)t^{-m}, \quad t,\tau\in(0,1], \tag{5}$$

$$\int_0^t\int_0^\tau \omega(u,v)u^{-1}v^{-1}\,du\,dv \leqslant C_4\omega(t,\tau), \quad t,\tau\in(0,1]. \tag{6}$$

Then $h\in H^{m,n}(\omega)$ *(see the next section) if and only if*

$$\sum_{i=[p/2]}^{p}\sum_{k=[q/2]}^{q} a_{ik} \leqslant C\omega(1/p,1/q), \quad p,q\in\mathbb{N}\cap[2,+\infty).$$

Similar results were obtained in [20] for double sine and cosine series. Fülöp (see [8] and [9]) gave the necessary and sufficient conditions for sums of sine, cosine, and mixed double series to belong the space $\Lambda_*(2) = H^{2,2}(\omega_{1,1})$ (see the next section), where $\omega_{\alpha,\beta}(u,v) = u^\alpha v^\beta$, $0<\alpha,\ \beta\leqslant 1$. Donskikh [5] proved some multidimesional analogues of Fülöp results. Results from [8] and [9] were generalized by Yu [28] for classes $HH^\omega = H^{1,1}(\omega)$ (see the next section), where ω satisfies the conditions similar to (3)–(6) in the case $m=n=1$. Han, Li, and Yu [11] considered mixed modulus of smoothness of natural orders and obtained

Theorem 3. *Let* $\omega(u,v)$ *be a conitinuous on* $[0,2\pi]^2$*, increasing and subadditive in each variable function, such that* $\omega(0,0)=0$*,* $a_{jk}\geqslant 0$ *for all* $j,k\in\mathbb{N}$*,* $\sum\limits_{j=1}^{\infty}\sum\limits_{k=1}^{\infty} a_{jk}<\infty$*,* $r,s\in\mathbb{N}$*.*

(i) If the following relations

$$\sum_{j=1}^{m}\sum_{k=1}^{n} j^r k^s a_{jk} \;=\; O(m^r n^s\omega(1/m,1/n)), \quad m,n\in\mathbb{N}; \tag{7}$$

$$\sum_{j=1}^{m}\sum_{k=n+1}^{\infty} j^r a_{jk} = O(m^r\omega(1/m,1/n)), \quad m,n\in\mathbb{N}; \tag{8}$$

$$\sum_{j=m+1}^{\infty}\sum_{k=1}^{n} k^s a_{jk} = O(n^s\omega(1/m,1/n)), \quad m,n\in\mathbb{N}; \tag{9}$$

$$\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty} a_{jk} = O(\omega(1/m,1/n)), \quad m,n\in\mathbb{N}; \tag{10}$$

are valid, then $g(x,y)=\sum_{k=1}^{\infty}\sum_{k=1}^{\infty} a_{jk}\sin jx\sin ky$ *belongs to* $H^{m,n}(\omega)$.

(ii) If $g\in H^{m,n}(\omega)$, *then*

$$\sum_{j=1}^{m}\sum_{k=1}^{n} j^{r^*}k^{s^*}a_{jk} = O(m^{r^*}n^{s^*}\omega(1/m,1/n)),$$

where $m,n\in\mathbb{N}$, $r^*=r+1$ *for even* r *and* $r^*=r$ *for odd* r.

Using several conditions on ω, the authors of [11] obtained some criteria for $g\in H^{m,n}(\omega)$ in terms of Fourier coefficients of g, but the confusion in formula numeration in [11, Theorem C] makes understanding of statements hard. The conditions (7)–(10) are not independent (see Lemmas 2 and 4 below). Using these facts, results of Theorem 3 were rewritten in a new form and extended to the Nikol'skii classes in [27].

In the present study, we extend Theorem 3 and its counterpart from [27] to the case of differentiable even with respect to the first argument and odd with respect to the second one functions, using the mixed modulus of smoothness and derivatives of arbitrary natural orders (Theorem 4). In the case of weak monotone Fourier coefficients with a given rate of decreasing, we give the sharp conditions for h from Theorem 2 to belong to the Nikol'skii classes $W^{r,s}H^{m,n}(\omega)$ (Theorem 5). Finally, we obtain an o-analogue of Theorem 4 (Theorem 6). The results are dependent on the evenness of $m+r$ and $n+s$.

