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CONSTRUCTIVE DESCRIPTION OF FUNCTION CLASSES ON SURFACES IN $\mathbb{R}^3$ AND $\mathbb{R}^4$

Dedicated to the memory of the brilliant mathematician N.A. Lebedev

Abstract. Functional classes on a curve in a plane (a partial case of a spatial curve) can be described by the approximation speed by functions that are harmonic in three-dimensional neighbourhoods of the curve. No constructive description of functional classes on rather general surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$ has been presented in literature so far. The main result of the paper is Theorem 1.

Key words: *constructive description, rational functions, harmonic functions, pseudoharmonic functions*

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1. Introduction. Describing functional classes on sets in terms of speed of approximation of the functions from the class by polynomials, rational functions, harmonic functions, and other types of functions, has been an important branch of real and complex analysis. For example, many works have been devoted to such description of classes of functions that are analytical in a domain and continuous in its closure (see, e. g., references within the book [4]). It turned out that classes of functions on a flat curve has its own peculiarity [2, 3, 5]. Up to the work [1], there were no constructive description of function classes on a curve in $\mathbb{R}^3$ with using polynomials, rational functions, harmonic functions, or any other functions used for approximation. In [1] functions harmonic in a neighbourhood of the curve were used; the approximation speed or estimation of their gradient depended on this neighbourhood. A curve on a plane can be treated as a partial case of a spatial curve; then functional classes on this curve can be described not only in terms of approximation speed by analytical polynomials, but also in terms of functions harmonic in three-dimensional neighbourhoods of the curve.

To our knowledge, no constructive description of functional classes on rather general surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$ has been yet presented. Our work is devoted to this question.

2. The results. Let Q be the square $\{(s,t)\colon 0 \leqslant s \leqslant 1,\, 0 \leqslant t \leqslant 1\}$, $F\colon Q \to \mathbb{R}^4$ be the mapping $(s,t) \to (x_1(s,t), x_2(s,t), x_3(s,t), x_4(s,t))$ with the properties

$$|x_k(s,t) - x_k(s_0,t_0)| \leqslant M_1\sqrt{(s-s_0)^2 + (t-t_0)^2}, 1 \leqslant k \leqslant 4,$$

$(s,t), (s_0,t_0) \in Q$ for some $M_1 > 0$ and

$$\|F(s,t) - F(s_0,t_0)\| \geqslant M_2\sqrt{(s-s_0)^2 + (t-t_0)^2} \text{ for some } M_2 > 0.$$

Let $S = F(Q)$ and denote by Ω_δ, $\delta > 0$, the set

$$\Omega_\delta = \{X \in \mathbb{R}^4,\, dist(X,S) < \delta\}. \tag{1}$$

By $\omega(x)$ we denote the modulus of continuity. Let for some $x > 0$ the following inequality hold:

$$\int_0^x \frac{\omega(t)}{t} dt + x\int_x^\infty \frac{\omega(t)}{t^2} dt \leqslant C_0\omega(x),$$

where $C_0 > 0$. Let $H^\omega(S)$ be a set of complex-valued functions f on S with the property

$$|f(x_2) - f(x_1)| \leqslant C_f\omega(\|x_2 - x_1\|),\ x_1, x_2 \in S.$$

Denote the set of all functions harmonic in Ω_δ by $H(\Omega_\delta)$.

The following statements hold.

Theorem 1. *Let $f \in H^\omega(S)$. Then for any δ, $0 < \delta < 1$, there exists a function $U_\delta \in H(\Omega_\delta)$, that satisfies the conditions*

$$|f(X) - U_\delta(X)| \leqslant C_1\omega(\delta),\ X \in S, \tag{2}$$

and

$$\|\mathbf{grad}\, U_\delta(X)\| \leqslant C_2\frac{\omega(\delta)}{\delta},\ X \in \Omega_\delta. \tag{3}$$

for some constant $C_1 = C_{1f}$ and $C_2 = C_{2f}$.

