DOI: 10.15393/j3.art.2020.8571

UDC 517.55

A. M. Kytmanov, E. K. Myshkina

ON FINDING THE RESULTANT OF TWO ENTIRE FUNCTIONS

Abstract. Using Newton's recurrent formulae, we find the product of values of an entire function of one variable in zeroes of another entire function. This allows to answer whether they have common zeros. By that, we propose an approach to construction of the resultant of two entire functions. We also give examples illustrating the main result.

Key words: *resultant, entire function, Newton formulae*

2010 Mathematical Subject Classification: *13P15, 32A15*

1. Introduction. The classical resultant $R\,(f,g)$ of two polynomials f and g may be defined variously: using the Sylvester determinant (see e.g. [1], [10], [16]), the product formula $R\,(f,g) = \prod\limits_{\{x\colon f(x)=0\}} g\,(x)$ (see [1], [10], [16]), or the Bezout-Caley approach (see [9]).

In this paper, we give a constructive approach to define a resultant of two entire functions of one complex variable using the product formula. Our choice is justified by the fact that entire functions are the direct generalization of polynomials.

In a series of papers (see [4–6], [8], [15]) various authors proposed generalizations for the resultant of analytic functions in the ring of matrix-valued functions, meromorphic functions on Riemann surfaces, for systems of algebraic and transcendent equations. In all these investigations, it is assumed that the number of zeroes and poles is finite. In our case, functions may have infinite number of zeroes but instead we need to employ the limit procedure.

The interest to this problem is explained by the fact that many mathematical models require studying non-algebraic equations and systems, for example, equations of chemical kinetics often use exponential polynomials [2].

The first step in defining the resultant of two entire functions was done in [11], where authors studied the case of an entire function and a polynomial (or an entire function with a finite number of zeroes). The conditions for an entire function to have a finite number of zeroes were studied in [12]. The paper [13] generalizes the results of [11] to the case when one of entire functions satisfies some strict conditions, but may have infinite number of zeroes.

Our approach allows to determine if entire functions have common zeroes without computing the zeroes. Our formulae for the resultant involve power sums of roots, which may be found by Newton's formulae without finding the zeroes.

2. A cubic polynomial and an entire function. To begin with, we consider a cubic polynomial $f(z)$ and a polynomial $g(z)$ of degree n:

$$\begin{cases} f(z) = a_0 + a_1 z + a_2 z^2 + z^3, \\ g(z) = b_0 + b_1 z + b_2 z^2 + \ldots + b_n z^n. \end{cases} \tag{1}$$

Denote the roots of $f(z)$ by z_1, z_2, z_3. Then the resultant $R(f, g)$ of f and g is computed as

$$\begin{aligned} R(f,g) = \prod_{i=1}^{3} g(z_i) = g(z_1)\cdot g(z_2)\cdot g(z_3) = \sum_{k=0}^{n} b_k^3 (z_1 z_2 z_3)^k + \\ + \sum_{s=0}^{n} \sum_{t=s+1}^{n} (z_1 z_2 z_3)^s (b_s b_t^2 [(z_1 z_2)^{t-s} + (z_1 z_3)^{t-s} + (z_2 z_3)^{t-s}] + \\ + b_s^2 b_t [z_1^{t-s} + z_2^{t-s} + z_3^{t-s}]) + \\ + \sum_{s=0}^{n} \sum_{t=s+1}^{n} \sum_{p=t+1}^{n} b_s b_t b_p (z_1 z_2 z_3)^s [z_1^{p-s} z_2^{t-s} + z_2^{p-s} z_1^{t-s} + z_1^{p-s} z_3^{t-s} + \\ + z_3^{p-s} z_1^{t-s} + z_2^{p-s} z_3^{t-s} + z_3^{p-s} z_2^{t-s}]. \end{aligned} \tag{2}$$