2. Definitions. Let $r,s\in\mathbf{Z}_+=\{0,1,\ldots\}$, $\{a_{jk}\}_{j,k\in\mathbb{N}}\subset\mathbb{R}$ and

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} j^r k^s|a_{jk}|<\infty. \tag{11}$$

It follows From (11) that the series

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk}\cos jx\sin ky \tag{12}$$

and

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} j^r k^s a_{jk} \cos(jx + \pi r/2) \sin(ky + \pi s/2) \tag{13}$$

converge absolutely and uniformly to functions $h(x, y)$ and $\psi(x, y)$, respectively. By the classical theorem on differentiabilty of function series, we have

$$h^{(r,s)}(x, y) := \frac{\partial^{r+s} h(x, y)}{\partial x^r \partial y^s} = \psi(x, y)$$

everywhere. Let

$$\dot{\triangle}_{t,\tau}^{m,n} f(x, y) = \sum_{j=0}^{m}\sum_{k=0}^{n}(-1)^{j+k}\binom{m}{j}\binom{n}{k} f(x+(m-2j)t/2, y+(n-2k)\tau/2)$$

be the mixed difference of orders m, n with steps t, τ. For $\dot{\triangle}_{t,\tau}^{m,n} h^{(r,s)}(x, y)$ see Lemma 5. Let us consider the class $\Phi^{(2)}$ of positive on $[0, 2\pi]^2 \setminus \{(0, 0)\}$ functions ω, for which $\omega(0, 0) = 0$, $\omega(x_1, y_1) \leqslant \omega(x_2, y_1)$, $\omega(x_1, y_1) \leqslant \omega(x_1, y_2)$ if $x_2 \geqslant x_1$, $y_2 \geqslant y_1$, $x_i, y_i \in [0, 2\pi]$, $i = 1, 2$.

If $\omega \in \Phi^{(2)}$ is such that

$$\sum_{i=m}^{\infty}\sum_{j=n}^{\infty}(ij)^{-1}\omega\left(\frac{2\pi}{i}, \frac{2\pi}{j}\right) = O\left(\omega\left(\frac{2\pi}{m}, \frac{2\pi}{n}\right)\right), \quad m, n \in \mathbb{N},$$

then ω belongs to the class BB.

If $m, n > 0$ and for $\omega \in \Phi^{(2)}$ the inequality

$$\sum_{i=1}^{j}\sum_{k=1}^{l} i^{m-1}k^{n-1}\omega\left(\frac{2\pi}{i}, \frac{2\pi}{k}\right) = O\left(j^m l^n \omega\left(\frac{2\pi}{j}, \frac{2\pi}{l}\right)\right), \quad j, l \in \mathbb{N},$$

is valid, then ω belongs to the class $B_m B_n$. One-dimensional analogues of these classes were introduced by Bary and Stechkin [1]; for the two-dimensional case see, for example, [28]. For $m, n \in \mathbb{N}$ and $\omega \in \Phi^{(2)}$, we will write $f \in H^{m,n}(\omega)$, if for all $\delta_1, \delta_2 \in [0, 2\pi]$ the inequality

$$\omega_{mn}(f, \delta_1, \delta_2) := \sup\{|\dot{\triangle}_{t,\tau}^{m,n} f(x, y)| : 0 \leqslant t \leqslant \delta_1, 0 \leqslant \tau \leqslant \delta_2\} \leqslant C\omega(\delta_1, \delta_2)$$

holds and $f \in W^{r,s}H^{m,n}(\omega)$, $r, s \in \mathbb{Z}_+$, if $f^{(r,s)}$ exists everywhere and belongs to $H^{m,n}(\omega)$. Learn more about these classes in L^p setting, e. g., in [18]. We will also consider

$$h^{m,n}(\omega) = \{f \in H^{m,n}(\omega) : \omega_{mn}(f, \delta_1, \delta_2) = o(\omega(\delta_1, \delta_2))\},$$

where $\delta_1, \delta_2 \to 0+$, and $W^{r,s}h^{m,n}(\omega)$ are defined similarly to $W^{r,s}H^{m,n}(\omega)$. In the case $r = s = 0$, $m = n = 1$ and $\omega_{\alpha,\beta}(u, v) = u^\alpha v^\beta$, $0 < \alpha, \beta \leqslant 1$, we denote $W^{r,s}H^{m,n}(\omega_{\alpha,\beta})$ by $Lip(\alpha, \beta)$.

We shall write $\omega \in \Delta_2$, if $\omega(2t, \tau) \leqslant C_1\omega(t, \tau)$ for all $2t, \tau \in [0, 2\pi]$ and $\omega(t, 2\tau) \leqslant C_1\omega(t, \tau)$ for all $t, 2\tau \in [0, 2\pi]$.

