DOI: 10.15393/j3.art.2020.7390

UDC 517.521.1, 517.518.13

Rashid A. Aliev, Aynur N. Ahmadova

DISCRETE AHLFORS–BEURLING TRANSFORM AND ITS PROPERTIES

Abstract. The Ahlfors–Beurling transform has been well studied on classical Lebesgue, Morrey, Sobolev, Besov, Campanato, etc. spaces. However, its discrete version is still not studied well. In this paper, we study the properties of the discrete Ahlfors–Beurling transform on discrete Lebesgue spaces.

Key words: *discrete Ahlfors–Beurling transform, Lebesgue spaces, boundedness*

2010 Mathematical Subject Classification: *44A15, 40A05, 46B45, 42B35*

1. Introduction. The Ahlfors–Beurling transform of a function $f \in L_p(C)$, $1 \le p < \infty$, is defined as the following singular integral:

$$(Bf)(z) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \int\limits_{\{w \in C\,:\, |z-w| > \varepsilon\}} \frac{f(w)}{(z-w)^2} dm(w).$$

The Ahlfors–Beurling transform is one of the important operators in complex analysis. It has been shown in [1], [5], [8], [13], [20] that this transform plays an essential role in applications to the theory of quasiconformal mappings and to the Beltrami equation with discontinuous coefficients.

From the theory of singular integrals (see [7], [16]), it is known that the Ahlfors–Beurling transform is a bounded operator in the space L_p, $1 < p < \infty$; that is, if $f \in L_p$, then $Bf \in L_p$ and the inequality

$$\|Bf\|_{L_p} \le C_p \|f\|_{L_p} \tag{1}$$

holds. In the case $f \in L_1$, only the weak inequality holds:

$$m\{z \in C :\ |(Bf)(z)| > \lambda\} \le \frac{C_1}{\lambda} \|f\|_{L_1}, \quad \lambda > 0, \tag{2}$$

where m stands for the Lebesgue measure, C_p, C_1 are constants independent of f.

Note that the Ahlfors–Beurling transform of a function $f \in L_1$ is not Lebesgue integrable. In [4], the authors considered a modified version of the Ahlfors–Beurling transform and, using the notion of A-integrability, proved an analogue of the Riesz equality.

In [6], [9], [10], [11], [12], [14], [18], [19], the boundedness of the operator B in other function spaces (in the spaces of Sobolev, Besov, Campanato, Morrey, etc.) was studied. But the discrete version of the Ahlfors–Beurling transform has not been studied. In this paper, we study the properties of the discrete Ahlfors–Beurling transform on discrete Lebesgue spaces.

2. Discrete Ahlfors–Beurling transform and its boundedness on discrete Lebesgue spaces.

Denote by $l_p := l_p(Z_C)$, $p \geq 1$, the class of sequences $h = \{h_n\}_{n \in Z_C}$ satisfying the condition

$$\|h\|_{l_p} := \Big(\sum_{n \in Z_C} |h_n|^p\Big)^{1/p} < \infty,$$

where $Z_C := \{m + in \in C : \ m, n \in Z\}$ and Z is the set of integers.

Let $h = \{h_n\}_{n \in Z_C} \in l_p$, $p \geq 1$. Namely, the sequence $\tilde{B}(h) = \{(\tilde{B}h)_n\}_{n \in Z_C}$ is called the Ahlfors–Beurling transform of the sequence h, where

$$(\tilde{B}h)_n = \sum_{m \in Z_C,\ m \neq n} \frac{h_m}{(n-m)^2}, \quad n \in Z_C.$$

Note that if $h \in l_p$, $1 \leq p < \infty$, then from the Holder inequality it follows that the series $\sum\limits_{m \in Z_C,\ m \neq n} \dfrac{h_m}{(n-m)^2}$ absolutely converges, and, therefore, the Ahlfors-Beurling transform of the sequence h exists. In [7], A.P. Calderon and A. Zygmund noted that the discrete Ahlfors–Beurling transform is of special interest among discrete analogues of singular integrals. In this work, it was also noted (without proof) that the discrete analogues of singular integrals, including the discrete Ahlfors–Beurling transform, is bounded in L_p. For $n = 1$, this remark is due to M. Riesz [15] (see also [17]), and the proof in the case of general n follows a similar pattern. For the sake of readability of the article, we first give a proof of this fact.

