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ON SOLVABILITY OF THE BOUNDARY VALUE PROBLEMS FOR HARMONIC FUNCTION ON NONCOMPACT RIEMANNIAN MANIFOLDS

Abstract. We study questions of existence and belonging to the given functional class of solutions of the Laplace-Beltrami equations on a noncompact Riemannian manifold M with no boundary. In the present work we suggest the concept of ϕ-equivalency in the class of continuous functions and establish some interrelation between problems of existence of solutions of the Laplace-Beltrami equations on M and off some compact $B \subset M$ with the same growth "at infinity". A new conception of ϕ-equivalence classes of functions on M develops and generalizes the concept of equivalence of function on M and allows us to more accurately estimate the rate of convergence of the solution to boundary conditions.

Key words: *Riemannian manifold, harmonic function, boundary-value problems, ϕ-equivalency*

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1. Introduction. This article is devoted to the investigation of the behavior of harmonic function in relation to the geometry of the manifold in question. Such problems originate in the classification theory of noncompact Riemannian surfaces and manifolds (see [15]). For a noncompact Riemann surface, the well-known problem of conformal type identification can be stated as follows: does a nontrivial positive superharmonic function exist on this surface? Exactly this property served as a basis for the extension of the parabolicity notion for arbitrary Riemannian manifolds. Namely, manifolds on which any lower-bounded superharmonic function is constant are called parabolic manifolds. In the paper [4] it is shown that parabolicity of the type of a complete Riemannian manifold is equivalent to the fact that the capacity of any compact set is zero. Moreover, the capacitive technique has shown high efficiency in studying the behavior of solutions of elliptic equations and inequalities on noncompact Riemannian manifolds (see [4–7, 11]).

Many questions of this kind fit into the pattern of a Liouville-type theorem saying that the space of bounded solutions of some elliptic equation is trivial.

In works of a number of the authors the conditions ensuring the validity of the Liouville property on noncompact Riemannian manifolds are adduced in terms of volume growth, or isoperimetric inequalities, and so on (see [4–6, 10, 14]). However, the class of manifolds admitting nontrivial solutions of some elliptic equations is wide. For example, conditions ensuring the solvability of the Dirichlet problem with continuous boundary conditions "at infinity" for several noncompact manifolds has been found in many papers (see, e. g., [1, 10, 14]). In particular, similar questions for bounded harmonic functions were studied in papers [2, 13, 16]).

Notice that even the formulation of boundary-value problems for elliptic differential equations (in particular, the Dirichlet problem) on noncompact Riemannian manifolds and in unbounded domains of that manifolds can be problematic, since it is unclear how we should interpret the boundary data. Geometric compactification enables us sometimes to define them analogously the classical statement of the Dirichlet problem in bounded domains of R^n (see, e. g., [1, 10, 14]).

In [12] the author suggested a new approach to the statement of boundary value problems for elliptic differential equation on noncompact Riemannian manifolds, which is based on the consideration of equivalence classes of bounded continuous functions on M.

Namely, let M be an arbitrary smooth connected noncompact Riemannian manifold without boundary and let $\{B_k\}_{k=1}^{\infty}$ be an exhaustion of M, i. e., a sequence of precompact open subsets of M such that $\overline{B_k} \subset B_{k+1}$ and $M = \cup_{k=1}^{\infty} B_k$. Throughout the sequel, we assume that boundaries ∂B_k are C^1-smooth submanifolds.

The continuous functions f_1 and f_2 are *equivalent on* M and write $f_1 \sim f_2$ if for some exhaustion $\{B_k\}_{k=1}^{\infty}$ of M we have

$$\lim_{k\to\infty} \sup_{M\setminus B_k} |f_1 - f_2| = 0.$$

Using this approach has been established the interrelation between the solvability of boundary value problems and solvability of exterior boundary problems for the stationary Schrödinger equation on noncompact Riemannian manifold (see e. g. [12]). A similar result for inhomogeneous elliptic equations was obtained in [11].

This equivalence relation characterizes the asymptotic behavior of solutions of elliptic equations at infinity and ensures that the solution converges to boundary conditions in a uniform norm.