3. Auxiliary propositions. Lemmas 1–4 are proved in [26].

Lemma 1. *Let $m, n > 0$, $\omega \in \Phi^{(2)}$.*

(i) If $\omega \in B_mB_n$, then $\omega \in \Delta_2$.

(ii) If $\omega \in BB \cap \Delta_2$, then $\omega(\cdot, t) \in B$ for any fixed $t \in [0, 2\pi]$.

Lemma 2. *Let $\{a_{jk}\}_{j,k\in\mathbb{N}} \subset \mathbb{R}_+ = [0, +\infty)$, $\omega \in \Phi^{(2)}$, $m, n > 0$.*

(i) If $\omega \in B_mB_n$, then the condition

$$\sum_{j=M}^{\infty}\sum_{k=N}^{\infty} a_{jk} = O\left(\omega\left(\frac{2\pi}{M}, \frac{2\pi}{N}\right)\right), \quad M, N \in \mathbb{N}, \tag{14}$$

implies

$$\sum_{j=1}^{M}\sum_{k=1}^{N} j^m k^n a_{jk} = O\left(M^m N^n \omega\left(\frac{2\pi}{M}, \frac{2\pi}{N}\right)\right), \quad M, N \in \mathbb{N}. \tag{15}$$

(ii) If $\omega \in BB \cap \Delta_2$, then (14) follows from (15).

Lemma 3. *Let $\{a_{jk}\}_{j,k\in\mathbb{N}} \subset \mathbb{R}_+$, $\omega \in \Phi^{(2)}$, $m, n > 0$.*

(i) If $\omega \in B_mB_n$, $\{a_{jk}\}_{j,k\in\mathbb{N}}$ satisfies (14) and

$$\sum_{j=M}^{\infty}\sum_{k=N}^{\infty} a_{jk} = o\left(\omega\left(\frac{2\pi}{M}, \frac{2\pi}{N}\right)\right), \quad M, N \to \infty, \tag{16}$$

then

$$\sum_{j=1}^{M}\sum_{k=1}^{N} j^m k^n a_{jk} = o\left(M^m N^n \omega\left(\frac{2\pi}{M}, \frac{2\pi}{N}\right)\right), \quad M, N \to \infty. \tag{17}$$

(ii) If $\omega \in BB \cap \Delta_2$, $\{a_{jk}\}_{j,k\in\mathbb{N}}$ satisfies (17), then (16) is valid.

Lemma 4. *(i) Let $\{a_{jk}\}_{j,k\in\mathbb{N}} \subset \mathbb{R}_+$, $\omega \in \Phi^{(2)}$, $\omega(\cdot, t) \in B$ for all $t \in [0, 2\pi]$, $m, n > 0$ and the relation (15) is valid. Then*

$$\sum_{j=M}^{\infty}\sum_{k=1}^{N} k^n a_{jk} = O\left(N^n \omega\left(\frac{2\pi}{M}, \frac{2\pi}{N}\right)\right), \quad M, N \in \mathbb{N}.$$

(ii) If, instead of (15) *in (i) we have* (17), *then*

$$\sum_{j=M}^{\infty}\sum_{k=1}^{N} k^n a_{jk} = o\left(N^n\omega\left(\frac{2\pi}{M},\frac{2\pi}{N}\right)\right), \quad M,N\to\infty.$$

Lemma 5. *If* $m,k\in\mathbb{N}$, $\dot{\Delta}_t^m f(x)=\sum_{j=0}^{m}(-1)^j\binom{m}{j}f(x+(m-2j)t/2)$, *then we have*

$$\dot{\Delta}_t^m \cos kx = (2\sin kt/2)^m\cos(kx+\frac{m\pi}{2}),$$

$$\dot{\Delta}_t^m \sin kx = (2\sin kt/2)^m\sin(kx+\frac{m\pi}{2}).$$

In particular, for h *defined as the sum of* (12) *and* ψ *defined as the sum of* (13), *respectively, under condition* (11), *we obtain*

$$\dot{\Delta}_{t,\tau}^{m,n}h(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk}\cos\left(jx+\frac{m\pi}{2}\right)\sin\left(ky+\frac{n\pi}{2}\right)\times \times(2\sin jt/2)^m(2\sin k\tau/2)^n. \quad (18)$$

$$\dot{\Delta}_{t,\tau}^{m,n}\psi(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} j^r k^s a_{jk}\times \times\cos\left(jx+\frac{(m+r)\pi}{2}\right)\sin\left(ky+\frac{(n+s)\pi}{2}\right)(2\sin jt/2)^m(2\sin k\tau/2)^n. \quad (19)$$