Theorem 1. *Let $1 < p < \infty$. For any $h \in l_p$, we have $\tilde{B}h \in l_p$, and there exists $c_p > 0$ such that*

$$\|\tilde{B}h\|_{l_p} \leq c_p \cdot \|h\|_{l_p} \tag{3}$$

holds for all $h \in l_p$.

Proof. Define the function $f(z)$ to be $(-4\pi h_n)$ for $z \in P(n, 1/4)$, $n \in Z_C$, and 0 elsewhere, where

$$P(n, \delta) := \{w \in C : \ -\delta \leq \Re(w - n) < \delta, \ -\delta \leq \Im(w - n) < \delta\}.$$

It follows from $h \in l_p$, that $f \in L_p$ and

$$\|f\|_{L_p} = \Big(\sum_{n \in Z_C} \int\limits_{P(n,\, 1/4)} |4\pi h_n|^p dm(z) \Big)^{1/p} = 4^{1-1/p} \pi \|h\|_{l_p}.$$

Then, from inequality (1) it follows that $Bf \in L_p$, and

$$\|Bf\|_{L_p} \leq C_p 4^{1-1/p} \pi \|h\|_{l_p}.$$

Define the function $F(z)$ to be $(\tilde{B}h)_n$ for $z \in P(n, 1/2)$, $n \in Z_C$ and

$$G(z) = (Bf)(z) - F(z).$$

We first prove that $G(z) \in L_p$.

For every $z \in P(n, 1/2)$, $|\Re(z - n)| \neq 1/4$, $|\Im(z - n)| \neq 1/4$, $n \in Z_C$, we have

$$\begin{aligned} G(z) = \sum_{m \in Z_C} 4h_m \int\limits_{P(m,1/4)} \frac{dm(w)}{(z-w)^2} - \sum_{m \in Z_C,\ m \neq n} \frac{h_m}{(n-m)^2} = \\ = \sum_{m \in Z_C,\ m \neq n} 4h_m \int\limits_{P(m,1/4)} \Big(\frac{1}{(z-w)^2} - \frac{1}{(n-m)^2} \Big) dm(w) + \\ + 4h_n \int\limits_{P(n,1/4)} \frac{dm(w)}{(z-w)^2} = G_1(z) + G_2(z), \end{aligned} \tag{4}$$

where

$$\int\limits_{P(n,1/4)} \frac{dm(w)}{(z-w)^2} := \lim_{\varepsilon \to 0+} \int\limits_{\{w \in P(n,1/4):\ |w-z| \geq \varepsilon\}} \frac{dm(w)}{(z-w)^2}.$$

Let $m \neq n$. Since for every $z \in P(n,1/2)$ and $w \in P(m,1/4)$

$$|n-m|-3/4 \leq |z-w| \leq |n-m|+3/4,$$

then we get

$$\Big|\frac{1}{(z-w)^2} - \frac{1}{(n-m)^2}\Big| = \frac{|n-m-z+w|\cdot|n-m+z-w|}{|z-w|^2\cdot|n-m|^2} \leq$$
$$\leq \frac{3/4(2|n-m|+3/4)}{|n-m|^2\cdot(|n-m|-3/4)^2} \leq \frac{33}{|n-m|^3}.$$

Therefore, for every $z \in P(n,1/2)$

$$|G_1(z)| \leq \sum_{m\in Z_C,\, m\neq n} 4|h_m| \int\limits_{P(m,1/4)} \Big|\frac{1}{(z-w)^2} - \frac{1}{(n-m)^2}\Big| dm(w) \leq$$
$$\leq \sum_{m\in Z_C,\, m\neq n} \frac{33|h_m|}{|n-m|^3}. \quad (5)$$