This approach has been developed in a number of works. In particular, the concept of weak equivalence of solutions of elliptic equations was introduced in [8,9], and an estimate was obtained of the rate of convergence of solutions to boundary conditions using the capacitive potential.

In this article we study questions of existence and belonging to given functional class of solutions of the Laplace-Beltrami equation

$$\Delta u = 0, \tag{1}$$

on a noncompact Riemannian manifold M with no boundary.

In our research developing the approach described above, we introduce a new conception of ϕ-equivalence classes of functions on M. This concept generalizes the concept of weak equivalence and allows us to more accurately estimate the rate of convergence of the solution to boundary conditions.

Let $B \subset M$ be an arbitrary connected compact subset and the boundary of B be a C^1-smooth submanifold. Assume that the interior of B is non-empty and $B \subset B_k$ for all k.

Let $\phi > 0$ be continuous function on M such that

$$\lim_{k\to\infty} \|\phi\|_{C(M\setminus B_k)} = 0,$$

where $\|\phi\|_{C(G)} = \sup_G |\phi(x)|$.

Let f_1 and f_2 be arbitrary bounded continuous functions on M.

Definition 1. *Say that f_1 and f_2 are ϕ-equivalent on M and write $f_1 \overset{\phi}{\sim} f_2$ if for some constant $C > 0$ and for all $x \in M \setminus B$ we have*

$$|f_1(x) - f_2(x)| \leqslant C\phi(x).$$

It is easy to verify that the relation " $\overset{\phi}{\sim}$ " is an equivalence and so partitions the set of all bounded continuous functions on M into equivalence classes. Denote the ϕ-equivalence class of a function f by $[f]_\phi$.

It is clear that if $f_1 \overset{\phi}{\sim} f_2$ then $f_1 \sim f_2$. More importantly we have $\phi \sim 0$. The concept of ϕ-equivalence establishes not only inclusion into the class but also determines approach rate of function f_1 and f_2.

Definition 2. *We say that on M the boundary-value problem for equation (1) is solvable with boundary data from the class $[f]_\phi$, if there is a harmonic function $u(x)$ on M such that $u \in [f]_\phi$.*

Definition 3. *Let $\Phi(x) \in C(\partial B)$ be any function continuous on ∂B. We say that on $M \setminus B$ the boundary-value problem for equation (1) is solvable with boundary data $(\Phi, [f]_\phi)$ if there is a harmonic function $u(x)$ on $M \setminus B$ such that $u \in [f]_\phi$ and $u|_{\partial B} = \Phi|_{\partial B}$.*

Definition 4. *We call a function w ϕ-asymptotically nonnegative on M if there exists a continuous and bounded on M function $f \geqslant 0$ such that $w \overset{\phi}{\sim} f$.*

Let us formulate the main results.

Theorem 1. *Suppose that for every constant A there is a harmonic function $v(x)$ on $M \setminus B$ such that $v \in [f]_\phi$ and $v|_{\partial B} = A$. Then there is a harmonic function $u(x)$ on M such that $u \in [f]_\phi$.*

Theorem 2. *Suppose that there is a harmonic function $u(x)$ on M such that $u \in [f]_\phi$. Then for any continuous function Φ on ∂B there is a harmonic function $v(x)$ on $M \setminus B$ such that $v \in [f]_\phi$ and $v|_{\partial B} = \Phi$.*

Theorem 3. *On $M \setminus B$, for any continuous function $\Phi(x) \in C(\partial B)$, the boundary-value problem for equation (1) is solvable with boundary data $(\Phi, [f]_\phi)$ if and only if on M the boundary-value problem for equation (1) with boundary data from the class $[f]_\phi$ is solvable too.*

Remark 1. *The connections between solvability of boundary-value and exterior boundary-value problems for linear and quasilinear elliptic equations in terms of equivalent functions is investigated in detail, for example, in [11, 12].*

2. The auxiliaries. We formulate and prove some auxiliary assertions. Analogy statements for class of equivalence functions were proved in [11, 12]. The proof of all results is based on classical propositions of the theory of equations with partial derivatives: the Maximum Principle, the Comparison and Uniqueness Theorems for solutions to linear elliptic differential equations. Their validity on precompact subsets of manifold M can be shown in just the same way as for bounded domains in R^n (see [3, pp. 39–40]).