In the partial case of $x_4(s,t) \equiv 0$, S can be treated as a surface embedded into $\mathbb{R}^3$; however, the approximating functions U_δ are assumed to be harmonic in four-dimensional domains Ω_δ defined in (1) with $S = F((s,t)) = (x_1(s,t), x_2(s,t), x_3(s,t), 0)$. The estimation (3) remains valid in Ω_δ.

In the proof of the Theorem, we will use the continuation of f on $\mathbb{R}^4$ called pseudo-harmonic, similarly to continuation of a function from a curve in [1].

Theorem 2. *Let $f \in H^\omega(S)$. Then there exists a function f_0, defined on $\mathbb{R}^4$, with the following properties:*

$$f_0 \,|_s = f, \quad f_0(X) = 0, \quad X \in \mathbb{R}^4 \setminus \Omega_C, \tag{4}$$

$$f_0 \in C(\mathbb{R}^4), \quad \|\mathbf{grad} f_0(X)\| \leqslant C_{3f} \frac{\omega(dist(X,S))}{dist(X,S)}, \tag{5}$$

$$|\Delta f_o(X)| \leqslant C_{4f} \frac{\omega(dist(X,S))}{dist^2(X,S)}, \tag{6}$$

where C is any fixed value; $X \in \mathbb{R}^4 \setminus S$ in (5) and (6).

3. Preliminary constructions.

Let

$$\Lambda_n = 2\sqrt{2} M_1 2^{-n}, \; n \in \mathbb{Z},$$

$$T^n_{k\,l} \stackrel{def}{=} (s^n_{k\,l}, t^n_{k\,l}) = \left(2^{-n}\left(k - \frac{1}{2}\right), 2^{-n}\left(-\frac{1}{2}\right)\right)$$

and let $X^n_{k\,l} = F\left(T^n_{k\,l}\right)$. Then, define the ball and the sphere

$$B_r(X) = \left\{Y \in \mathbb{R}^4 \colon \|Y - X\| < r\right\}, \; \Sigma_r(X) = \partial B_r(X).$$

Also, let $\Xi_n = \bigcup\limits_{1 \leqslant k,\, l \leqslant 2^n} \overline{B}_{\Lambda_n}(X^n_{k\,l})$. Assumptions on the mapping F imply existence of constant $C_1^\star = C_1^\star(M_1, M_2) > 0$ and $C_2^\star = C_2^\star(M_1, M_2)$, such that the following inclusions hold:

$$\Omega_{C_1^\star \cdot 2^{-n}} \subset \Xi_n \subset \Omega_{C_2^\star \cdot 2^{-n}} \tag{7}$$

Choose $n_0 \in \mathbb{Z}$ so, that the estimation $C_1^\star \cdot 2^{-n_0} \geqslant \max(1, C)$ is valid; here C is from (4). Then Theorems 1 and 2 follow from the similar statements with Ω_1, Ω_C replaced by the set Ξ_{n_0}.

Let $d(X) = dist(X, S)$. The following statement holds:

Lemma 1. *There exists a twice smooth and co-measurable function $d_0(X)$ and constants C_5, C_6, C_7, C_8, such that the following estimations hold:*

$$C_5 d(X) \leqslant d_0(X) \leqslant C_6 d(X), \tag{8}$$

$$\|\mathbf{grad}\, d_0(X)\| \leqslant C_7,\ X \notin S, \tag{9}$$

$$\|\mathbf{grad}^2\, d_0(X)\| \leqslant C_8 d^{-1}(X). \tag{10}$$

A proof of this Lemma is in [6, chapter 6]; it also can be proved similarly to the arguments in [1].