Note that the expressions in z_j's in these sums are symmetric polynomials. This means that they can be expressed via the coefficients of $f(z)$, however, we rewrite them using the power sums of the roots:

$$S_k = z_1^k + z_2^k + z_3^k, \quad k \in \mathbb{N}.$$

If necessary, S_k's may be rewritten via the elementary symmetric polynomials e_j's of the roots:

$$\begin{cases} e_1 = z_1 + z_2 + z_3, \\ e_2 = z_1 \cdot z_2 + z_1 \cdot z_3 + z_2 \cdot z_3, \\ e_3 = z_1 \cdot z_2 \cdot z_3, \end{cases}$$

using the Newton-Girard formula (see, e. g., [3], [7, formula 8]):

$$S_k = \sum_{\substack{r_1+2r_2+\ldots+kr_k=k, \\ r_1,r_2,\ldots,r_k \geqslant 0}} (-1)^k \frac{k(r_1+\ldots+r_k-1)!}{r_1! \cdot \ldots \cdot r_k!} \prod_{i=1}^{k} (-e_i)^{r_i}. \tag{3}$$

Viete's formulae allow to rewrite these in terms of the coefficients of $f(z)$.

It is easy to see that

$$z_1^k z_2^l + z_1^l z_2^k + z_1^k z_3^l + z_1^l z_3^k + z_2^k z_3^l + z_2^l z_3^k = S_k \cdot S_l - S_{k+l}, \quad k, l \in \mathbb{N}. \tag{4}$$

and

$$(z_1 z_2)^k + (z_1 z_3)^k + (z_2 z_3)^k = \frac{1}{2}\left(S_k^2 - S_{2k}\right).$$

Therefore, the expression (2) becomes

$$\begin{aligned} R = \prod_{i=1}^{3} g(z_i) = \sum_{k=0}^{n} b_k^3 (-a_0)^k + \\ + \sum_{s=0}^{n} \sum_{t=s+1}^{n} (-a_0)^s \left(\frac{1}{2} b_s b_t^2 \left(S_{t-s}^2 - S_{2t-2s}\right) + b_s^2 b_t S_{t-s} \right) + \\ + \sum_{s=0}^{n} \sum_{t=s+1}^{n} \sum_{p=t+1}^{n} b_s b_t b_p (-a_0)^s \left[S_{t-s} \cdot S_{p-s} - S_{t+p-2s}\right]. \end{aligned} \tag{5}$$

Thus, we arrive at the first result:

Theorem 1. *The resultant $R(f, g)$ of the polynomials (1) is given by (5).*

Passing to the limit as $n \to \infty$ in (5), we get the result for a system consisting of a cubic polynomial and an entire function.

Theorem 2. *Let $g(z)$ be an entire function in one complex variable and*

$$g(z) = b_0 + b_1 z + b_2 z^2 + \ldots + b_n z^n + \ldots,$$

be its series expansion, and let $f(z)$ be a cubic polynomial as in (1). Then the resultant $R(f,g)$ is given by

$$R(f,g) = \sum_{k=0}^{\infty} b_k^3(-a_0)^k + \\ + \sum_{s=0}^{\infty}\sum_{t=s+1}^{\infty}(-a_0)^s\left(\frac{1}{2}b_s b_t^2\left(S_{t-s}^2 - S_{2t-2s}\right) + b_s^2 b_t S_{t-s}\right) + \\ + \sum_{s=0}^{\infty}\sum_{t=s+1}^{\infty}\sum_{p=t+1}^{\infty} b_s b_t b_p(-a_0)^s\left[S_{t-s}\cdot S_{p-s} - S_{t+p-2s}\right],$$

provided the series in the right-hand side converges absolutely.

3. A case of two polynomials. Consider a system of two equations consisting of two polynomials of degrees m and n

$$\begin{cases} f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_{m-1}z^{m-1} + z^m, \\ g(z) = b_0 + b_1 z + b_2 z^2 + \ldots + b_n z^n. \end{cases} \tag{6}$$

The case $m = 2$ has been considered in [14], and the case $m = 3$ has been considered above.