Proof. Let $\dot{\Delta}_t^m f(x)$ be as above. It is known that $\dot{\Delta}_t^m f(x)=\dot{\Delta}_t^1(\dot{\Delta}_t^{m-1}f(x))$ and that $\dot{\Delta}_t^1 e^{ikx}=(2i\sin kt/2)e^{ikx}$; therefore,

$$\dot{\Delta}_t^m\cos kx = \mathrm{Re}(\dot{\Delta}_t^m e^{ikx}) = \mathrm{Re}[(2i\sin kt/2)^m e^{ikx}].$$

The last expression equals to $(-1)^{m/2}\cos kx(2\sin kt/2)^m$ for even m and to $(-1)^{(m+1)/2}\sin kx(2\sin kt/2)^m$ for odd m. Thus, $\dot{\Delta}_t^m\cos kx=(2\sin kt/2)^m\times \times\cos(kx+m\pi/2)$. The second formula is proved in a similar manner. Since $\dot{\Delta}_{t,\tau}^{m,n}h(x,y)$ is the composition of $\dot{\Delta}_t^m$ with respect to x and $\dot{\Delta}_\tau^n f$ with respect to y, we obtain (18). But differentiation and the m-th difference commute, also the equalities $\cos^{(r)}(x)=\cos(x+r\pi/2)$, $\sin^{(r)}(x)=\sin(x+r\pi/2)$ hold, hence (19) is valid. □

Lemma 6. *Let $r, s \in \mathbb{Z}_+$, f be 2π-periodic in each variable continuous function. If $f^{(r,s)}$ exists everywhere and is continuous, then for any $m, n \in \mathbb{N}$ one has*

$$\omega_{m+r,\,n+s}(f, \delta_1, \delta_2) \leqslant \delta_1^r \delta_2^s \omega_{mn}(f^{(r,s)}, \delta_1, \delta_2), \quad \delta_1, \delta_2 \in [0, 2\pi].$$

Lemma 6 may be proved in the same way as similar one-dimensional result (see [25, Ch.3, § 3.3, (1)]) using representation of higher order difference by means of derivative (see [25, Ch.3, § 3.3, (4)]) and the equality of type $\dot{\Delta}_h^{m+r} = \dot{\Delta}_h^r(\dot{\Delta}_h^m)$ with respect to both variables. For this lemma in the case of mixed L^p moduli of smoothness see [17] or [18].

4. Main results

Theorem 4. *(i) Assume that $r, s \in \mathbb{Z}_+$, $\{a_{jk}\}_{j,\,k\in\mathbb{N}}$ satisfies the condition (11) and $h(x, y)$ is the sum of (12). If $m, n \in \mathbb{N}$, $\omega \in BB \cap \Delta_2$ and the condition*

$$\sum_{j=1}^{M}\sum_{k=1}^{N} j^{m+r}k^{n+s}|a_{jk}| = O\left(M^m N^n \omega\left(\frac{2\pi}{M}, \frac{2\pi}{N}\right)\right), \quad M, N \in \mathbb{N}, \quad (20)$$

holds, then $h \in W^{r,s}H^{m,n}(\omega)$.

(ii) Let $\{a_{jk}\}_{j,\,k\in\mathbb{N}} \subset \mathbb{R}_+ = [0, \infty)$ satisfy the condition (11), $h(x, y)$ be the sum of (12), m, n, r, s be as in the part (i). If $\omega \in BB \cap \Delta_2$ and $m+r$ is even, $n+s$ is odd, then from $h \in W^{r,s}H^{m,n}(\omega)$ it follows that the condition (20) is valid. If $m+r$ is odd or $n+s$ is even and $\omega \in B_m B_n \cap BB$, then $h \in W^{r,s}H^{m,n}(\omega)$ also implies (20).

Proof. (i) Let the condition (20) be valid. By (19) from Lemma 5, we have for t, $\tau > 0$ an upper estimate for $|\dot{\Delta}_{t,\tau}^{m,n} h^{(r,s)}(x, y)|$ of the type

$$2^{m+n}\left(\sum_{j=1}^{M}\sum_{k=1}^{N} + \sum_{j=M+1}^{\infty}\sum_{k=1}^{N} + \sum_{j=1}^{M}\sum_{k=N+1}^{\infty} + \sum_{j>M}\sum_{k>N}\right) j^r k^s |a_{jk}| \times$$

$$\times \left|\sin\frac{jt}{2}\right|^m \left|\sin\frac{k\tau}{2}\right|^n =: I_{MN}^{(1)} + I_{MN}^{(2)} + I_{MN}^{(3)} + I_{MN}^{(4)}, \quad (21)$$

where $M = [2\pi/t]$, $N = [2\pi/\tau]$. By virtue of the inequality $|\sin x| \leqslant |x|$, $x \in \mathbb{R}$, and (20), we find that for $t, \tau \in (0, 2\pi]$

$$I_{MN}^{(1)} \leqslant t^m \tau^n \sum_{j=1}^{M}\sum_{k=1}^{N} j^{m+r}k^{n+s}|a_{jk}| = O(\omega(t, \tau)).$$

