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ON THE PROBLEM OF MEAN PERIODIC EXTENSION

Abstract. This paper is devoted to a study of the following version of the mean periodic extension problem:

(i) Suppose that $T \in \mathcal{E}'(\mathbb{R}^n)$, $n \geq 2$, and E is a non-empty subset of $\mathbb{R}^n$. Let $f \in C(E)$. What conditions guarantee that there is an $F \in C(\mathbb{R}^n)$ coinciding with f on E, such that $F * T = 0$ in $\mathbb{R}^n$?

(ii) If such an extension F exists, then estimate the growth of F at infinity.

In this paper, we present a solution of this problem for a broad class of distributions T in the case when E is a segment in $\mathbb{R}^n$.

Key words: *convolution equation, mean periodicity, continuous extension, spherical transform*

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1. Introduction. Let $\mathbb{R}^n$ be the real Euclidean space of dimension n with Euclidean norm $|\cdot|$. By $\mathcal{D}'(\mathcal{O})$ (respectively, $\mathcal{E}'(\mathcal{O})$), we denote the space of distributions (respectively, the space of compactly supported distributions) on a domain $\mathcal{O} \subset \mathbb{R}^n$, and by $\mathcal{D}(\mathcal{O})$, the space of compactly supported infinitely differentiable functions on $\mathcal{O}$. Given $T \in \mathcal{E}'(\mathbb{R}^n)$, we denote the support of T by supp T, and put

$$\mathcal{O}^T = \{x \in \mathbb{R}^n : x - y \in \mathcal{O} \quad \text{for each} \quad y \in \operatorname{supp} T\}.$$

If $\mathcal{O}^T \neq \varnothing$, then, for any $f \in \mathcal{D}'(\mathcal{O})$, the convolution $f * T$ is defined on $\mathcal{O}^T$ by the formula

$$\langle f * T, \varphi \rangle = \langle f_y, \langle T_x, \varphi(x + y) \rangle \rangle, \quad \varphi \in \mathcal{D}(\mathcal{O}^T)$$

(the subscripts of f and T mean the action with respect to the variable indicated). We set

$$\mathcal{D}'_T(\mathcal{O}) = \{f \in \mathcal{D}'(\mathcal{O}) : f * T = 0 \ \text{ in } \ \mathcal{O}^T\}$$

in the case where $\mathcal{O}^T \neq \varnothing$, and let $\mathcal{D}'_T(\mathcal{O}) = \mathcal{D}'(\mathcal{O})$ if $\mathcal{O}^T = \varnothing$.

Distributions from the class $\mathcal{D}'_T(\mathcal{O})$ are said to be mean periodic in $\mathcal{O}$ with respect to T. If $\mathfrak{M}(\mathcal{O})$ is a subset of $\mathcal{D}'(\mathcal{O})$, then we denote the intersection $\mathcal{D}'_T(\mathcal{O}) \cap \mathfrak{M}(\mathcal{O})$ by $\mathfrak{M}_T(\mathcal{O})$. For example, $C_T(\mathcal{O}) = \mathcal{D}'_T(\mathcal{O}) \cap C(\mathcal{O})$.

The general mean periodic extension problem is stated as follows.

Problem 1.

(i) *Suppose that $f \in \mathfrak{M}_T(\mathcal{O})$ and a domain $\mathcal{O}_1$ contains $\mathcal{O}$. Under what conditions does there exist an $F \in \mathfrak{M}_T(\mathcal{O}_1)$ coinciding with f on $\mathcal{O}$?*

(ii) *If such an extension F exists, is it unique?*

Even the one-dimensional case of Problem 1 is profound and deeply related to various branches of analysis. It was studied by J.-P. Kahan, V. D. Golovin, A. F. Leont'ev, A. M. Sedletskii, and other authors (see [3], [5]– [10] and the references therein). The answer to Problem 1 (ii) is positive for every $T \in \mathcal{E}'(\mathbb{R})$ by Titchmarsh's support theorem (see [2, Theorem 4.3.3] as well as [3, Chapter 5, Section 1], [7, Part 3, Chapter 1, Theorem 1.1]. The answer to Problem 1 (i) depends essentially on properties of T related to the distribution of zeros of its Fourier transform, that is, the entire function

$$\widehat{T}(z) = \left\langle T, e^{-izt} \right\rangle, \quad z \in \mathbb{C}.$$

The strongest results in this direction for the classes $\mathcal{D}'_T$ and C^∞_T were obtained by the first author ([7, Part 3]). In particular, the extension in Problem 1 (i) for these classes was shown to exist under the following conditions.