Denote the roots of $f(z)$ by $z_1, z_2, \ldots, z_m$ (there may be multiple roots among them). And S_k are there power sums of order k. Elementary but cumbersome calculations show that

$$\prod_{i=1}^{m} g(z_i) = G_1 + G_2 + \ldots + G_i + \ldots + G_m, \tag{7}$$

where each G_i consists of several multiple sums. Namely,

$$G_1 = \sum_{s=0}^{n}(-1)^{ms}a_0^s b_s^m,$$

$$G_2 = \sum_{s=0}^{n}\sum_{t=s+1}^{n}(-1)^{ms}a_0^s b_s^{m-1}b_t S_{t-s} + \\ + \sum_{s=0}^{n}\sum_{t=s+1}^{n}(-1)^{ms}a_0^s b_s^{m-2}b_t^2\left(\frac{S_{t-s}^2 - S_{2t-2s}}{2}\right) + \\ + \sum_{s=0}^{n}\sum_{t=s+1}^{n}(-1)^{ms}a_0^s b_s^{m-3}b_t^3\left(\frac{S_{t-s}^3 - 3S_{2t-2s}\cdot S_{t-s} + 2S_{3t-3s}}{4}\right) +$$

$$+\sum_{l=4}^{m-1}\sum_{s=0}^{n}\sum_{t=s+1}^{n}(-1)^{ms}a_0^s b_s^{m-l} b_t^l\left(\frac{S_{t-s}^l - lS_{t-s}\cdot S_{(l-1)(t-s)} + (l-1)S_{l(t-s)}}{2^{l-1}}\right),$$

$$G_3 = \sum_{s=0}^{n}\sum_{t=s+1}^{n}\sum_{p=t+1}^{n}(-1)^{ms}a_0^s b_s^{m-2} b_t b_p\left(S_{t-s}\cdot S_{p-s} - S_{t+p-2s}\right) +$$

$$+\sum_{s=0}^{n}\sum_{t=s+1}^{n}\sum_{p=t+1}^{n}(-1)^{ms}a_0^s b_s^{m-3} b_t^2 b_p \times$$

$$\times\left(\frac{S_{t-s}^2\cdot S_{p-s} - S_{2t-2s}\cdot S_{p-s} - 2S_{t+p-2s}\cdot S_{t-s} + 2S_{2t+p-3s}}{2}\right) +$$

$$+\sum_{s=0}^{n}\sum_{t=s+1}^{n}\sum_{p=t+1}^{n}(-1)^{ms}a_0^s b_s^{m-3} b_t b_p^2 \times$$

$$\times\left(\frac{S_{p-s}^2\cdot S_{t-s} - S_{2p-2s}\cdot S_{t-s} - 2S_{t+p-2s}\cdot S_{p-s} + 2S_{2p+t-3s}}{2}\right) +$$

$$+\sum_{l=4}^{m-1}\sum_{\substack{\beta_1+\beta_2=l\\ \beta_1\geqslant 1,\,\beta_2\geqslant 1}}\sum_{s=0}^{n}\sum_{t=s+1}^{n}\sum_{p=t+1}^{n}(-1)^{ms}a_0^s b_s^{m-l} b_t^{\beta_1} b_p^{\beta_2}\times$$

$$\times\left(\frac{S_{p-s}^{\beta_2}\cdot S_{t-s}^{\beta_1} - \beta_2 S_{p-s}\cdot S_{\beta_1(t-s)+(p-s)(\beta_2-1)} -}{2^{l-2}}\right.$$

$$\left.\frac{-\beta_1 S_{t-s}\cdot S_{\beta_2(p-s)+(t-s)(\beta_1-1)} + (\beta_1+\beta_2-1)S_{\beta_1(t-s)+\beta_2(p-s)}}{}\right),$$

$$G_4 = \sum_{s=0}^{n}\sum_{t=s+1}^{n}\sum_{p=t+1}^{n}\sum_{r=p+1}^{n}(-1)^{ms}a_0^s b_s^{m-3} b_t b_p b_r \times$$

$$\times\left(S_{t-s}\cdot S_{p-s}\cdot S_{r-s} - S_{t+p-2s}\cdot S_{r-s} - S_{t+r-2s}\cdot S_{p-s} -\right.$$

$$\left. - S_{p+r-2s}\cdot S_{t-s} + 2S_{p+t+r-3s}\right) +$$

$$+\sum_{l=4}^{m-1}\sum_{\substack{\beta_1+\beta_2+\beta_3=l\\ \beta_1\geqslant 1,\,\beta_2\geqslant 1,\,\beta_3\geqslant 1}}\sum_{s=0}^{n}\sum_{t=s+1}^{n}\sum_{p=t+1}^{n}\sum_{r=p+1}^{n}(-1)^{ms}a_0^s b_s^{m-l} b_t^{\beta_1} b_p^{\beta_2} b_r^{\beta_3}\times$$