Using Lemma 5, inequalities $|\sin x| \leqslant 1$ and $|\sin x| \leqslant |x|$, $x \in \mathbb{R}$, we have

$$I_{MN}^{(2)} \leqslant 2^m \tau^n \sum_{j=M+1}^{\infty} \sum_{k=1}^{N} j^r k^{n+s} |a_{jk}|. \tag{22}$$

One may rewrite (20) in the form

$$\sum_{j=1}^{M} \sum_{k=1}^{N} j^m k^n (j^r k^s |a_{jk}|) = O\left(M^m N^n \omega\left(\frac{2\pi}{M}, \frac{2\pi}{N}\right)\right), \quad M, N \in \mathbb{N}, \tag{23}$$

and by Lemma 1(ii) and Lemma 4(i) we obtain

$$\sum_{j=M+1}^{\infty} \sum_{k=1}^{N} k^n (j^r k^s |a_{jk}|) = O\left(N^n \omega\left(\frac{2\pi}{M}, \frac{2\pi}{N}\right)\right), \quad M, N \in \mathbb{N}. \tag{24}$$

From (22) and (24) we deduce that $I_{MN}^{(2)} = O(\omega(t,\tau))$, $t,\tau \in [0, 2\pi]$. Similarly, we estimate $I_{MN}^{(3)}$. Finally, by the condition $\omega \in BB \cap \Delta_2$ and part (ii) of Lemma 2 (we again write (20) in the form (23))

$$I_{MN}^{(4)} \leqslant \sum_{j=M+1}^{\infty} \sum_{k=N+1}^{\infty} 2^{m+n} j^r k^s |a_{jk}| = O(\omega(t,\tau)), \;\; t, \tau \in [0, 2\pi].$$

Combining the obtained estimates, we see that

$$|\Delta_{t,\tau}^{m,n} h^{(r,s)}(x, y)| = O(\omega(t,\tau)), \quad t, \tau \in [0, 2\pi],$$

whence the statement $h \in W^{r,s} H^{m,n}(\omega)$ follows.

(ii) In Steps I-IV, we assume that $r = s = 0$ and set $t = M^{-1}$, $\tau = N^{-1}$ for $M, N \in \mathbb{N}$.

Step I. Let m be even, n be odd. Then, by Lemma 5, we have

$$C_1 \omega(t,\tau) \geqslant |\dot{\Delta}_{t,\tau}^{m,n} h(0,v)| = \left| \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk} (2 \sin jt/2)^m (2 \sin k\tau/2)^n \cos kv \right|. \tag{25}$$

Since the series in the right-hand side of (25) converges uniformly in v, it may be integrated term by term over $v \in [-\tau/2, \tau/2]$, and we obtain

$$C_1 \tau \omega(t,\tau) \geqslant \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} k^{-1} (2 \sin jt/2)^m (2 \sin k\tau/2)^{n+1} a_{jk} \geqslant$$

$$\geqslant C_2 \sum_{j=1}^{M} \sum_{k=1}^{N} (jt)^m (k\tau)^{n+1} k^{-1} a_{jk}$$

and

$$\sum_{j=1}^{M} \sum_{k=1}^{N} j^m k^n a_{jk} \leqslant C_3 M^m N^n \omega(t,\tau) \leqslant C_3 M^m N^n \omega(2\pi/M, 2\pi/N).$$

Step II. Let m, n be even. Then, by Lemma 5, we have

$$C_1\omega(t,\tau) \geqslant |\dot{\Delta}_{t,\tau}^{m,n} h(0,v)| = \Big| \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk} (2\sin jt/2)^m (2\sin k\tau/2)^n \sin kv \Big|.$$

The series in the right-hand side may be integrated term by term over $v \in [0,\tau]$, and we obtain

$$C_1\tau\omega(t,\tau) \geqslant \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} k^{-1} (2\sin jt/2)^m (2\sin k\tau/2)^n a_{jk} (1-\cos k\tau) =$$

$$= 2^{-1} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} k^{-1} (2\sin jt/2)^m (2\sin k\tau/2)^{n+2} a_{jk} \geqslant$$

$$\geqslant C_4 \sum_{j=1}^{M} \sum_{k=1}^{N} (jt)^m (k\tau)^{n+2} k^{-1} a_{jk}$$

and

$$\sum_{j=1}^{M} \sum_{k=1}^{N} j^m k^{n+1} a_{jk} \leqslant C_5 M^m N^{n+1} \omega(2\pi/M, 2\pi/N), \quad M, N \in \mathbb{N}.$$

By the condition $\omega \in BB \cap \Delta_2$ (see Lemma 1) and Lemma 2 (ii), we obtain (14), while the condition $\omega \in B_m B_n$, (14) and Lemma 2 (i) imply (20).

