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GREEN ENERGY AND EXTREMAL DECOMPOSITIONS

Abstract. We give two precise estimates for the Green energy of a discrete charge, concentrated in an even number of points on the circle, with respect to the concentric ring. The lower estimate for the Green energy is attained for the points with a nonstandard symmetry. The well-known Pólya-Schur inequality for the logarithmic energy is a special case of this estimate. The proof is based on the application of dissymmetrization and an asymptotic formula for the conformal capacity of a generalized condenser in the case when some of its plates contract to given points. The upper bound is established for a charge that takes values of opposite signs. Its proof reduces to solving a problem on the so-called extremal decomposition of a circular ring with free poles on a circle.

Key words: *Green energy, discrete charge, dissymmetrization, extremal decompositions*

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1. Introduction and statement of results. There are many studies related to the extremal problems for different kinds of energy of a discrete charge (see, e. g., the papers [2–4], [9], [10], [12], and the references therein). In contrast to the previous research, we consider the Green energy (see also the recent articles [1] and [5]). In addition, an extremal problem with an alternating charge is studied.

Let θ_k, $k = 1, \ldots, 2n$, be real numbers, such that

$$\theta_1 < \theta_2 < \ldots < \theta_{2n} < \theta_1 + 2\pi$$

($n \geqslant 2$), and let $Z = \{z_k\}_{k=1}^{2n}$, $z_k = \exp(i\theta_k)$, $k = 1, \ldots, 2n$. Denote by

$$E(Z, B) = \sum_{k=1}^{2n} \sum_{\substack{l=1 \\ l \neq k}}^{2n} g_B(z_k, z_l)$$

the Green energy of the collection Z with respect to the ring $B := \{z : \ R_1 < |z| < R_2\}$, $0 < R_1 < 1 < R_2 < \infty$. Here $g_B(z, z_k)$ is the Green function of the domain B with poles at z_k, $k = 1, \ldots, 2n$.

The following statement is valid.

Theorem 1. *For any collection of points Z, and for any numbers R_1, R_2, the inequality*

$$E(Z, B) \geqslant E(Z^*, B) \tag{1}$$

holds, where $Z^* = \{z_k^*\}_{k=1}^{2n}$, and the symmetrically located points z_k^* are defined by the relations $|z_k^*| = 1$, $\arg z_{2j-1}^* = -\eta/(2n) + 2\pi j/n$,

$$\arg z_{2j}^* = \eta/(2n) + 2\pi j/n, \ j = 1, \ldots, n, \ \eta = \sum_{j=1}^{n}(\theta_{2j} - \theta_{2j-1}).$$

The proof of Theorem 1 is based on the theory of the condenser capacities and dissymmetrization. These ideas go back to the book [4]. By taking the limit $\eta \to 0$ in (1), we obtain an inequality for the Green energy of a charge concentrated in an unrestricted number of points (not necessary even). Passing to the limit $R_1 \to 0$, $R_2 \to \infty$, inequality (1) gives the classical Pólya-Schur inequality

$$\prod_{k=1}^{n}\prod_{\substack{l=1 \\ l \neq k}}^{n}|z_k - z_l| \leqslant \prod_{k=1}^{n}\prod_{\substack{l=1 \\ l \neq k}}^{n}|z_k^* - z_l^*| = n^n,$$

where z_k, $k = 1, \ldots, n$, are some points on the unit circle $|z| = 1$ and $z_k^* = \exp(2\pi i k/n)$, $k = 1, \ldots, n$.

Theorem 2. *Let Z and B be as above. Then, for the discrete Green energy*

$$\mathcal{E}(Z, B) := \sum_{k=1}^{2n}\sum_{\substack{l=1 \\ l \neq k}}^{2n}(-1)^{k+l}g_B(z_k, z_l),$$

we have

$$\mathcal{E}(Z, B) \leqslant \mathcal{E}(\mathcal{Z}^*, B), \tag{2}$$

where $\mathcal{Z}^* = \{z_k^*\}_{k=1}^{2n}$, $z_k^* = \exp(\pi i k/n)$, $k = 1, \ldots, 2n$.