Let us demonstrate how the function f_0 from Theorem 2 is constructed. For an $X \in \mathbb{R}^4 \setminus S$, set $X_5 = \{X_0 \in S\colon \|X_0 - X\| = \min\limits_{Y \in S} \|Y - X\|\}$ and let

$$f_1(X) = \begin{cases} f(X),\ X \in S, \\ f(X_5),\ X \in \Omega_{\frac{C}{2C_6}} \setminus S, \\ 0,\ X \in \mathbb{R}^4 \setminus \Omega_{\frac{C}{2C_6}}, \end{cases} \tag{11}$$

$$d_1(X) = \frac{1}{4C_6} d_0(X),$$

$$f_2(X) = \frac{1}{\left|B_{d_1(X)}(X)\right|} \int\limits_{B_{d_1(X)}(X)} f_1(Y)\,dm_4(Y),\ X \notin S, \tag{12}$$

$$f_0(X) = \frac{1}{\left|B_{d_1(X)}(X)\right|} \int\limits_{B_{d_1(X)}(X)} f_2(Y)\,dm_4(Y). \tag{13}$$

After that, consider the function f_0 defined by relations (11)–(13). Apply to f_0 the arguments similar to those from [1] for the space $\mathbb{R}^4$ instead of $\mathbb{R}^3$. This gives the desired relations (4)–(6).

4. Construction of the function $\mathbf{U_\delta(X)}$ for $\boldsymbol{\delta = 2^{-n}}$. According to the Note in section 3, it is enough to construct the function $U_\delta(X)$ for $\delta = 2^{-n}$, $n \gg n_0$, and to obtain estimations (2) and (3) (in the latter we assume that $X \in \Xi_n \setminus S$). Rearrange the vectors (k, l), $1 \leqslant k,\ l \leqslant 2^n$ in the following order: $(1,1)$, $(2,1)$, $(1,2)$, $(1,3)$, $(2,2)$, $(3,1)$, and so forth. Let $\nu(k,l)$ be the number of the vector (k,l). Set $D_1^n = \overline{B}_{\Lambda_n}\left(X_{11}^{n-1}\right)$,

$$\mathcal{D}_\nu^n = \overline{B}_{\Lambda_n}\left(X_{\nu(k,l)}^{n-1}\right) \setminus \bigcup_{\mu<\nu} \overline{B}_{\Lambda_n}\left(X_\mu^{n-1}\right),\ 2 \leqslant \nu \leqslant 2^{2n-2}, \tag{14}$$

where $\nu = \nu(k,l)$ in (14), while $\mu = \nu(k_1,l_1) < \nu(k,l)$.

The case $\mathcal{D}_\nu^n = \varnothing$ for some ν is not excluded. Define the following functions $\varphi_\nu^n(X)$, $1 \leqslant \nu \leqslant 2^{2n-4}$:

$$\varphi_\nu^n(X) = \gamma_\nu^n \chi_{E_\nu^n}(X), \tag{15}$$

where $\chi_{E_\nu^n}$ is a characteristic function of the set $E_\nu^n = \overline{B}_{\Lambda_{n-2}}\bigl(X_{\nu(k,l)}^{n-2}\bigr) \setminus \Xi_{n-1}$ in $\mathbb{R}^4$, (k,l), $1 \leqslant k,\ l \leqslant 2^{n-2}$, are such that $\nu(k,l) = \nu$, and the constant γ_ν^n is such that

$$\int\limits_{\overline{B}_{\Lambda_{n-2}}\left(X_{\nu(k,l)}^{n-2}\right)} \gamma_\nu^n \chi_{E_\nu^n}\, dm_4(X) = \int\limits_{\mathcal{D}_\nu^{n-1}} \Delta f_0(X)\, dm_4(X). \tag{16}$$

Estimations from section 5 imply that for γ_ν^n the following relations hold:

$$\gamma_\nu^n \leqslant C_9 2^{2n} \omega\left({}^1\!/_{2^n}\right) \tag{17}$$

for some constant C_9.