Step III. Let m, n be odd. Then, by Lemma 5, we have

$$C_1\omega(t,\tau) \geqslant |\dot{\Delta}_{t,\tau}^{m,n} h(u,v)| =$$

$$= \Big| \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk} (2\sin jt/2)^m (2\sin k\tau/2)^n \sin ju \cos kv \Big| \qquad (26)$$

for $u, v \in \mathbb{R}$. The series in the right-hand side of (26) uniformly converges in v and may be integrated term by term over $v \in [-\tau/2, \tau/2]$. Therefore,

$$C_1\tau\omega(t,\tau) \geqslant \Big|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} k^{-1}a_{jk}(2\sin jt/2)^m(2\sin k\tau/2)^{n+1}\sin ju\Big|, \quad u \in \mathbb{R}. \tag{27}$$

Substituting $u = t/2$ into (27), we obtain

$$\begin{aligned} C_1\tau\omega(t,\tau) &\geqslant 2^{-1}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} k^{-1}a_{jk}(2\sin jt/2)^{m+1}(2\sin k\tau/2)^{n+1} \geqslant \\ &\geqslant C_6\sum_{j=1}^{M}\sum_{k=1}^{N}(jt)^{m+1}(k\tau)^{n+1}k^{-1}a_{jk} \end{aligned}$$

and

$$\sum_{j=1}^{M}\sum_{k=1}^{N} j^{m+1}k^n a_{jk} \leqslant C_7 M^{m+1}N^n\omega(2\pi/M, 2\pi/N), \quad M,N \in \mathbb{N}.$$

By Lemmas 2 and 1 and the condition $\omega \in BB \cap B_mB_n$, we deduce, similarly to Step II, that (20) holds.

Step IV. Let m be odd, n be even. By Lemma 5, we have

$$C_1\omega(t,\tau) \geqslant \Big|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk}(2\sin jt/2)^m(2\sin k\tau/2)^n\sin ju\sin kv\Big|, \ u, v \in \mathbb{R}. \tag{28}$$

Integrating (28) over $v \in [0, \tau]$, we obtain

$$C_1\tau\omega(t,\tau) \geqslant \Big|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} k^{-1}a_{jk}(2\sin jt/2)^m(2\sin k\tau/2)^n\sin ju(1-\cos k\tau)\Big|, \tag{29}$$

where $u \in \mathbb{R}$. Substituting $u = t/2$ into (29), we obtain

$$\begin{aligned} C_1\tau\omega(t,\tau) &\geqslant 4^{-1}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} k^{-1}a_{jk}(2\sin jt/2)^{m+1}(2\sin k\tau/2)^{n+2} \geqslant \\ &\geqslant C_8\sum_{j=1}^{M}\sum_{k=1}^{N}(jt)^{m+1}(k\tau)^{n+2}k^{-1}a_{jk} \end{aligned}$$

and

$$\sum_{j=1}^{M}\sum_{k=1}^{N} j^{m+1}k^{n+1}a_{jk} \leqslant C_7 M^{m+1}N^{n+1}\omega(2\pi/M, 2\pi/N), \quad M, N \in \mathbb{N}.$$

Using Lemmas 2 and 1 and the condition $\omega \in BB \cap B_m B_n$, we deduce, again, that (20) holds.

Step V. Secondly, we consider the general case $r, s \in \mathbb{Z}_+$.
If $h \in W^{r,s}H^{m,n}(\omega)$, then, by Lemma 6,

$$\omega_{m+r,\, n+s}(h, \delta_1, \delta_2) \leqslant \delta_1^r \delta_2^s \omega_{m,n}(h^{(r,s)}, \delta_1, \delta_2) \leqslant C_5 \delta_1^r \delta_2^s \omega(\delta_1, \delta_2),$$