$$\times\left(\frac{S_{t-s}^{\beta_1}\cdot S_{p-s}^{\beta_2}\cdot S_{r-s}^{\beta_3} - \beta_1 S_{t-s}\cdot S_{\beta_2(p-s)+\beta_3(r-s)+(\beta_1-1)(t-s)} -}{2^{l-3}}\right.$$

$$\frac{-\beta_2 S_{p-s}\cdot S_{\beta_1(t-s)+\beta_3(r-s)+(\beta_2-1)(p-s)} - \beta_3 S_{r-s}\cdot S_{\beta_1(t-s)+\beta_2(p-s)+(\beta_3-1)(r-s)} +}{}$$

$$\left.\frac{+(\beta_1+\beta_2+\beta_3-1) S_{\beta_1(t-s)+\beta_2(p-s)+\beta_3(r-s)}}{}\right).$$

If we denote the summation indices by $\alpha_1, \alpha_2, \ldots, \alpha_m$ and differences of indices by $j_2 = \alpha_2 - \alpha_1$, $j_3 = \alpha_3 - \alpha_1$, $\ldots$, $j_m = \alpha_m - \alpha_1$, then

$$G_m = \sum_{\alpha_1=0}^{n} \sum_{\alpha_2=\alpha_1+1}^{n} \sum_{\alpha_3=\alpha_2+1}^{n} \ldots \sum_{\alpha_m=\alpha_{m-1}+1}^{n} (-1)^{m\alpha_1} a_0^{\alpha_1} b_{\alpha_1}^{m-(m-1)} b_{\alpha_2} \ldots b_{\alpha_m} \times$$
$$\times \left(\prod_{k=2}^{m} S_{j_k} - \sum_{k=2}^{m} S_{j_k} \cdot S_{j_2+\ldots+j_{k-1}+j_{k+1}+\ldots+j_m} + (m-2) S_{j_2+j_3+\ldots+j_m} \right).$$

Remark. *If $f(z)$ is a polynomial of degree m, then G_m consists of one m-tuple sum.*

Theorem 3. *The resultant $R(f,g)$ of the system of polynomials (6) is given by (7).*

The proof follows easily by induction over m with the base for $m = 3$.

4. A general case. For the general case, it is enough to pass to the limit as n and m tend to infinity.

Theorem 4. *Let $f(z)$ and $g(z)$ be entire functions of one complex variable*

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_m z^m + \ldots,$$
$$g(z) = b_0 + b_1 z + b_2 z^2 + \ldots + b_n z^n + \ldots.$$

Then the resultant $R(f,g)$ is given by formula (7), where we pass to the limit as n and m tend to infinity. The function $f(z)$ must be transformed to the form (6) by dividing it on a_m.

We suggest also another approach to obtain these formulae, which is more convenient for algorithmic realization. Let

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + z^m,$$
$$g(z) = b_0 + b_1 z + b_2 z^2 + \ldots + b_n z^n + \ldots.$$

Denote the roots of $f(z)$ by $z_1, \dots, z_m$. Then the resultant $R(f, g)$ of f and g is

$$R = \prod_{i=1}^{m} g(z_i) = \Big(\sum_{j_1=0}^{\infty} b_{j_1} z_1^{j_1}\Big) \dots \Big(\sum_{j_m=0}^{\infty} b_{j_m} z_m^{j_m}\Big). \tag{8}$$