Let $\Phi^n(X) = \sum\limits_{\nu=1}^{2^{2n-4}} \varphi_\nu^n(X)$. Set

$$U_{2^{-n}}(X) = \\ = -\frac{1}{4\pi^2} \int\limits_{\mathbb{R}^4 \setminus \Xi_{n-1}} \frac{\Delta f_0(Y)}{\|X-Y\|^2} dm_4(Y) - \frac{1}{4\pi^2} \int\limits_{\mathbb{R}^4} \frac{\Phi^n(Y)}{\|X-Y\|^2} dm_4(Y). \tag{18}$$

5. The main lemma. The proof of Theorem 1 follows, step by step, that of Theorem 1 from [1]. Using the main Lemma below, one can get the following representation for an $X \in S$:

$$f(X) = -\frac{1}{4\pi^2} \int\limits_{\mathbb{R}^4} \frac{\Delta f_0(Y)}{\|X-Y\|^2} dm_4(Y).$$

Together with the definition of the function $U_{2^{-n}}(X)$ in (18), this gives the inequality

$$f(X) - U_{2^{-n}}(X) = \\ = -\frac{1}{4\pi^2} \int\limits_{\Xi_{n-1}} \frac{\Delta f_0(Y)}{\|X-Y\|^2} dm_4(Y) + \frac{1}{4\pi^2} \int\limits_{\mathbb{R}^4} \frac{\Phi^n(Y)}{\|X-Y\|^2} dm_4(Y),\ X \in S. \tag{19}$$

The difference $f(X) - U_{2^{-n}}(X)$ is estimated in the same way as in [1], taking into account (15)–(19) and using the following statement:

The main Lemma. The relation

$$\int\limits_{\overline{B}_{\Lambda_n(X_0)}} \frac{\omega(d(Y))}{d^2(Y)} \frac{dm_4(Y)}{\|X_0 - Y\|^2} \leqslant C_{10}\, \omega\left(2^{-n}\right) \tag{20}$$

holds for some constant C_{10}.

Proof. Let $\mathcal{G}_m = \overline{B}_{\Lambda_m(X_0)} \setminus \overline{B}_{\Lambda_{m+1}(X_0)}$, $I_m = \int\limits_{\mathcal{G}_m} (\cdot)$, where $(\cdot)$ stands for the integrand of (20). Then $\int\limits_{\overline{B}_{\Lambda_n(X_0)}} (\cdot) = \sum\limits_{m=n}^{\infty} I_m$. While estimating I_m, we meet various constants, all denoted by C. Inequality $\|Y - X_0\| \geqslant C \cdot 2^{-m}$ holds for $Y \in \mathcal{G}_m$. So,

$$\begin{aligned} I_m \leqslant C \cdot 2^{2m} \int\limits_{\mathcal{G}_m} \frac{\omega(d(Y))}{d^2(Y)}\, dm_4(Y) \leqslant \\ \leqslant C \cdot 2^{2m} \int\limits_{\overline{B}_{\Lambda_m(X_0)}} \frac{\omega(d(Y))}{d^2(Y)}\, dm_4(Y). \end{aligned} \tag{21}$$

Then, $B_{\Lambda_m(X_0)} \subset \Xi_{m+r_0}$ for some $\widehat{C}_0 \in \mathbb{Z}$, so

$$\begin{aligned} \int\limits_{\overline{B}_{\Lambda_m(X_0)}} \frac{\omega(d(Y))}{d^2(Y)}\, dm_4(Y) \leqslant \\ \leqslant \sum_{\nu=m}^{\infty} \int\limits_{\left(\Xi_{\nu+\widehat{C}_0} \setminus \Xi_{\nu+1+\widehat{C}_0}\right) \bigcap B_{\Lambda_m}(X_0)} \frac{\omega(d(Y))}{d^2(Y)}\, dm_4(Y) \stackrel{def}{=} \\ \stackrel{def}{=} \sum_{\nu=m}^{\infty} J_{\nu,m}. \end{aligned} \tag{22}$$