(a)

$$\underset{\lambda \in \mathcal{Z}(\widehat{T})}{sup} \frac{|\mathrm{Im}\lambda|}{\ln(2 + |\lambda|)} < +\infty,$$

where $\mathcal{Z}(\widehat{T}) = \{z \in \mathbb{C} : \widehat{T}(z) = 0\}$.

(b) The sequence of multiplicities m_λ of the zeros $\lambda \in \mathcal{Z}(\widehat{T})$ is bounded.

(c) For every $\lambda \in \mathcal{Z}(\widehat{T})$ we have

$$\sum_{j=0}^{m_\lambda - 1} \left| \frac{1}{j!} \left(\frac{(z-\lambda)^{m_\lambda}}{\widehat{T}(\lambda)} \right)^{(j)} \Bigg|_{z=\lambda} \right| \leq (2 + |\lambda|)^\gamma,$$

where the constant $\gamma > 0$ is independent of λ.

We point out that the set of distributions $T \in \mathcal{E}'(\mathbb{R}^n)$ satisfying conditions (a)–(c) is quite large and contains many of the distributions used in applications ([7, Part 3]). On the other hand, there are examples of distributions $T \in \mathcal{E}'(\mathbb{R})$ for which any two of the conditions (a) – (c) hold but the third fails. The question of the necessity and unimprovability of these conditions in extension theorems was studied in [10]. First, the counterexamples showing that condition (a) is unimprovable and (c) is necessary, were constructed (see [10, Theorems 3, 5]). Second, condition (b) can be weakened as follows:

$$\mathop{sup}_{\lambda \in Z(\widehat{T})} \frac{m_\lambda}{\ln(2+|\lambda|)} < +\infty,$$

and this estimate is already unimprovable even if the extension in not required to be mean periodic (see [10, Theorems 1, 2, 4]). Finally, the investigation of the one-dimensional case in [10] contains a result on non-existence of a continuous extension of a continuous mean periodic function for which m_λ grows faster than $|\mathrm{Im}\lambda|$ (see [10, Theorem 6]).

Some results on the possibility of a mean periodic extension of functions from the classes C and L^p for distributions T of the form

$$\langle T, \varphi \rangle = \int_{-r}^{r} \varphi(t)\, d\sigma(t), \quad \varphi \in \mathcal{D}(\mathbb{R}),$$

where σ is a function of bounded variation on $[-r, r]$ with a jump at one of the points $\pm r$, were obtained in [3, Chapter 5], [5]. The presence of jumps of σ at the points $\pm r$ imposes a number of constraints on the zeros of $\widehat{T}$. Among them are the following conditions (see [3, Lemma 5.1.1]):

(a) All zeros λ of the function $\widehat{T}$ are contained in some horizontal strip.

(b) The sequence of multiplicities $\{m_\lambda\}$ of the zeros λ is bounded.

(c) For any $\delta > 0$, there exists a constant $c_\delta > 0$, such that

$$|\widehat{T}(z)| \geq c_\delta e^{r|\mathrm{Im}\, z|}$$

outside disks of radius δ centered at λ.

The multidimensional case of Problem 1 is more complicated and more specific. For example, the continuation is not generally unique. This can be seen from the example of linear hyperbolic differential equations with constant coefficients. Moreover, when speaking of the existence of an extension, one must impose additional conditions on f (see [10, Proposition 10]). The main results in this direction in dimensions $n \geq 2$ were obtained by the authors (see [7] – [10]). This is done using the technique of transmutation operators related to expansions in eigenfunctions of the Laplacian (see [8]).

This paper is devoted to a study of the following version of the mean periodic extension problem.

Problem 2.