Lemma 1. *Suppose that $\Delta w \leqslant 0$ on $M \backslash B$, $w|_{\partial B} \geqslant 0$, and w is ϕ-asymptotically nonnegative. Then $w \geqslant 0$ on $M \backslash B$.*

Proof. Assume that there is a point $x^* \in M\backslash B$ such that $w(x^*) < 0$. Since the sequence $\{B_k\}_{k=1}^{\infty}$ is monotone increasing, we may assume that $x^* \in B_k\backslash B$ for all k. Since w is ϕ-asymptotically nonnegative, there is a function $f \geqslant 0$ such that $|w - f| \leqslant C\phi(x)$ for all $x \in M \setminus B$ and some constant $C > 0$. Here is $\lim\limits_{k\to\infty} \|\phi\|_{C(M\backslash B_k)} = 0$. Then, for every $\varepsilon > 0$ there is $K = K(\varepsilon)$ such that $\sup\limits_{M\backslash B_k} |\phi| < \varepsilon$ for all $k \geqslant K(\varepsilon)$ and so $|\phi| < \varepsilon$ on ∂B_k.

Take $\varepsilon = \frac{|w(x^*)|}{C}$. Then for $k \geqslant K(\varepsilon)$ we have the following inequality

$$|w(x) - f(x)| \leqslant C\phi(x) < \frac{|w(x^*)|}{C} \cdot C < |w(x^*)|$$

for $x \in \partial B_k$.

Hence, $w(x) > f(x) - |w(x^*)| \geqslant -|w(x^*)| = w(x^*)$ for all $x \in \partial B_k$. Moreover, from the condition $w|_{\partial B} \geqslant 0$ we find that $w(x) > w(x^*)$ for all $x \in \partial B$.

Furthermore, the function $w(x)$, being continuous on the compact set $\overline{B_K\backslash B}$ attains its minimal value, where $K = K(\varepsilon)$. Suppose that $w(x^{**}) = \min\limits_{\overline{B_K\backslash B}} w(x)$. Moreover, we find that $w(x^{**}) \leqslant w(x^*) < 0$ and $x^{**} \in B_K\backslash\overline{B}$. Such x^{**} is an interior point of the minimum of harmonic function $w(x)$ in the domain $B_K\backslash B$. Hence, the harmonic function $w(x)$ is a constant. What is more, we have $w(x) \geqslant 0$ since $w|_{\partial B} \geqslant 0$. This result contradicts our assumption above. The proof of the lemma is over. □

Lemma 2. *Suppose that $\Delta w \leqslant 0$ on M and w is ϕ-asymptotically nonnegative. Then $w \geqslant 0$ on M.*

Proof. Firstly, let w be constant. Then, clearly, we have $w \geqslant 0$ on M.

Further, let $w \not\equiv \text{const}$ and there be a point $x^* \in M\backslash B$ such that $w(x^*) < 0$. Since the sequence $\{B_k\}_{k=1}^{\infty}$ is monotone increasing, we may assume that $x^* \in B_k$ for all k. Since w is ϕ-asymptotically nonnegative, there is a function $f \geqslant 0$ such that $|w - f| \leqslant C\phi(x)$ for all $x \in M \setminus B$ and some constant $C > 0$, where $\lim\limits_{k\to\infty} ||\phi||_{C(M\backslash B_k)} = 0$.

Repeating the arguments similar to Lemma 1, we conclude that B_K has an interior point of minimum of the harmonic function $w(x)$, where $K = K(\varepsilon)$ and B_K is defined as in the Lemma 1. Hence, the harmonic function $w(x)$ is a constant. Besides, we have $w(x) \equiv w(x^*) < 0$, which contradicts our assumption above. The proof of the lemma is over. □

Lemmas 1 and 2 readily imply the fulfillment of the Comparison Principle and so the Uniqueness Theorem for solutions to boundary value and exterior boundary problems for equation (1) with boundary data in the class $[f]_\phi$.