From this and from the Holder inequality, it follows that

$$\|G_1\|_{L_p} \leq 33\Big(\sum_{n\in Z_C}\Big[\sum_{m\in Z_C,\, m\neq n}\frac{|h_m|}{|n-m|^3}\Big]^p\Big)^{1/p} \leq$$
$$\leq 33\Big(\sum_{n\in Z_C}\Big[\sum_{m\in Z_C,\, m\neq n}\frac{|h_m|^p}{|n-m|^3}\Big]\cdot\Big[\sum_{m\in Z_C,\, m\neq n}\frac{1}{|n-m|^3}\Big]^{p-1}\Big)^{1/p} =$$
$$= 33 d_0^{1-1/p}\Big(\sum_{n\in Z_C}\sum_{m\in Z_C,\, m\neq n}\frac{|h_m|^p}{|n-m|^3}\Big)^{1/p} =$$
$$= 33 d_0^{1-1/p}\Big(\sum_{m\in Z_C}|h_m|^p\sum_{n\in Z_C, n\neq m}\frac{1}{|n-m|^3}\Big)^{1/p} = 33 d_0\|h\|_{l_p}, \quad (6)$$

where $d_0 = \sum\limits_{n\in Z_C, n\neq 0}\frac{1}{|n|^3}$.

Let us show that $G_2 \in L_p$. If $z \in P(n,1/2)$, $|\Re(z-n)| \neq 1/4$, $|\Im(z-n)| \neq 1/4$, $n \in Z_C$, then

$$|G_2(z)| \leq 4|h_n| \int\limits_{U(n,2)\setminus U(n,\rho(z))} \frac{dm(w)}{|z-w|^2} = 8\pi|h_n|\cdot\ln\frac{2}{\rho(z)}, \quad (7)$$

where $\rho(z) := \rho(z; \overline{P(n,\, 1/4)}) = \min\{|z - w|\colon\, w \in \overline{P(n,\, 1/4)}\}$ is the distance

from the point z to the set $\overline{P(n,\, 1/4)}$ and $U(z, \varepsilon) = \{w \in C : |w - z| < \varepsilon\}$. Therefore, for every $n \in Z_C$,

$$\int\limits_{P(n,\,1/2)} |G_2(z)|^p dm(z) \le (8\pi)^p |h_n|^p \cdot \int\limits_{P(n,\,1/2)} \Big(\ln \frac{2}{\rho(z)}\Big)^p dm(z) =$$

$$= (8\pi)^p |h_n|^p \cdot 8 \int\limits_0^{1/2} dx \int\limits_x^{1/2} \Big(\ln \frac{2}{1/2 - y}\Big)^p dy \le (8\pi)^p |h_n|^p \cdot 4 d_1,$$

where $d_1 := \displaystyle\int\limits_0^{1/2} \Big(\ln \frac{2}{1/2 - y}\Big)^p dy.$

From this, we have

$$\|G_2\|_{L_p} = \Big(\sum_{n \in Z_C} \int\limits_{P(n,\,1/2)} |G_1(z)|^p dm(z)\Big)^{1/p} \le 8\pi \cdot (4d_1)^{1/p} \|h\|_{l_p} \qquad (8)$$

It follows from (4), (6), (8), that $G \in L_p$.

Since $F(z) = (Bf)(z) - G(z)$, then we have from $Bf \in L_p$ and $G \in L_p$ that $F \in L_p$,

$$\|F\|_{L_p} \le (C_p 4^{1-1/p} \pi + 33 d_0 + 8\pi \cdot (4d_1)^{1/p}) \|b\|_{l_p}.$$

Therefore,

$$\|\tilde{B}h\|_{l_p} = \Big(\sum_{n \in Z_C} |(\tilde{B}h)_n|^p\Big)^{1/p} = \Big(\sum_{n \in Z_C} \int\limits_{P(n,\,1/2)} |F(z)|^p dm(z)\Big)^{1/p} =$$

$$= \|F\|_{L_p} \le (C_p 4^{1-1/p} \pi + 33 d_0 + 8\pi \cdot (4d_1)^{1/p}) \|b\|_{l_p}.$$