(i) *Suppose that $T \in \mathcal{E}'(\mathbb{R}^n)$, $n \geq 2$ and E is a non-empty subset of $\mathbb{R}^n$. Let $f \in C(E)$. What conditions guarantee that there exists an $F \in C_T(\mathbb{R}^n)$ coinciding with f on E?*

(ii) *If such an extension F does exist, then estimate the growth of F at infinity.*

In this paper, we present a solution of Problem 2 for a broad class of distributions T in the case when E is a segment in $\mathbb{R}^n$ (see Theorem 1 below). The precise statement of the main result is given in the next section. In Section 3, we prove some auxiliary lemmas. The proof of Theorem 1 is contained in Section 4.

2. The main results. Throughout, we assume that $n \geq 2$ and

$$E = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : -L \leq x_1 \leq L, x_2 = \ldots = x_n = 0\}$$

for some $L > 0$. Let $SO(n)$ be the group of rotations of $\mathbb{R}^n$. A distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is said to be radial if

$$\langle T, \varphi(x) \rangle = \langle T, \varphi(\tau x) \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^n)$$

for all $\tau \in SO(n)$. We write $\mathcal{E}'_\natural(\mathbb{R}^n)$ for the set of all radial distributions from the class $\mathcal{E}'(\mathbb{R}^n)$. Given $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$, we denote the spherical transform of T by $\widetilde{T}$, that is,

$$\widetilde{T}(z) = 2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)\left\langle T_x, \frac{J_{\frac{n}{2}-1}(z|x|)}{(z|x|)^{\frac{n}{2}-1}}\right\rangle, \quad z \in \mathbb{C}$$

(here, Γ is the gamma-function and $J_{\frac{n}{2}-1}$ is the Bessel function of the first kind). We set

$$\mathcal{Z}(T) = \{z \in \mathbb{C} : \widetilde{T}(z) = 0\}.$$

Let $a > 0$. We say that a distribution T from $\mathcal{E}'_\natural(\mathbb{R}^n)$ belongs to $\mathcal{Z}_a$ if there is a sequence $\{\lambda_m\}_{m=1}^\infty$ of points in $\mathcal{Z}(T)$ such that

$$\lambda_m = am + \varepsilon_m, \quad m = 1, 2, \dots, \tag{1}$$

where

$$\sum_{m=1}^{\infty} |\varepsilon_m|^2 < +\infty. \tag{2}$$

We emphasize that for each $a > 0$ the class $\mathcal{Z}_a$ is fairly large. It is often quite easy to verify whether a given distribution T belongs to $\mathcal{Z}_a$, using asymptotic expansions of $\widetilde{T}$, which are known under very general assumptions on T (see [4, Chapter 2, Theorem 10.2]). We note also that if $T \in \mathcal{Z}_a$, then $T * U \in \mathcal{Z}_a$ for each $U \in \mathcal{E}'_\natural(\mathbb{R}^n)$.

For $f \in C(E)$, $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$, let $C_{T,f}(\mathbb{R}^n)$ denote the set of all $F \in C_T(\mathbb{R}^n)$ coinciding with f on E.

Theorem 1. *The following assertions are valid.*

(i) Let $a > 0$, $T \in \mathcal{Z}_a$, and $f \in C(E)$. Then, for each $\varepsilon > 0$ there is $F \in C_{T,f}(\mathbb{R}^n)$, such that

$$F(x) = O\left(e^{\varepsilon |x_2|}\right), \quad x \in \mathbb{R}^n. \tag{3}$$

(ii) For each $a > 0$, there is a $T \in \mathcal{Z}_a$ with the following property: if $f \in C(E)$, $f \neq 0$, then for each $F \in C_{T,f}(\mathbb{R}^n)$ there is an $\varepsilon > 0$, such that

$$\overline{\lim_{x\to\infty}} |F(x)| e^{-\varepsilon |x|} > 0. \tag{4}$$

Several remarks are in order here. The proof of assertion (i) shows (see Section 4) that the function F in (i) depends only on x_1 and x_2. It is unclear whether the condition for T in (i) can be relaxed, but it is easy to see that this condition cannot be removed. For example, if $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$, supp $T = \{0\}$, and $\mathcal{Z}(T) \neq \varnothing$, then the equation $F * T = 0$ is an elliptic differential equation with constant coefficients. Hence, each solution F must be real-analytic in $\mathbb{R}^n$, and we see that $C_{T,f}(\mathbb{R}^n) = \varnothing$ if $f \in C(E)$ is not real-analytic in E. If, in addition, $\widetilde{T}(\lambda) = \widetilde{T}'(\lambda) = 0$ for some $\lambda \in \mathbb{C}$ and