The proof of inequality (2) is carried out by reduction to the extremal decomposition problems, which have a long history and many applications [6], [11]. A significant contribution to the solution of such problems was

made by N. A. Lebedev [13]. By taking the limit $R_1 \to 0$, $R_2 \to \infty$ in (2), we have

$$\prod_{k=1}^{n}\prod_{\substack{l=1 \\ l \neq k}}^{n} |z_k - z_l|^{(-1)^{k+l}} \geqslant \left(\frac{n}{2}\right)^{2n}.$$

This inequality has been proved in [7] (see also [6, p. 127]).

2. Proof of Theorem 1. The notions and notation from the book [6] will be used in the proof below. For sufficiently small $r > 0$ the condenser

$$C^*(r) := (B, \{E_k^*\}_{k=0}^{2n}, \Delta^*),$$

where $E_0^* = \partial B$, $E_k^* = \{z : |z - z_k^*| \leqslant r\}$, $k = 1, \ldots, 2n$, and $\Delta^* = \{0, 1, \ldots, 1\}$ is well defined [6, p. 33]. Let Φ be the group of symmetries of $\overline{\mathbb{C}}$, formed by the compositions of the reflections with respect to the rays $\{z : \arg z = 2\pi k/n\}$, $k = 1, \ldots, n$ and the bisectors of the angles formed by these rays ("dihedral group"). The condenser $C^*(r)$ is symmetric with respect to the group Φ (Φ-symmetric). By Lemma 4.3 from [6], there exist a dissymmetrization Dis, such that $\text{Dis}\, E^* = E$. Here

$$E^* = \bigcup_{j=1}^{n}\left\{z : |z| = 1,\ -\frac{\eta}{2n} + \frac{2\pi j}{n} \leqslant \arg z \leqslant \frac{\eta}{2n} + \frac{2\pi j}{n}\right\},$$

$$E = \bigcup_{j=1}^{n}\{z : |z| = 1, \theta_{2j-1} \leqslant \arg z \leqslant \theta_{2j}\}.$$

According to Theorem 4.14 [6], we obtain the inequality for the capacities:

$$\text{cap}\, C^*(r) \geqslant \text{cap}\, \text{Dis}\, C^*(r). \tag{3}$$

It is clear that

$$\text{Dis}\, C^*(r) = (B, \{E_k\}_{k=0}^{2n}, \Delta^*),$$

where $E_0 = \partial B$ and $E_k = \{z : |z - z_k| \leqslant r\}$, $k = 1, \ldots, 2n$.

In view of Theorem 2.1 from [6], the following asymptotic equalities hold as $r \to 0$:

$$\text{cap}\, C^*(r) = -\frac{4\pi n}{\log r} -$$
$$- 2\pi \left\{\sum_{k=1}^{2n} \log r(B, z_k^*) + E(Z^*, B)\right\}\left(\frac{1}{\log r}\right)^2 + o\left(\left(\frac{1}{\log r}\right)^2\right),$$

$$\operatorname{cap} \operatorname{Dis} C^*(r) = -\frac{4\pi n}{\log r} -$$

$$- 2\pi \Big\{ \sum_{k=1}^{2n} \log r(B, z_k) + E(Z, B) \Big\} \Big(\frac{1}{\log r} \Big)^2 + o\Big(\Big(\frac{1}{\log r} \Big)^2 \Big),$$

where $r(B, z_k)$ is the inner radius of B with respect to the point z_k, $k = 1, \ldots, n$. Note that

$$r(B, z_k^*) = r(B, z_k), \ k = 1, \ldots, 2n.$$

Substituting these equalities in (3), we obtain the required inequality (1).