This completes the proof of Theorem. □

Theorem 2. *There exists $c_1 > 0$, such that for any $h \in l_1$ and for any $\lambda > 0$ the distribution function $(\tilde{B}h)(\lambda) = |\{n \in Z_C : |(\tilde{B}h)_n| > \lambda\}| := \sum_{\{n \in Z_C:\, |(\tilde{B}h)_n| > \lambda\}} 1$ of the Ahlfors-Beurling transform of the sequence h satisfies the inequality*

$$|(\tilde{B}h)(\lambda)| \le \frac{c_1}{\lambda} \|h\|_{l_1}. \qquad (9)$$

Proof. We define the functions $f(z)$, $F(z)$, $G(z)$, $G_1(z)$, $G_2(z)$ as in the proof of Theorem 1. It follows from $h \in l_1$ that $f \in L_1$,

$$\|f\|_{L_1} = \sum_{n\in Z_C} \int\limits_{P(n,\,1/4)} |4\pi h_n| dm(z) = \pi \|h\|_{l_1}.$$

Then, from (2) we have

$$m\{z \in C : \ |(Bf)(z)| > \lambda\} \le \frac{C_1}{\lambda}\pi\|h\|_{l_1}, \lambda > 0. \tag{10}$$

It follows from (5) and (7) that $G_1 \in L_1$ and $G_2 \in L_1$,

$$\|G_1\|_{L_1} = \sum_{n\in Z_C} \int\limits_{P(n,\,1/2)} |G_1(z)| dm(z) \le 33 \sum_{n\in Z_C} \sum_{m\in Z_C,\ m\ne n} \frac{|h_m|}{|n-m|^3} =$$

$$= 33 \sum_{m\in Z_C} |h_m| \sum_{n\in Z_C,\, n\ne m} \frac{1}{|n-m|^3} = 33 d_0 \|h\|_{l_1}, \tag{11}$$

$$\|G_2\|_{L_1} = \int\limits_C |G_2(z)| dm(z) \le 8\pi \cdot \sum_{n\in Z_C} |h_n| \int\limits_{P(n,\,1/2)} \ln \frac{2}{\rho(z)} dm(z) \le$$

$$\le 32\pi \int\limits_0^{1/2} \ln \frac{2}{1/2 - y} dy \cdot \|h\|_{l_1} = 16\pi[\ln 4 + 1] \cdot \|h\|_{l_1}. \tag{12}$$

It follows from (4), (11), (12) that $G \in L_1$,

$$\|G\|_{L_1} \le [16\pi(\ln 4 + 1) + 33 d_0] \cdot \|h\|_{l_1}.$$

Therefore, by Chebyshev's inequality, we have

$$m\{z \in C : \ |G(z)| > \lambda\} \le \frac{16\pi(\ln 4 + 1) + 33 d_0}{\lambda} \|h\|_{l_1}. \tag{13}$$

Since $F(z) = (Bf)(z) - G(z)$, then it follows from (10) and (13) that

$$\{z \in C : |F(z)| > \lambda\} \le m\{z \in C : |(Bf)(z)| > \frac{\lambda}{2}\} +$$

$$+ m\{z \in C : |G(z)| > \frac{\lambda}{2}\} \le \frac{2C_1\pi + 32\pi(\ln 4 + 1) + 66 d_0}{\lambda} \|h\|_{l_1}.$$

Therefore,

$$(\tilde{B}h)(\lambda) = |\{n \in Z_C : |(\tilde{B}h)_n| > \lambda\}| = m\{z \in C : |F(z)| > \lambda\} \leq$$
$$\leq \frac{2C_1\pi + 32\pi(\ln 4 + 1) + 66d_0}{\lambda}\|h\|_{l_1}.$$

This completes the proof. □

3. Asymptotic behavior of the distribution function of the discrete Ahlfors-Beurling transform.

The following theorem shows the main reason why the discrete Ahlfors-Beurling transform does not belong to the class l_1.

Theorem 3. *Let $h \in l_1$. Then the equation*

$$\lim_{\lambda \to 0+} \lambda \cdot (\tilde{B}h)(\lambda) = \pi \Big| \sum_{n \in Z_C} h_n \Big| \tag{14}$$

holds.

At first, we prove the auxiliary lemma.

Lemma 1. *Let $h \in l_1$ and $\sum_{n \in Z_C} h_n = 0$. Then the equation*

$$(\tilde{B}h)(\lambda) = o(1/\lambda), \lambda \to 0+ \tag{15}$$

holds.