Each term in this product is in one-to-one correspondence with a point of $\mathbb{Z}^n_{\geqslant}$. To group all the terms into symmetric polynomials, we subdivide $\mathbb{Z}^n_{\geqslant}$ into cones. In fact, due to symmetry, it is enough to describe only one cone containing exactly one monomial from each symmetric polynomial in (8). For this, we denote the standard base vectors of $\mathbb{Z}^n$ by $e_1, \dots, e_n$ and define vectors

$$v_k = \sum_{j=1}^{k} e_j, \quad j = 1, \dots, n.$$

These vectors generate the required simplicial cone. The faces of this cone always intersect; to avoid this, we consider the following sets of integer points lying in partially closed cones:

$$\sigma_0 = \mathbb{Z}_{\geqslant} v_n;$$
$$\sigma_1^k = \mathbb{Z}_{\geqslant} v_n + \mathbb{Z}_{>} v_k, \quad k = 1, \dots, n-1;$$
$$\sigma_2^{\mathbf{k}} = \mathbb{Z}_{\geqslant} v_n + \mathbb{Z}_{>} v_{k_1} + \mathbb{Z}_{>} v_{k_2}, \mathbf{k} = (k_1, k_2), k_1, k_2 = 1, \dots, n-1, k_1 \neq k_2;$$
$$\dots$$
$$\sigma_n = \mathbb{Z}_{\geqslant} v_n + \mathbb{Z}_{>} v_1 + \dots + \mathbb{Z}_{>} v_{n-1}.$$

A point ξ of any of these sets encodes a monomial

$$b_\xi z^\xi = b_{\xi_1} \dots b_{\xi_n} z_1^{\xi_1} \dots z_n^{\xi_n}.$$

Each set σ from the list above produces the series

$$\sum_{\alpha \in \sigma} b_\alpha \operatorname{Sym}(z^\alpha), \tag{9}$$

where $\operatorname{Sym}(z^\alpha)$ is the sum of elements of the orbit of z^α under the action of the symmetric group

$$\operatorname{Sym}(z^\alpha) = \sum_{p \text{ is a permutation}} z^{p(\alpha)}.$$

For example, σ_0 gives the series

$$\sum_{s=0}^{\infty} b_{s\cdot v_n} \operatorname{Sym}(z^{s\cdot v_n}) = \sum_{s=0}^{\infty} b_s^n (z_1 \dots z_n)^s$$

and σ_1^1

$$\sum_{s=0}^{\infty}\sum_{t=1}^{\infty} b_{s\cdot v_n + t\cdot v_1} \operatorname{Sym}(z^{s\cdot v_n + t\cdot v_1}) =$$
$$= \sum_{s=0}^{\infty}\sum_{t=1}^{\infty} b_{s+t} b_s^{n-1} (z_1^{s+t} z_2^s \dots z_n^s + \dots + z_1^s z_2^s \dots z_n^{s+t}) =$$
$$= \sum_{s=0}^{\infty}\sum_{t=1}^{\infty} b_{s+t} b_s^{n-1} (z_1 \dots z_n)^s (z_1^t + \dots + z_n^t).$$

The symmetric polynomials in each series (9) can be expressed via coefficients of f and power sums of its roots by standard procedures of computer algebra systems or by formulae from [3]. Summing up over all σ from the list and passing to the limit as m tends to infinity, we obtain the result of Theorem 4, provided that all the series converge absolutely.

5. Examples. In this section, we consider examples that demonstrate the main results of the paper, as well as examples leading to computation of some earlier unknown sums of multiple series. These results are of independent interest. We begin with a simple example illustrating Theorem 1, where we compute all the quantities in formula (5).

Example 1. Consider the system of equations ($n = 2$)

$$\begin{cases} f(z) = z(z-1)(z+1) = z^3 - z, \\ g(z) = (z-2)(z+2) = z^2 - 4. \end{cases}$$

In this case,

$$a_0 = 0, \quad a_1 = -1, \quad a_2 = 0, \quad b_0 = -4, \quad b_1 = 0, \quad b_2 = 1.$$

Then formula (5) takes the form

$$R(f,g) = \sum_{k=0}^{2} b_k^3 (-a_0)^k + \sum_{s=0}^{2}\sum_{t=s+1}^{2} (-a_0)^s \Big(\frac{1}{2} b_s b_t^2 \left(S_{t-s}^2 - S_{2t-2s}\right) + b_s^2 b_t S_{t-s}\Big) +$$

$$+\sum_{s=0}^{2}\sum_{t=s+1}^{2}\sum_{p=t+1}^{2} b_s b_t b_p (-a_0)^s \left[S_{t-s}\cdot S_{p-s} - S_{t+p-2s}\right].$$