i. e., $W^{r,s}H^{m,n}(\omega) \subset H^{m+r,\, n+s}(\Omega_{r,s})$, where $\Omega_{r,s}(\delta_1, \delta_2) = \delta_1^r \delta_2^s \omega(\delta_1, \delta_2)$. If $\omega \in BB \cap \Delta_2$, then $\Omega_{r,s}$ also belongs to $BB \cap \Delta_2$, while if $\omega \in B_m B_n$, then, by definition, $\Omega_{r,s}$ belongs to $B_{m+r,\, n+s}$. Applying the results obtained in Steps I-IV in all cases, we have

$$\sum_{j=1}^{M}\sum_{k=1}^{N} j^{m+r}k^{n+s}a_{jk} = O\Big(M^{m+r}N^{n+s}\Omega_{r,s}\Big(\frac{2\pi}{M}, \frac{2\pi}{N}\Big)\Big) = \\ = O\Big(M^m N^n \omega\Big(\frac{2\pi}{M}, \frac{2\pi}{N}\Big)\Big).$$

Thus, (20) holds under conditions of part (ii) of Theorem 1. □

Corollary 1. *If $m = n = 1$, $\omega \in BB \cap B_1 B_1$, $\{a_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{R}_+$ satisfies (11) and $h(x, y)$ is the sum of (12), then the conditions*

$$\sum_{j=1}^{M}\sum_{k=1}^{N} jka_{jk} = O(MN\omega(2\pi/M, 2\pi/N)), \quad M, N \in \mathbb{N},$$

and $h \in H^{1,1}(\omega)$ are equivalent.

Corollary 2. *If $m = 2$, $n = 1$, $\omega \in BB \cap \Delta_2$, $\{a_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{R}_+$ satisfies (11) and $h(x, y)$ is the sum of (12), then the conditions*

$$\sum_{j=1}^{M}\sum_{k=1}^{N} j^2ka_{jk} = O(M^2N\omega(2\pi/M, 2\pi/N)), \quad M, N \in \mathbb{N},$$

and $h \in H^{2,1}(\omega)$ are equivalent. In particular, $\omega_{2,1}(f, \delta_1, \delta_2) = O(\delta_1^2\delta_2)$, $\delta_1, \delta_2 \in [0, 2\pi]$, if and only if the series $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} j^2ka_{jk}$ converges.

Corollary 3. *Let $\{a_{jk}\}_{j,k=1}^{\infty}$ and h be as in Corollary 2. Then, $h \in \Lambda_*(2)$ (see Introduction) if and only if the condition*

$$\sum_{j=M}^{\infty}\sum_{k=N}^{\infty} a_{jk} = O(M^{-1}N^{-1}), \quad M,N \in \mathbb{N},$$

holds.

Remark. *Corollary 3 is obtained by Fülöp [9] together with its "small" analogue that may be derived from Theorem 6. Let us note that a particular case $\omega(\delta_1,\delta_2) = \delta_1^2\delta_2$ in Corollary 2 corresponds to the exclusive case (ii) in Theorem 1.*

5. Concluding remarks. We say that $\{a_{jk}\}_{j,k=1}^{\infty}$ is weak monotone if $a_{jk} \geqslant 0$ for all j,k and $a_{ij} \leqslant Ca_{kl}$ for all $i \in [k,2k-1]$, $j \in [l,2l-1]$. The famous Lorentz theorem [13] states that if $\{a_n\}_{n=1}^{\infty}$ decreases to zero and $0 < \alpha < 1$, then the assertions (i) $a_n = O(n^{-\alpha-1})$, $n \in \mathbb{N}$, (ii) $f(x) = \sum\limits_{n=1}^{\infty} a_n \cos nx \in Lip(\alpha)$ and (iii) $g(x) = \sum\limits_{n=1}^{\infty} a_n \sin nx \in Lip(\alpha)$ are equivalent. Several one-dimensional generalizations of the Lorentz theorem to generalized Lipschitz or Nikol'skii spaces and classes of general monotone sequences may be found in papers of Tikhonov [23] and [24]. The following theorem is an extension of the Lorentz result and results from [23] to the two-dimensional mixed case. Applications of one-dimensional weak monotonicty can be found in [12]. Also, we note the paper by Dyachenko and Tikhonov [7], where the estimates of Fourier coefficients satisfying another definition of weak monotonicity are given.

Theorem 5. *Let $\{a_{jk}\}_{j,k=1}^{\infty}$ be weak monotone, $m,n \in \mathbb{N}$, $r,s \in \mathbb{Z}_+$, $\omega \in BB \cap B_mB_n$. If $\{a_{jk}\}_{j,k=1}^{\infty}$ satisfies (11), $h(x,y)$ is the sum of (12), then the conditions*

(i) $h \in W^{r,s}H^{m,n}(\omega)$;
and
(ii) $a_{jk} = O(j^{-r-1}k^{-s-1}\omega(2\pi/j, 2\pi/k))$, $j,k \in \mathbb{N}$, are equivalent.