Proof. At first, assume that the sequence $h \in l_1$ is concentrated on some finite ball $\{w \in C : |w| \leq m\}$, that is, $h_n = 0$ for $|n| > m$. In this case, from the equality

$$(\tilde{B}h)_n = \sum_{|k| \leq m} \frac{h_k}{(n-k)^2} - \frac{1}{n^2} \sum_{|k| \leq m} h_k = \sum_{|k| \leq m} \frac{2n-k}{(n-k)^2 n^2} k h_k, \ |n| > m$$

we get that

$$|(\tilde{B}h)_n| \leq \frac{8}{|n|^3} \sum_{|k| \leq m} |k h_k|$$

for large values of n, whence it follows asymptotic equation (15).

Now let us consider the general case. From the condition $\sum_{n \in Z_C} h_n = 0$ it follows that for any $\varepsilon > 0$ there exist the sequences $h' = \{h'_n\}_{n \in Z} \in l_1$ and $h'' = \{h''_n\}_{n \in Z} \in l_1$ satisfying the condition: $h = h' + h''$; the

sequence $h' \in l_1$ is concentrated on some finite ball $\{w \in C : |w| \leq m\}$ and $\sum_{n \in Z_C} h'_n = 0$; $h'' \in l_1$ satisfies the inequality $\sum_{n \in Z_C} |h''_n| < \frac{\varepsilon}{4c_1}$, where c_1 is the constant in estimation (9). Since the sequence $h' \in l_1$ is concentrated on $\{w \in C : |w| \leq m\}$ and $\sum_{n \in Z_C} h'_n = 0$, then for the sequence $h' \in l_1$ equation (15) is satisfied, and, therefore, there exists $\lambda(\varepsilon) > 0$ such that for $0 < \lambda < \lambda(\varepsilon)$ the inequality

$$\lambda(\tilde{B}h')(\lambda/2) < \varepsilon/2 \tag{16}$$

holds, where $(\tilde{B}h')(\lambda) = \sum_{\{n \in Z_C : |(\tilde{B}h')_n| > \lambda\}} 1$. On the other hand, from inequality (9) it follows that for any $\lambda > 0$

$$\lambda(\tilde{B}h'')(\lambda/2) < 2c_1 \|h''\|_{l_1} < \varepsilon/2 \tag{17}$$

where $(\tilde{B}h'')(\lambda) = \sum_{\{n \in Z_C : |(\tilde{B}h'')_n| > \lambda\}} 1$. From inequalities (16), (17) and from the inclusion

$$\{n \in Z_C : |(\tilde{B}h)_n| > \lambda\} \subset \\ \subset \{n \in Z_C : |(\tilde{B}h')_n| > \lambda/2\} \cup \{n \in Z_C : |(\tilde{B}h'')_n| > \lambda/2\}$$

we get

$$\lambda \cdot (\tilde{B}h)(\lambda) \leq \lambda(\tilde{B}h')(\lambda/2) + \lambda(\tilde{B}h'')(\lambda/2) < \varepsilon$$

for $0 < \lambda < \lambda(\varepsilon)$. This shows that equality (15) was satisfied for all $h \in l_1$ satisfying a condition $\sum_{n \in Z_C} h_n = 0$. This completes the proof of the lemma. □

Proof of theorem 3. In the case $\sum\limits_{n \in Z_C} h_n = 0$ the assertion of the theorem follows from Lemma 1. Let us consider the case $\sum\limits_{n \in Z_C} h_n = \alpha \neq 0$. Denote by $h'_n = h_n$ for $n \neq 0$, $h'_0 = h_0 - \alpha$ and $h''_n = 0$ for $n \neq 0$, $h''_0 = \alpha$. Then $h = h' + h''$, where $h' = \{h'_n\}_{n \in Z} \in l_1$ and $h'' = \{h''_n\}_{n \in Z} \in l_1$. Since $\sum\limits_{n \in Z_C} h'_n = 0$, then from Lemma 1 we get

$$(\tilde{B}h')(\lambda) = o(1/\lambda), \ \lambda \to 0+. \tag{18}$$

Since $(\tilde{B}h'')_n = \frac{\alpha}{n^2}$ for $n \neq 0$, $(\tilde{B}h'')_0 = 0$, then

$$(\tilde{B}h'')(\lambda) \sim \frac{\pi|\alpha|}{\lambda}, \ \lambda \to 0+. \tag{19}$$