For a $Y \in \Xi_{\nu+\widehat{C}_0} \setminus \Xi_{\nu+1+\widehat{C}_0}$ we have $d(Y) \geqslant C \cdot 2^{-\nu}$, $d(Y) \leqslant C' \cdot 2^{-\nu}$, therefore,

$$\begin{aligned} J_{\nu,m} \leqslant C \cdot 2^{2\nu} \omega(2^{-\nu}) m_4\left(\left(\Xi_{\nu+\widehat{C}_0} \setminus \Xi_{\nu+1+\widehat{C}_0}\right) \bigcap B_{\Lambda_m}(X_0)\right) \leqslant \\ \leqslant C \cdot 2^{2\nu} \omega(2^{-\nu}) m_4\left(\Xi_{\nu+\widehat{C}_0} \bigcap B_{\Lambda_m}(X_0)\right). \end{aligned} \tag{23}$$

Let $X_0 = F((s_0, t_0))$ and the point $s^m_{k_0\, l_0}$ be the nearest (or one of those) to the point (s_0, t_0). Conditions for the surface S imply the following estimation for any point $X^{\nu+\widehat{C}_0}_{kl}$ such that $\overline{B}_{\Lambda_{\nu+\widehat{C}_0}(X_0)}\left(X^{\nu+\widehat{C}_0}_{kl}\right) \bigcap B_{\Lambda_m}(X_0) \neq \varnothing$:

$$\begin{aligned} &\left\|\left(s^{\nu+\widehat{C}_0}_{kl}, t^{\nu+\widehat{C}_0}_{kl}\right) - \left(s^m_{k_0l_0}, t^m_{k_0l_0}\right)\right\| \leqslant C \cdot 2^{-m}, \\ &F\left(\left(s^{\nu+\widehat{C}_0}_{kl}, t^{\nu+\widehat{C}_0}_{kl}\right)\right) = X^{\nu+\widehat{C}_0}_{kl}, \ \nu \geqslant m. \end{aligned} \tag{24}$$

Denote the number of such points $\left(s^{\nu+\widehat{C}_0}_{kl}, t^{\nu+\widehat{C}_0}_{kl}\right)$ by $N_{m,\nu}$. From estimation (24) we get $N_{m,\nu} \leqslant C \cdot 2^{2(\nu-m)}$. Therefore, measures of the set $\Xi_{\nu+\widehat{C}_0} \bigcap B_{\Lambda_m}(X_0)$ follow the relation

$$m_4\left(\Xi_{\nu+\widehat{C}_0} \bigcap B_{\Lambda_m}(X_0)\right) \leqslant C \cdot 2^{-4\nu} N_{m,\nu} \leqslant C \cdot 2^{-2(\nu+m)}. \tag{25}$$

Now (24) and (25) imply the relation

$$J_{\nu,m} \leqslant C \cdot 2^{2\nu}\omega(2^{-\nu}) \cdot 2^{-2(\nu+m)} = C \cdot \omega(2^{-\nu}) \cdot 2^{-2m}. \tag{26}$$

After that, we get an estimation from (21), (22), and (26):

$$\begin{aligned} I_m \leqslant C \cdot 2^{2m} \sum_{\nu=m}^{\infty} J_{\nu,m} \leqslant C \cdot 2^{2m} \sum_{\nu=m}^{\infty} \omega(2^{-\nu}) \cdot 2^{-2m} = \\ = \sum_{\nu=m}^{\infty} \omega(2^{-\nu}) \leqslant C\omega(2^{-m}). \end{aligned} \tag{27}$$

Here, in (27) we have taken into account the condition on the continuity modulus $\omega(t)$. Finally, from (27) we get

$$\begin{aligned} \int\limits_{\overline{B}_{\Lambda_n(X_0)}} \frac{\omega(d(Y))}{d^2(Y)} \frac{dm_4(Y)}{\|X_0 - Y\|^2} = \sum_{m=n}^{\infty} I_m \leqslant \\ \leqslant \sum_{m=n}^{\infty} C\omega(2^{-m}) \leqslant C_{10}\, \omega\left(2^{-n}\right). \end{aligned} \tag{28}$$

This is what we needed to prove. The main Lemma is thus proved. □

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