Corollary 1. *(Comparison Principle) Suppose that $\Delta w \leqslant \Delta u$ on $M \backslash B$, $w|_{\partial B} \geqslant u|_{\partial B}$ and $w \overset{\phi}{\sim} u$. Then $w \geqslant u$ on $M \backslash B$.*

Suppose that $\Delta w \leqslant \Delta u$ on M and $w \overset{\phi}{\sim} u$. Then $w \geqslant u$ on M.

Corollary 2. *(Uniqueness Theorem) Let $\Delta w = \Delta u$ on $M \backslash B$, $w|_{\partial B} = u|_{\partial B}$ and $w \overset{\phi}{\sim} u$, then $w = u$ on $M \backslash B$. Let $\Delta w = \Delta u$ on M and $w \overset{\phi}{\sim} u$, then $w = u$ on M.*

3. The proof of the main results.

Theorem 1. *Suppose that for every constant A there is a harmonic function $v(x)$ on $M \setminus B$ such that $v \in [f]_\phi$ and $v|_{\partial B} = A$. Then there is a harmonic function $u(x)$ on M such that $u \in [f]_\phi$.*

Proof. Let u_0 be a harmonic function on $M \setminus B$ such that $u_0 \in [f]_\phi$ and $u_0|_{\partial B} = 0$. It is clear that u_0 is a bounded function on $M \setminus B$.

Consider the sequence of functions u_k that are solutions of the problems

$$\begin{cases} \Delta u_k = 0 \quad \text{in} \quad B_k, \\ u_k\,|_{\partial B_k} = u_0|_{\partial B_k}\,. \end{cases}$$

By the maximum principle, we have for all k

$$|u_k| \leqslant \max_{B_k} |u_k| = \max_{\partial B_k} |u_k| = \max_{\partial B_k} |u_0| \leqslant \max_{M \backslash B} |u_0|,$$

which implies the uniform boundedness of the family of functions $\{u_k\}_{k=1}^{\infty}$ on M. From the uniform boundedness of the family of harmonic functions, we obtain the existence of the limit harmonic function $u = \lim\limits_{k\to\infty} u_k$ on M. Next, we will show that $u \in [f]_\phi$.

Since ∂B is a compact subset, there exists $A = \max_{\partial B} |u|$ and we have

$$-A \leqslant u|_{\partial B} \leqslant A$$

and also

$$-(A+1) \leqslant u_k|_{\partial B} \leqslant A+1$$

for sufficiently large values k.

Since $u_0|_{\partial B} = 0$, it follows that $-(A+1) \leqslant u_0|_{\partial B} \leqslant A+1$ and $-(A+1) \leqslant u_k|_{\partial B} \leqslant A+1$ for k large enough. Under the assumption of the theorem, there is a harmonic function $\underline{u}(x) \in [f]_\phi$ and $\overline{u}(x) \in [f]_\phi$ on $M \backslash B$ such that

$$\underline{u}|_{\partial B} = -(A+1), \quad \overline{u}|_{\partial B} = A+1.$$

Hence, we find that

$$\underline{u}|_{\partial B} \leqslant u_0|_{\partial B} = 0 \leqslant \overline{u}|_{\partial B}.$$

Moreover, we have $\Delta \underline{u} = \Delta u = \Delta \overline{u}$ on $M \backslash B$ and $\underline{u} \overset{\phi}{\sim} u_0 \overset{\phi}{\sim} \overline{u}$. Then, applying the Corollary 1 (Comparison Principle) we get $\underline{u} \leqslant u_0 \leqslant \overline{u}$ on $M \backslash B$. The last inequality implies that next relations are true:

$$\underline{u}|_{\partial B_k} \leqslant u_k\,|_{\partial B_k} = u_0|_{\partial B_k} \leqslant \overline{u}|_{\partial B_k}, \qquad \underline{u}|_{\partial B} \leqslant u_k\,|_{\partial B} \leqslant \overline{u}|_{\partial B}.$$