The proof of Theorem 5 is similar to the proof of Theorem 2 from [27].

The last theorem is the o-analog of Theorem 4.

Theorem 6. *(i) Let $m,n \in \mathbb{N}$, $r,s \in \mathbb{Z}_+$, $\omega \in BB \cap \Delta_2$. If $\{a_{jk}\}_{j,k=1}^{\infty}$ satisfies (11), h is the sum of (12) and the conditions (20) and*

$$\sum_{j=1}^{M}\sum_{k=1}^{N} j^{m+r}k^{n+s}|a_{jk}| = o(M^mN^n\omega(2\pi/M, 2\pi/N)), \quad M,N \to \infty, \quad (30)$$

are valid, then $h \in W^{r,s}h^{m,n}(\omega)$.

(ii) If $m,n \in \mathbb{N}$, $r,s \in \mathbb{Z}_+$, $a_{j,k} \geqslant 0$ *for all* $j,k \in \mathbb{N}$, $\{a_{jk}\}_{j,k=1}^{\infty}$ *satisfies* (11), h *is the sum of* (12), $\omega \in BB \cap \Delta_2$, $m+r$ *is even,* $n+s$ *is odd, and* $h \in W^{r,s}h^{m,n}(\omega)$, *then* (30) *is valid. If, in addition to the previous conditions of (ii), we have* $\omega \in B_m B_n$, *then also* (30) *is valid.*

Proof. Similarly to the proof of Theorem 4, we use the estimate (21) for $t,\tau \in (0,2\pi]$ and $M = [2\pi/t]$, $N = [2\pi/\tau]$. By (30) and the condition $\omega \in \Delta_2$, we find that

$$I_{MN}^{(1)} \leqslant \sum_{j=1}^{M}\sum_{k=1}^{N} j^r k^s |a_{jk}| (jt)^m (k\tau)^n \leqslant$$

$$\leqslant \varepsilon t^m \tau^n M^m N^n \omega(2\pi/M, 2\pi/N) \leqslant C_1 \varepsilon \omega(t,\tau), \quad 0 < t,\tau < \delta_1(\varepsilon). \tag{31}$$

By Lemma 4 (ii) applied to $\{j^r k^s a_{jk}\}_{j,k=1}^{\infty}$ and (30) for $M,N > n_0(\varepsilon)$, we obtain

$$I_{MN}^{(2)} \leqslant C_2 \tau^n \sum_{j=M+1}^{\infty}\sum_{k=1}^{N} j^r k^{n+s} a_{jk} < \varepsilon \tau^n N^n \omega\left(\frac{2\pi}{M}, \frac{2\pi}{N}\right) \leqslant C_3 \varepsilon \omega(t,\tau), \tag{32}$$

where $0 < t,\tau < \delta_2(\varepsilon)$. The quantity $I_{MN}^{(3)}$ is estimated in a similar manner for $0 < t,\tau < \delta_3(\varepsilon)$. By the condition $\omega \in BB \cap \Delta_2$ and Lemma 3 (ii), we have $I_{MN}^{(4)} = o(M^m N^n \omega(2\pi/M, 2\pi/N))$, $M,N \to \infty$, whence

$$I_{MN}^{(4)} \leqslant C_4 \varepsilon \omega(t,\tau), \quad 0 < t,\tau < \delta_4(\varepsilon). \tag{33}$$

From estimates (31)–(33) we find that $|\dot{\Delta}_{t,\tau}^{m,n} h^{(r,s)}(x,y)| \leqslant C_5 \varepsilon \omega(t,\tau)$ for all $t,\tau < \delta(\varepsilon) := \min_{1\leqslant i\leqslant 4} \delta_i(\varepsilon)$, i. e. $h \in W^{r,s}h^{m,n}(\omega)$.

(ii) The assertion is proved similarly to the proof of (ii) from Theorem 4. We may substitute in all steps of this proof C_1 instead of η and get at the end of Steps I-IV an estimate of type $K\eta M^m N^n \omega(2\pi/M, 2\pi/N)$, where K depends on m,n. Setting $K\eta = \varepsilon$, we finish the proof. □

Remark. *It is interesting to obtain a variant of Theorem 4 without the condition* $\omega \in B_m B_n$. *The first attempt in this direction may be found in [27, Theorem 3], but this result gives only sufficient conditions for an odd in each argument functions to belong to a Nikol'skii class.*

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