$$f(x) = x_1 e^{i\lambda x_1}, \quad x \in E,$$

then the assumption $F \in C_{T,f}(\mathbb{R}^n)$ implies that

$$F(x_1, 0, \dots, 0) = x_1 e^{i\lambda x_1}, \quad x_1 \in \mathbb{R}.$$

In this case, $C_{T,f}(\mathbb{R}^n) \neq \varnothing$, but condition (3) is not met.

Assertion (ii) shows, in particular, that condition (3) cannot be strengthen in the general case. However, if all the numbers λ_m in (1) are real, then (3) can be replaced by

$$F(x) = O(1 + \varepsilon|x_2|), \quad x \in \mathbb{R}^n$$

(see the proof of (i) in Section 4). This means, in particular, that assertion (ii) is not generally true for each $T \in \mathcal{Z}_a$.

3. Auxiliary results. The following lemmas are needed in the proof of Theorem 1.

Lemma 1. *Let $z_1, \dots, z_m$ be pairwise different complex numbers. Then, for all $a \in \mathbb{R}$, $b > a$, $c_j \in \mathbb{C}$ $(j = 1, \dots, m)$, there is a function $g \in C^\infty(\mathbb{R})$ with a support on $[a, b]$, such that*

$$\widehat{g}(z_j) = \int_{-\infty}^{\infty} g(t) e^{-iz_j t} dt = c_j, \quad j = 1, \dots, m. \tag{5}$$

Proof. For brevity, we set

$$\xi = \frac{a+b}{2}, \qquad \eta = \frac{b-a}{2}. \tag{6}$$

Suppose that $u \in C^\infty(\mathbb{R})$, $u \neq 0$ and $\operatorname{supp} u \subset [-\eta, \eta]$. Then, for each $N \in \mathbb{Z}_+$, there exists a positive constant γ_N, such that

$$|\widehat{u}(z)| \leq \gamma_N \frac{e^{\eta|\operatorname{Im} z|}}{(1+|z|)^N} \quad \text{for all} \quad z \in \mathbb{C}.$$

Consider the function

$$v(z) = \widehat{u}(z) \prod_{j=1}^{m} (z - z_j), \quad z \in \mathbb{C}.$$

Let $\{k_j\}_{j=1}^m$ be the sequence of multiplicities of the zeros $\{z_j\}_{j=1}^m$ of v. Then the function

$$w(z) = \sum_{j=1}^{m} c_j e^{iz_j\xi} \frac{k_j!\, v(z)}{(z - z_j)^{k_j} v^{(k_j)}(z_j)} \tag{7}$$

is entire and

$$w(z_j) = c_j e^{iz_j\xi}, \quad j = 1, \dots, m. \tag{8}$$

Assume now that $|z - z_j| \geq 1$ for all j. Then

$$\left|\frac{v(z)}{(z-z_j)^{k_j}}\right| \leq |v(z)| \leq \gamma_N \frac{e^{\eta|\operatorname{Im} z|}}{(1+|z|)^N} \prod_{j=1}^{m} |z - z_j| \quad \text{for all} \quad N \in \mathbb{Z}_+,$$

$j = 1, \dots, m$. Together with (7), this shows that for each $N \in \mathbb{Z}_+$ there exists a constant $\gamma'_N > 0$ such that

$$|w(z)| \leq \gamma'_N \frac{e^{\eta|\operatorname{Im} z|}}{(1+|z|)^N} \quad \text{for all} \quad z \in \mathbb{C}.$$

By the classical Paley-Wiener theorem, there exists a function $h \in C^\infty(\mathbb{R})$, such that $\operatorname{supp} h \subset [-\eta, \eta]$ and $\widehat{h}(z) = w(z)$. Putting $g(t) = h(t - \xi)$, we conclude from (8) and (6) that g satisfies the required conditions. $\square$

To continue, for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$, we set

$$(z, \zeta) = \sum_{j=1}^{n} z_j \zeta_j.$$

Also, let

$$S = \{z \in \mathbb{C}^n : (z, z) = 1\}, \quad \mathbb{S}^{n-1} = S \cap \mathbb{R}^n. \tag{9}$$

Lemma 2. *Let $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$, $\lambda \in \mathbb{C}$, $\widetilde{T}(\lambda) = 0$, $\xi \in S$. Then the function*

$$h_\xi(x) = e^{i\lambda(x,\xi)}, \quad x \in \mathbb{R}^n$$

is in the class $C^\infty(\mathbb{R}^n)$.