Further, we use the comparison principle to the harmonic functions u_k on $B_k \backslash B$ for k large enough and get $\underline{u} \leqslant u_k \leqslant \overline{u}$. Passing to the limit as $k \to \infty$ on $M \backslash B$, we obtain $\underline{u} \leqslant u \leqslant \overline{u}$. Since $\underline{u} \overset{\phi}{\sim} \overline{u}$, we arrive at the equivalence $u \overset{\phi}{\sim} u_0$. So $u \in [f]_\phi$, which completes the proof of Theorem 1. □

Theorem 2. *Suppose that there is a harmonic function $u(x)$ on M such that $u \in [f]_\phi$. Then for any continuous function Φ on ∂B there is a harmonic function $v(x)$ on $M \setminus B$ such that $v \in [f]_\phi$ and $v|_{\partial B} = \Phi$.*

Proof. We first prove that for every continuous function Φ on ∂B there is a harmonic fuction w on $M \setminus B$ such that $w|_{\partial B} = \Phi$ and $w \in [0]_\phi$. Consider the sequence of harmonic functions w_k that are solutions to the boundary value problems:

$$\begin{cases} \Delta w_k = 0 \quad \text{in} \quad B_k \setminus B, \\ w_k|_{\partial B} = \Phi, \\ w_k|_{\partial B_k} = \phi. \end{cases}$$

By the maximum principle, for every k we have

$$|w_k| \leqslant \max_{\partial(B_k \backslash B)} |w_k| \leqslant \max_{\partial B} |\Phi| + \max_{\partial B_k} |\phi| \leqslant \max_{\partial B} |\Phi| + \sup_{M \backslash B} |\phi|,$$

i. e., the sequence $\{w_k\}_{k=1}^{\infty}$ is uniformly bounded on $M \setminus B$ and so there is the limit harmonic function $w(x) = \lim\limits_{k\to\infty} w_k$. It is clear that $w|_{\partial B} = \Phi$.

We will show that $w \in [0]_\phi$, i. e., for all $x \in M \setminus B$ we have

$$|w(x)| \leqslant C\phi(x)$$

for some constant $C > 0$.

Consider the sequence of harmonic functions w_k and show that

$$|w_k(x)| \leqslant C\phi(x) \tag{2}$$

for all $x \in B_k \setminus B$ and for some constant $C > 0$.

Suppose that $x \in \partial B_k$; then $w_k(x) = \phi(x)$ and so the inequality (2) is true for any constant $C \geqslant 1$.

Further, suppose that $x \in \partial B$; then $w_k(x) = \Phi(x)$ and so $|w_k(x)| = |\Phi(x)| \leqslant C\phi(x)$, where $C = {}^{A_1}\!/_{A_2}$. The constants $A_1 = \max_{\partial B} |\Phi(x)|$ and $A_2 = \min_{\partial B} \phi(x)$ exist, since $\Phi(x) \in C(\partial B)$ and $\phi(x) \in C(\partial B)$. Put $C = \max\{1, {}^{A_1}\!/_{A_2}\}$. By the maximum principle, for every k we have inequality (2) in the whole set $B_k \setminus B$.

Taking the limit in (2) as $k \to \infty$, we obtain $|w(x)| \leqslant C\phi(x)$ for all $x \in M \setminus B$. Hence, we have $w \in [0]_\phi$.

Now, let $u \in [f]_\phi$ be a harmonic function on M and Φ be an arbitrary continuous function on ∂B. Consider the continuous function $\Phi^* = u - \Phi$ on ∂B. As shown above, there is a harmonic function w on $M \setminus B$ such that $w|_{\partial B} = \Phi^*$ and $w \in [0]_\phi$. Then the function $v = u - w$ is a sought harmonic function on $M \setminus B$ such that $v \in [f]_\phi$ and $v|_{\partial B} = \Phi$. $\square$

The following theorem is a direct consequence of the previous results.

Theorem 3. *On $M \setminus B$ for any continues function $\Phi(x) \in C(\partial B)$ the boundary-value problem for equation (1) is solvable with boundary data $(\Phi,[f]_\phi)$ if and only if on M the boundary-value problem for equation (1) with boundary data from the class $[f]_\phi$ is solvable too.*

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