Since $a_0 = 0$, we get only terms that correspond to $k = 0$ and $s = 0$ in these sums, that are:

$$R(f,g) = b_0^3 + \sum_{t=1}^{2}\left(\frac{1}{2}b_0 b_t^2\left(S_t^2 - S_{2t}\right) + b_0^2 b_t S_t\right) + \\ + \sum_{t=1}^{2}\sum_{p=t+1}^{2} b_0 b_t b_p \left[S_t \cdot S_p - S_{t+p}\right].$$

Since $b_1 = 0$, the one-dimensional sum has only summands for $t = 2$ and the double sum has no non-zero terms. Thus,

$$R(f,g) = b_0^3 + \frac{1}{2}b_0 b_2^2\left(S_2^2 - S_4\right) + b_0^2 b_2 S_2.$$

We find the power sums of the roots S_2 and S_4 without using the roots themselves, but just formulae (3) and Viete's formulae. In our case

$$S_2 = 2, \quad S_4 = 2.$$

Thus, $R(f,g) = -36$.

Example 2. Consider the system of equations

$$\begin{cases} f(z) = z^3 - a^3, \\ g(z) = b_0 + b_1 z + b_2 z^2 + \ldots + b_n z^n. \end{cases}$$

This example demonstrates Theorem 1. According to the notation introduced above,

$$z_1 = a, \quad z_{2,3} = \frac{-a \pm a\sqrt{3}i}{2}.$$

We have

$$\begin{cases} e_3 = z_1 z_2 z_3 = a^3, \\ e_2 = z_1 z_2 + z_1 z_3 + z_2 z_3 = 0, \\ e_1 = z_1 + z_2 + z_3 = 0. \end{cases}$$

The expressions S_k are distinct from zero only if k is a multiple of 3, and then they are equal to

$$S_k = z_1^k + z_2^k + z_3^k = 3a^k.$$

For such k, we get

$$(z_1z_2)^k + (z_1z_3)^k + (z_2z_3)^k = \frac{1}{2}\left(S_k^2 - S_{2k}\right) = 3a^{2k}.$$

Thus, by (5) we have

$$\prod_{i=1}^{3} g(z_i) = \sum_{s=0}^{n} b_s^3 a^{3s} + \sum_{s=0}^{n}\sum_{j=1}^{\left[\frac{n-s}{3}\right]} a^{3s}\left(3b_s b_{s+3j}^2 a^{6j} + 3b_s^2 b_{s+3j} a^{3j}\right) +$$
$$+\sum_{s=0}^{n}\sum_{t=s+1}^{n}\sum_{p=t+1}^{n} b_s b_t b_p a^{3s}\left[S_{t-s}\cdot S_{p-s} - S_{t+p-2s}\right].$$

In the last expression, the triple sum is distinct from zero only if either both lower indices are multiples of 3, or they are not but their sum is.

Example 3. Consider the system

$$\begin{cases} f(z) = z^3 - a^3, \\ g(z) = e^{bz} = \sum\limits_{n=1}^{\infty} \frac{(bz)^n}{n!} = 1 + bz + \frac{(bz)^2}{2!} + \ldots + \frac{(bz)^n}{n!} + \ldots. \end{cases}$$

Using Theorem 2, we get

$$\prod_{i=1}^{3} g\left(z_i\right) =$$
$$= \sum_{k=0}^{\infty} \frac{(ab)^{3k}}{(k!)^3} + \sum_{s=0}^{\infty}\sum_{j=1}^{\infty} a^{3s}\left(3\frac{b^{3s+6j}}{s!(s+3j)!^2}a^{6j} + 3\frac{b^{3s+3j}}{s!^2(s+3j)!}a^{3j}\right) +$$
$$+\sum_{s=0}^{\infty}\sum_{t=s+1}^{\infty}\sum_{p=t+1}^{\infty} b_s b_t b_p a^{3s}\left[S_{t-s}\cdot S_{p-s} - S_{t+p-2s}\right] = 1.$$

Acknowledgment. The first author was supported by Krasnoyarsk Mathematical Center and financed by the Ministry of Science and Higher Education of the Russian Federation in the framework of the establishment and development of regional Centers for Mathematics Research and Education (Agreement No. 075-02-2020-1534/1), the second author was supported by the RFBR (grant 19-31-60012).