3. Proof of Theorem 2. Consider at the condenser

$$\mathcal{C}(r) = (B, \{E_k\}_{k=0}^{2n}, \Delta),$$

where B, E_k, $k = 0, 1, \ldots, 2n$, as in Section 2, $\Delta = \{0, 1, -1, 1, \ldots, -1\}$. Let u be the potential function of $\mathcal{C}(r)$ [6, p. 13]. In view of the asymptotic formula (2.10) [6], for $r \to 0$

$$\int_{B \setminus \cup_{k=1}^{2n} E_k} |\nabla u|^2 = \operatorname{cap} \mathcal{C}(r) = -\frac{4\pi n}{\log r} -$$

$$- 2\pi \Big\{ \sum_{k=1}^{2n} \log r(B, z_k) + \mathcal{E}(Z, B) \Big\} \Big(\frac{1}{\log r} \Big)^2 + o\Big(\Big(\frac{1}{\log r} \Big)^2 \Big). \quad (4)$$

Consider the family of functions $\{u_k\}_{k=1}^{2n}$ that are defined in the sectors $\overline{D}_k$, $D_k = \{z \in B : \theta_k < \arg z < \theta_{k+1}\}$, $k = 1, \ldots, 2n$, respectively $(\theta_{2n+1} = \theta_1 + 2\pi)$. For every k, the function u_k is continuous in $\overline{D}_k$, harmonic in $D_k \setminus (E_k \cup E_{k+1})$,

$$u_k(z) = \begin{cases} (-1)^{k+1}, & z \in E_k \cap \overline{D}_k; \\ (-1)^{k+2}, & z \in E_{k+1} \cap \overline{D}_k; \\ 0, & z \in (\partial D_k) \cap (\partial B), \end{cases}$$

and satisfies conditions: $\partial u_k / \partial n = 0$ on $(\partial D_k) \cap B \setminus (E_k \cup E_{k+1})$, $k = 1, \ldots, 2n$. By Dirichlet's principle,

$$\int_{D_k \setminus (E_k \cup E_{k+1})} |\nabla u|^2 \geqslant \int_{D_k \setminus (E_k \cup E_{k+1})} |\nabla u_k|^2, \quad k = 1, \ldots, 2n.$$

The symmetry principle for the harmonic functions gives

$$\int\limits_{D_k\setminus(E_k\cup E_{k+1})} |\nabla u_k|^2 = \int\limits_{D_k\setminus H_k} |\nabla\omega_k|^2,$$

where $H_k = \{z : |z - \zeta_k| \leqslant r\}$, $\zeta_k = \exp(i(\theta_{k+1} - \theta_k)/2)$, and the function ω_k is continuous in $\overline{D}_k$, harmonic in $D_k\setminus H_k$, is equal to 1 on H_k and to zero on ∂D_k, $k = 1, \ldots, 2n$. Once again, using formula (2.10) [6], we conclude that

$$\int\limits_{D_k\setminus(E_k\cup E_{k+1})} |\nabla u|^2 \geqslant \int\limits_{D_k\setminus H_k} |\nabla\omega_k|^2 =$$

$$= -\frac{2\pi}{\log r} - 2\pi[\log r(D_k, \zeta_k)]\left(\frac{1}{\log r}\right)^2 + o\left(\left(\frac{1}{\log r}\right)^2\right), r \to 0, \quad (5)$$

$k = 1, \ldots, 2n$. Note that a suitable branch of the logarithm $w = \log(z/\zeta_k)$ maps the sector D_k conformally and univalently onto the rectangle $G_k = \{w : \log R_1 < \mathrm{Re}\, w < \log R_2, |\mathrm{Im}\, w| < (\theta_{k+1} - \theta_k)/2\}$, $k = 1, \ldots, 2n$. The result of the Marcus radial averaging transformation [14] of the family $\{G_k\}_{k=1}^{2n}$ (with weights $\alpha_k = 1/(2n)$, $k = 1, \ldots, 2n$) belongs to the rectangle $G = \{w : \log R_1 < \mathrm{Re}\, w < \log R_2, |\mathrm{Im}\, w| < \pi/(2n)$ (also, see [6, p. 83] and [8]). By the Marcus theorem,

$$\prod_{k=1}^{2n} r(D_k, \zeta_k) = \prod_{k=1}^{2n} r(G_k, 0) \leqslant r^{2n}(G, 0).$$

Taking into account (4) and (5), we find

$$\sum_{k=1}^{2n} \log r(B, z_k) + \mathcal{E}(Z, B) \leqslant 2n \log r(G, 0).$$

It is straightforward to see that in the case $z_k = \exp(\pi i k/n)$, $k = 1, \ldots, 2n$, we have the equality sign in the last relation. This yields the required inequality.

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