Proof. For each $y \in \mathbb{R}^n$, one has

$$(h_\xi * T)(y) = \langle T, h_\xi(-x)\rangle\, h_\xi(y). \tag{10}$$

Since T is radial, we obtain

$$\langle T, h_\xi(-x)\rangle = \langle T, h_\xi(-\tau x)\rangle \tag{11}$$

for all $\tau \in SO(n)$. Let $d\tau$ be the Haar measure on $SO(n)$ normalized by $\int\limits_{SO(n)} d\tau = 1$. Formula (11) ensures that

$$\langle T, h_\xi(-x)\rangle = \int\limits_{SO(n)} \langle T, h_\xi(-\tau x)\rangle\, d\tau = \left\langle T, \int\limits_{SO(n)} h_\xi(-\tau x) d\tau \right\rangle. \tag{12}$$

Using now [1, Introducion, Section 3.1, formula (9)], we see that

$$\int_{SO(n)} h_\xi(-\tau x)\,d\tau = \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} e^{-i\lambda(|x|\eta,\,\xi)}\,d\omega(\eta),$$

where $d\omega$ is the area measure on $\mathbb{S}^{n-1}$ and

$$\omega_{n-1} = \int_{\mathbb{S}^{n-1}} d\omega(\eta) = n\pi^{\frac{n}{2}}/\Gamma\left(1+\frac{n}{2}\right).$$

Together with [1, Introducion, Section 3.2, Lemma 3.6], relation (12) brings us to the formula

$$\langle T, h_\xi(-x)\rangle = \widetilde{T}(\lambda) = 0.$$

Now, the assertion of Lemma 2 follows from (10). $\square$

4. Proof of the main result. Let us prove assertion (i) of Theorem 1. We start by noting that there is no loss of generality in assuming that

$$f(-L,0,\dots,0) = f(L,0,\dots,0) = 0 \tag{13}$$

(otherwise, it is enough to enlarge the number L and consider an appropriate continuous extension of f). By the assumption on T, there is a sequence $\{\lambda_m\}_{m=1}^\infty$ of points in $\mathcal{Z}(T)$ satisfying (1) and (2).

According to what has been said above, there exist $d>0$, $M>0$, such that

$$d > \max\left\{\frac{L}{\pi}, \frac{1}{a}\right\}, \tag{14}$$

and

$$m < d|\lambda_m| \tag{15}$$

for all $m\in\mathbb{N}$, $m>M$. Using (13) and (14), define the function $f_1\in C(\mathbb{R})$ by the formula

$$f_1(t) = \begin{cases} f(t,0,\dots,0) & \text{if } t\in[-L,L], \\ 0 & \text{if } t\notin[-L,L]. \end{cases} \tag{16}$$

Lemma 1 ensures us that there is a function $g\in C(\mathbb{R})$, such that $\operatorname{supp} g\subset[L,\pi d]$ and

$$\int_{-\pi d}^{\pi d} g(t)e^{-i\frac{k}{d}t}\,dt = -\int_{-\pi d}^{\pi d} f_1(t)e^{-i\frac{k}{d}t}\,dt \tag{17}$$

for all $k \in \mathbb{Z}$, $|k|\leq 2M$. Now define

$$f_2(t) = f_1(t) + g(t), \quad t \in \mathbb{R}.$$

Since $\operatorname{supp} f_1 \subset [-L, L]$ and $\operatorname{supp} g \subset [L, \pi d]$, we obtain $f_2 = f_1$ on $[-L, L]$. Owing to (17), the Fourier series of the function f_2 on $[-\pi d, \pi d]$ has the form