For any $0 < \varepsilon < 1$, by the inclusions

$$\{n \in Z_C : |(\tilde{B}h'')_n| > (1+\varepsilon)\lambda\} \setminus \{n \in Z_C : |(\tilde{B}h')_n| > \varepsilon\lambda\} \subset$$
$$\subset \{n \in Z_C : |(\tilde{B}h)_n| > \lambda\} \subset$$
$$\subset \{n \in Z_C : |(\tilde{B}h')_n| > \varepsilon\lambda\} \cup \{n \in Z_C : |(\tilde{B}h'')_n| > (1-\varepsilon)\lambda\}$$

and from (18), (19) we have

$$\frac{\pi|\alpha|}{1+\varepsilon} \le \liminf_{\lambda\to 0+} \lambda \cdot (\tilde{B}h)(\lambda) \le \limsup_{\lambda\to 0+} \lambda \cdot (\tilde{B}h)(\lambda) \le \frac{\pi|\alpha|}{1-\varepsilon}.$$

This implies the equation (14) and completes the proof of Theorem 3.

Note that for the discrete Hilbert transform an analogue of this theorem was proved in [2].

4. A necessary condition and a sufficient condition for the summability of the discrete Ahlfors-Beurling transform.

Theorem 4. *Let $h \in l_1$. Then, to include $\tilde{B}h \in l_1$, it is necessary that the equation*

$$\sum_{n\in Z_C} h_n = 0 \tag{20}$$

holds.

Proof. First, we prove that if the sequence $b = \{b_n\}_{n\in Z} \in l_1$, then the distribution function $b(\lambda) = |\{n \in Z_C : |b_n| > \lambda\}|$ of the sequence b satisfies the condition

$$b(\lambda) = o(1/\lambda), \lambda \to 0+. \tag{21}$$

It follows, from the inequality

$$\sum_{n\in Z}|b_n| = \sum_{\{n\in Z_C:|b_n|>1\}} |b_n| + \sum_{k=0}^{\infty}[\sum_{\{n\in Z_C:|b_n|\in(2^{-k-1};2^{-k}]\}} |b_n|] \ge$$

$$\ge |\{n \in Z_C : |b_n| > 1\}| + \sum_{k=0}^{\infty}[2^{-k-1} \cdot |\{n \in Z_C : |b_n| \in (2^{-k-1}; 2^{-k}]\}|] =$$

$$= b(1) + \sum_{k=0}^{\infty}[2^{-k-1} \cdot (b(2^{-k-1}) - b(2^{-k}))] = \sum_{k=0}^{\infty}[2^{-k-1} \cdot b(2^{-k})]$$

that

$$\lim_{k\to\infty} 2^{-k}\cdot b(2^{-k}) = 0.$$

Taking into account that the function $b(\lambda)$ is decreasing, we obtain (21).

It follows from (21) that, if $\tilde{B}h \in l_1$, then

$$(\tilde{B}h)(\lambda) = o(1/\lambda), \lambda \to 0+,$$

and, therefore, by Theorem 3, we obtain that equation (20) holds. The proof of the theorem is complete. □

Theorem 5. *If the sequence $h \in l_1$ satisfies the conditions*

i) $\sum\limits_{n\in Z_C} h_n = 0;$

ii) $\sum\limits_{m\in Z_C} |h_m|\ln(e+|m|) < \infty,$

then $\tilde{B}h \in l_1$ and the inequality

$$\|\tilde{B}h\|_{l_1} \leq 150 \sum_{m\in Z_C} |h_m|\ln(e+|m|) \tag{22}$$

holds.