References

[1] Bourbaki N. *Algebra. Polynomials and Fields, Ordered Groups.* Nauka, Moscow, 1965 [Russian translation].

[2] Bykov V. I., Tsybenova S. B. *Nonlinear Models of Chemical Kinetics.* KRASAND, Moscow, 2011 (in Russian).

[3] Chamberlin S., Rafizadeh A. *A generalized Newton-Girard formula for monomial symmetric polynomials.* Rocky Mountain J. Math., 2020, vol. 50, no. 3, pp. 941–946. DOI: https://doi.org/10.1216/rmj.2020.50.941

[4] Gohberg I. C., Heinig G. *Resultant Matrix and its Generalization. I. Resultant Operator of Matrix Polynomial.* Acta Sci. Math., 1975, vol. 72, pp. 41–61.

[5] Gohberg I. C., Heinig G. *Resultant Matrix and its Generalization. II. Continual Analog of Resultant Matrix.* Acta Math. Acad. Sci. Hungar, 1976, vol. 28, pp. 189 – 209.

[6] Gohberg I. C., Lerer L. E. *Resultant Operators of a Pair of Analytic Functions.* Proc. Amer. Math. Soc., 1978, vol. 72, no. 1, pp. 65 – 73.

[7] Gould H. W. *The Girard-Waring power sum formulas for symmetric functions and Fibonacci sequences.* Fibonacci Quart., 1999, vol. 37, no. 2, pp. 135 – 140.

[8] Gustafsson B., Tkachev V. G. *The Resultant on Compact Riemann Surfaces.* Comm. Math. Physics, 2009, vol. 10, pp. 265 – 308. DOI: https://doi.org/DOI:10.1007/s00220-008-0622-2

[9] Krein M. G., Naimark M. A. *The Method of Symmetric and Hermitian Forms in the Theory of the Separation of the Roots of Algebraic Equation.* Linear and Multilinear Algebra, 1981, vol. 10, no. 4, pp. 265 – 308. DOI: https://doi.org/10.1080/03081088108817420

[10] Kurosh A. G. *Course in Higher Algebra.* Nauka, Moscow, 1968 (Russian translation).

[11] Kytmanov A. M., Naprienko Ya. M. *An Approach to Define the Resultant of Two Entire Functions.* Complex variables and elliptic equat., 2017, vol. 62, no. 2, pp. 269 – 286. DOI: https://doi.org/10.1080/17476933.2016.1218855

[12] Kytmanov A. M., Khodos O. V. *On Localization of Zeros of an Entire Function of Finite Order of Growth.* Complex analysis and operator theory, 2017, vol. 11, no. 2, pp. 393 – 416. DOI: https://doi.org/10.1007/s11785-016-0606-8

[13] Kytmanov A. M., Khodos O. V. *An Approach to the Determination of the Resultant of Two Entire Functions.* Russian Mathematics, 2018, vol. 62, no. 4, pp. 42 – 51. DOI: https://doi.org/10.3103/S1066369X18040059

[14] Kytmanov A. M., Myshkina E. K. *On Some Approach for Finding the Resultant of Two Entire Functions.* J. Siberian Federal Univ. Math. Phys.,

2019, vol. 12, no. 4, pp. 434 – 438.
DOI: https://doi.org/10.17516/1997-1397-2019-12-4-434-438

[15] Morozov A. Yu., Shakirov Sh. R. *New and Old Results in Resultant Theory.* Theor. math. phys., 2010, vol. 163, no. 2, pp. 587 – 617.
DOI: https://doi.org/10.1007/s11232-010-0044-0

[16] van der Waerden B. L. *Algebra.* Springer-Verlag, Berlin-Heidelderg-New York, 1966.

Received May 29, 2020.
In revised form, August 16, 2020.
Accepted August 16, 2020.
Published online September 3, 2020.

A. M. Kytmanov
Siberian Federal University,
660041, 79, Svobodny pr., Krasnoyarsk, Russia
E-mail: akytmanov@sfu-kras.ru

E. K. Myshkina
Institute of Computational Modelling, SB RAS
660036, 50/44, Akademgorodok, Krasnoyarsk, Russia
E-mail: elfifenok@mail.ru