$$f_2(t) = \sum_{|m|>M} c_m e^{i\frac{m}{d}t}, \tag{18}$$

where the sequence $c_m \in \mathbb{C}$ satisfies the condition

$$\sum_{|m|>M} |c_m|^2 < +\infty.$$

Relation (2) shows that

$$\varepsilon_m \to 0 \quad \text{as} \quad m \to +\infty. \tag{19}$$

For $m > M$, define

$$\zeta_m = \frac{m}{\lambda_m d}.$$

For $z \in \mathbb{C}$, $z \neq 0$, we set $\sqrt{z} = \sqrt{|z|}e^{i(\arg z)/2}$, where $-\pi < \arg z \leq \pi$. Bearing (19) in mind, we see from (15) and (14) that

$$\sqrt{1-\zeta_m^2} = \sqrt{1 - \frac{1}{(ad)^2} + O\left(\frac{\varepsilon_m}{m}\right)} \quad \text{as} \quad m \to +\infty.$$

Hence,

$$\lambda_m \sqrt{1-\zeta_m^2} = \frac{m}{d}\gamma + \delta_m, \quad m > M, \tag{20}$$

where

$$\gamma = \sqrt{(ad)^2 - 1}, \quad \delta_m \in \mathbb{C}$$

and

$$\delta_m = O(\varepsilon_m) \quad \text{as} \quad m \to +\infty. \tag{21}$$

Next, for $m \in \mathbb{Z}$, $|m|> M$ define the function $h_m \in C^\infty(\mathbb{R}^n)$ by the formula

$$h_m(x) = \begin{cases} \exp(i\lambda_m(x_1\zeta_m + x_2\sqrt{1-\zeta_m^2})) & \text{if } m > M, \\ \exp\left(-i\lambda_{-m}\left(x_1\zeta_{-m} + x_2\sqrt{1-\zeta_{-m}^2}\right)\right) & \text{if } m < -M. \end{cases} \tag{22}$$

Since

$$\left(\zeta_m, \sqrt{1-\zeta_m^2}, 0, \dots, 0\right) \in S$$

for each $m > M$ (see (9)), we infer, by the definition of the sequence $\{\lambda_m\}_{m=1}^{\infty}$ and Lemma 2, that

$$h_m \in C_T^{\infty}(\mathbb{R}^n) \quad \text{for all} \quad |m| > M. \tag{23}$$

For $m \in \mathbb{Z}$, $|m| > M$, we set

$$\eta_m = \delta_{|m|} \tag{24}$$

(see (20)). In view of (15), (20) and (24), relation (22) yields

$$h_m(x) = e^{i\frac{m}{d}(x_1+\gamma x_2)} e^{i\eta_m x_2} \tag{25}$$

for all $m \in \mathbb{Z}$, $|m| > M$.

Let $\varphi \in C^{\infty}(\mathbb{R})$ be a non-negative function, such that

$$\operatorname{supp} \varphi \subset [-1, 1] \quad \text{and} \quad \int_{\mathbb{R}} \varphi(t)dt = 1. \tag{26}$$

Setting $\varphi_k(t) = k\varphi(kt)$, $k = 1, 2, \dots$ we conclude from (26) that

$$|\widehat{\varphi}_k(u)| \le 1 \quad \text{for all} \quad u \in \mathbb{R}, \tag{27}$$

$$\widehat{\varphi}_k(u) = O\left(|u|^{-2}\right) \quad \text{as} \quad u \to \infty, \tag{28}$$

and

$$\lim_{k\to\infty} \widehat{\varphi}_k(u) = 1 \quad \text{for each} \quad u \in \mathbb{R}. \tag{29}$$

Consider now the sequence of functions

$$F_k(x) = \sum_{|m|>M} c_m \widehat{\varphi}_k\left(\frac{m}{d}\right) h_m(x), \quad x \in \mathbb{R}^n. \tag{30}$$