Proof. From the definition of the discrete Ahlfors-Beurling transform, it follows that

$$|(\tilde{B}h)_0| = \Big|\sum_{m\neq 0} \frac{h_m}{m^2}\Big| \leq \|h\|_{l_1}. \tag{23}$$

From condition i) for $n \neq 0$, we have

$$|(\tilde{B}h)_n| = \Big|\sum_{m\in Z_C,\, m\neq n} \frac{h_m}{(n-m)^2} - \sum_{m\in Z_C} \frac{h_m}{n^2}\Big| \leq$$

$$\leq \Big|\frac{h_n}{n^2}\Big| + \sum_{m\in Z_C,\, m\neq n} \Big|\frac{h_m}{(n-m)^2} - \frac{h_m}{n^2}\Big|. \tag{24}$$

It follows from inequalities (23) and (24) that

$$\|\tilde{B}h\|_{l_1} = \sum_{n\in Z_C} |(\tilde{B}h)_n| \leq 2\|h\|_{l_1} + \sum_{n\in Z_C, n\neq 0} \sum_{m\in Z_C, m\neq n} \Big|\frac{h_m}{(n-m)^2} - \frac{h_m}{n^2}\Big| =$$

$$= 2\|h\|_{l_1} + \sum_{m\in Z_C, m\neq 0} |h_m| \sum_{n\in Z_C, n\neq 0, n\neq m} \Big|\frac{1}{(n-m)^2} - \frac{1}{n^2}\Big| =$$

$$= 2\|h\|_{l_1} + \sum_{m\in Z_C, m\neq 0} |h_m|\cdot J_m, \quad (25)$$

where

$$J_m = \sum_{|n|\le 3|m|, n\neq 0, n\neq m} \Big|\frac{1}{(n-m)^2} - \frac{1}{n^2}\Big| + \sum_{|n|>3|m|} \Big|\frac{1}{(n-m)^2} - \frac{1}{n^2}\Big| =$$
$$= J_m^{(1)} + J_m^{(2)}, \quad m \neq 0. \quad (26)$$

Estimate the summands $J_m^{(i)}$, $m \neq 0$, $i = 1, 2$. Defining $k = [\log_2(4|m|)] + 1$, where $[\log_2(4|m|)]$ is the integer part of the number $\log_2(4|m|)$, we have

$$J_m^{(1)} \le \sum_{|n|\le 3|m|, n\neq m} \frac{1}{|n-m|^2} + \sum_{|n|\le 3|m|, n\neq 0} \frac{1}{|n|^2} \le$$
$$\le 2 \sum_{|n|\le 4|m|, n\neq 0} \frac{1}{|n|^2} \le 2\sum_{p=1}^{k} \sum_{2^{p-1}\le|n|<2^p} \frac{1}{|n|^2} \le 2\sum_{p=1}^{k} 4\cdot 2^{2p}\cdot \frac{1}{2^{2p-2}} =$$
$$= 32k \le 96\ln(e+|m|),$$

$$J_m^{(2)} = \sum_{|n|>3|m|} \frac{|m||2n-m|}{|n-m|^2|n|^2} \le 6|m| \sum_{|n|>3|m|} \frac{1}{|n|^3} =$$
$$= 6|m| \sum_{p=1}^{\infty} \sum_{3|m|2^{p-1}<|n|\le 3|m|2^p} \frac{1}{|n|^3} \le$$
$$\le 6|m| \sum_{p=1}^{\infty} \frac{4\cdot(3|m|\cdot 2^p)^2}{27|m|^3\cdot 2^{3p-3}} = 64 \le 52\ln(e+|m|).$$

From this and from (25), (26), we obtain (22). The proof of the theorem is complete. □

Note that analogues of these theorems were proved for the discrete Hilbert transform in [3].

Acknowledgments. The authors are very grateful to the referees for many useful comments and suggestions that improved the original manuscript.

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Received December 06, 2019.
In revised form, April 05, 2020.
Accepted April 06, 2020.
Published online April 13, 2020.

Rashid A. Aliev
Baku State University, Baku, AZ 1148, Azerbaijan
Institute of Mathematics and Mechanics, NAS of Azerbaijan, Baku, AZ 1141, Azerbaijan
E-mail: aliyevrashid@mail.ru

Aynur N. Ahmadova
Sumgait State University, Sumgait, AZ 5008, Azerbaijan
E-mail: dissertant.aynur@gmail.com