Relations (25), (24), (21) and (28) show that the series in (30) converges locally uniformly on $\mathbb{R}^n$. Bearing (25) in mind, we have

$$F_k(x) = F_{k,1}(x) + F_{k,2}(x),$$

where

$$F_{k,1}(x) = \sum_{|m|>M} c_m \widehat{\varphi}_k\left(\frac{m}{d}\right) e^{i\frac{m}{d}(x_1+\gamma x_2)},$$

$$F_{k,2}(x) = \sum_{|m|>M} c_m \widehat{\varphi}_k\left(\frac{m}{d}\right) e^{i\frac{m}{d}(x_1+\gamma x_2)}\left(e^{i\eta_m x_2} - 1\right).$$

Let $\Phi \in C(\mathbb{R})$ be $2\pi d$ - periodic and assume that

$$\Phi(t) = f_2(t) \quad \text{for} \quad t \in [-\pi d, \pi d]. \tag{31}$$

It is easy to verify the relation

$$F_{k,1}(x) = (\Phi * \varphi_k)(x_1 + \gamma x_2) = \int_{\mathbb{R}} \Phi(t)\varphi_k(x_1 + \gamma x_2 - t)\, dt$$

(see (18)). This implies that the sequence $F_{k,1}$ converges locally uniformly on $\mathbb{R}^n$ and

$$\lim_{k\to\infty} F_{k,1}(x) = \Phi(x_1 + \gamma x_2).$$

Next, using (27) and the estimate

$$\left|e^{i\eta_m x_2} - 1\right| = \left|\int_0^{\eta_m x_2} e^{iz}\, dz\right| \le |\eta_m x_2| e^{|x_2 \mathrm{Im}\, \eta_m|},$$

we now get

$$\sum_{|m|>M} \left|c_m \widehat{\varphi}_k\left(\frac{m}{d}\right)\left(e^{i\eta_m x_2} - 1\right)\right| \le |x_2| \sum_{|m|>M} |c_m \eta_m| e^{|x_2 \mathrm{Im}\, \eta_m|}.$$

Hence,

$$|F_{k,2}(x)| \le |x_2| e^{\omega|x_2|} \sum_{|m|>M} \left(|c_m|^2 + |\eta_m|^2\right), \tag{32}$$

where

$$\omega = \max_{|m|>M} |\mathrm{Im}\, \eta_m|.$$

In addition, the sequence $F_{k,2}$ converges locally uniformly on $\mathbb{R}^n$ and

$$\Psi(x) = \lim_{k\to\infty} F_{k,2}(x) = \sum_{|m|>M} c_m e^{i\frac{m}{d}(x_1+\gamma x_2)}\left(e^{i\eta_m x_2} - 1\right)$$

(see (29)). Estimate (32) yields

$$|\Psi(x)| = O\left(|x_2| e^{\omega|x_2|}\right), \quad x \in \mathbb{R}^n. \tag{33}$$

We now define the function $F \in C(\mathbb{R}^n)$ by the formula

$$F(x) = \Phi(x_1 + \gamma x_2) + \Psi(x), \quad x \in \mathbb{R}^n.$$

Together with (30), condition (23) shows that $F_k \in C_T(\mathbb{R}^n)$ for each k, whence $F \in C_T(\mathbb{R}^n)$. Since $\Psi(x) = 0$ for $x \in E$, by the definition of f_2 and (31), we obtain $F = f$ on E. Using now (33), (21), and (19), we see that for a sufficiently large M, the function F is the required extension of f.

Now we prove (ii). Let $\{\varepsilon_m\}_{m=1}^{\infty}$ be a sequence of positive numbers satisfying (2). Consider the function

$$u(z) = \prod_{m=1}^{\infty} \left(1 - \frac{z^2}{\lambda_m^2}\right), \quad z \in \mathbb{C}, \tag{34}$$

where $\lambda_m = am + i\varepsilon_m$. Setting $\alpha_m = \lambda_m / a$, from expansion of function $\frac{\sin \pi z}{\pi z}$ in infinite product, we have

$$u(az) = \frac{\sin \pi z}{\pi z} v(z), \quad z \notin \mathbb{Z}, \tag{35}$$

where

$$v(z) = \prod_{m=1}^{\infty} \left(1 - \frac{z^2}{m^2}\right)^{-1} \left(1 - \frac{z^2}{\alpha_m^2}\right).$$

Let us estimate $|v(z)|$ for the case when

$$|z - l| \geq \frac{1}{2} \quad \text{for each} \quad l \in \mathbb{Z}. \tag{36}$$

One has

$$v(z) = \prod_{m=1}^{\infty} \left(\frac{m}{\alpha_m}\right)^2 \left(1 - \frac{m - \alpha_m}{m - z}\right) \left(1 - \frac{m - \alpha_m}{m + z}\right),$$

so that

$$\ln|v(z)| \leq \sum_{m=1}^{\infty} \left(\ln\left(\frac{m}{\alpha_m}\right)^2 + \ln\left(1 + \left|\frac{m - \alpha_m}{m - z}\right|\right) + \ln\left(1 + \left|\frac{m - \alpha_m}{m + z}\right|\right)\right).$$

Using the inequality $\ln(1 + t) \leq t$ for $t \geq 0$, we infer that

$$\ln|v(z)| \leq \frac{1}{a} \sum_{m=1}^{\infty} \varepsilon_m \left(\frac{2}{|\alpha_m|} + \frac{1}{|m - z|} + \frac{1}{|m + z|}\right) \leq$$

$$\leq \frac{1}{a}\sum_{m=1}^{\infty}\left(2\varepsilon_m^2+\frac{1}{|\alpha_m|^2}+\frac{1}{|m-z|^2}+\frac{1}{|m+z|^2}\right).$$

Hence, from (36) it follows that

$$|v(z)|\leq \gamma_1,$$

where the constant $\gamma_1>0$ is independent of z. Then, owing to (35) and the maximum modulus principle,

$$|u(z)|\leq \gamma_2 e^{\frac{\pi}{a}|\mathrm{Im}\, z|},\quad z\in\mathbb{C},$$

where $\gamma_2>0$ does not depend on z. Due to the Paley-Wiener theorem for the spherical transform (see [7, Part 1, Theorem 6.5]), there exists $T\in\mathcal{E}_\natural'(\mathbb{R}^n)$, such that $\widetilde{T}=u$. Moreover, from the definition of $\{\lambda_m\}_{m=1}^{\infty}$, we see that $T\in\mathcal{Z}_a$. We now estimate $|\widetilde{T}'(\lambda_q)|$ for sufficiently large $q\in\mathbb{N}$. Assume that $\varepsilon_q<a/8$ and $|z-\lambda_q|=a/8$. This yields

$$\left|1+\frac{z}{\lambda_q}\right|\geq 2-\frac{|z-\lambda_q|}{|\lambda_q|}>1 \tag{37}$$

and

$$\left|\frac{|z|}{a}-q\right|\geq \frac{1}{8}+\frac{\varepsilon_q}{a}<\frac{1}{4}. \tag{38}$$

Then, from (34) and (37) it follows that

$$\left|\frac{u(z)}{z-\lambda_q}\right|=\frac{|z+\lambda_q|}{|\lambda_q|^2}\prod_{\substack{m=1\\ m\neq q}}^{\infty}\left|1-\frac{z^2}{\lambda_q^2}\right|>\frac{1}{|\lambda_q|}\prod_{\substack{m=1\\ m\neq q}}^{\infty}\left|1-\frac{|z|^2}{(am)^2}\right|.$$

Together with (38), this shows that there exist $\gamma_3,\gamma_4>0$, such that

$$\left|\frac{u(z)}{z-\lambda_q}\right|>\frac{\gamma_3}{|\lambda_q|^2}\frac{\left|\sin\frac{\pi}{a}|z|\right|}{\left|1-\frac{|z|^2}{(aq)^2}\right|}>\frac{\gamma_4}{|\lambda_q|^2}. \tag{39}$$

Thus,

$$|\widetilde{T}'(\lambda_q)|\geq \min_{|z-\lambda_q|=\frac{a}{8}}\left|\frac{u(z)}{z-\lambda_q}\right|>\frac{\gamma_4}{|\lambda_q|^2}. \tag{40}$$

Assume now that $f\in C_{T,f}(\mathbb{R}^n)$ for some $f\in C(E)$, $f\neq 0$. Using (40) and the definition of $\{\lambda_m\}_{m=1}^{\infty}$, we see from [7, Part 3, Theorem 3.2] that condition (4) is true for some $\varepsilon>0$. Thus, Theorem 1 is completely proved.

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