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COMPARISON BETWEEN SOME SIXTH CONVERGENCE ORDER SOLVERS UNDER THE SAME SET OF CRITERIA

Abstract. Different set of criteria based on the seventh derivative are used for convergence of sixth order methods. Then, these methods are compared using numerical examples. But we do not know: if the results of those comparisons are true if the examples change; the largest radii of convergence; error estimates on distance between the iterate and solution, and uniqueness results that are computable. We address these concerns using only the first derivative and a common set of criteria. Numerical experiments are used to test the convergence criteria and further validate the theoretical results. Our technique can be used to make comparisons between other methods of the same order.

Key words: *Banach space, sixth convergence order methods, local convergence.*

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1. Introduction.

In this study, we compare some sixth-order methods for approximating a solution x_* of the nonlinear equation

$$F(x) = 0.$$

Here $F : \Omega \subset B_1 \to B_2$ is a continuously differentiable nonlinear operator between the Banach spaces B_1 and B_2, and Ω stands for an open non-empty convex compact set of B_1. The sixth-order method we are interested in is defined as follows [1]:

$$\begin{aligned} y_n &= x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n) \\ z_n &= x_n - A(V_n)F'(x_n)^{-1}F(x_n), \end{aligned} \tag{1}$$

$$x_{n+1} = z_n - 2(3B_n^{-1} - F'(x_n)^{-1})F(z_n),$$

where, $A : B_1 \longrightarrow L(B_2, B_1)$, $B_n = F'(x_n) + F'(y_n)$ and $V_n = B_n^{-1}F'(x_n)$.

These methods use similar information; derived based on different techniques, whose convergence has been shown using Taylor expansions involving the seventh-order derivative not on these methods, of F. The assumptions involving the seventh derivatives limit the applicability of these methods. For example: Let $B_1 = B_2 = \mathbb{R}$, $\Omega = [-\frac{1}{2}, \frac{3}{2}]$. Define f on Ω by

$$f(s) = \begin{cases} s^3 \log s^2 + s^5 - s^4, & s \neq 0 \\ 0, & s = 0. \end{cases}$$

Then, we get

$$f'(s) = 3s^2 \log s^2 + 5s^4 - 4s^3 + 2s^2,$$
$$f''(s) = 6s \log s^2 + 20s^3 - 12s^2 + 10s,$$
$$f'''(s) = 6 \log s^2 + 60s^2 - 24s + 22.$$

Obviously $f'''(s)$ is not bounded on Ω. Hence, the convergence of methods (1) is not guaranteed by the earlier analysis.

Moreover, in the case of the last three methods, no computable convergence radii, upper error estimates on $\|x_n - x_*\|$, nor results on the uniqueness of x_* are given. Furthermore, their performance is compared by numerical examples. Hence, we do not know in advance, having the same set of assumptions, which method provides the largest radius of convergence (i. e., more initial points x_0); the tightest error estimates on $\|x_n - x_*\|$ (i. e., needs fewer iterations to obtain a desired error tolerance); and the best information on the location of the solution.

In this paper, we address these concerns. The same convergence order is obtained using COC or ACOC (to be precised in Remark 1); it depends only on the first derivative and the iteration. Hence, we also extend the applicability of these methods. Our technique can be used to compare other methods [1–12] in the same way.

The rest of the paper is organized as follows. The convergence analysis of schemes (1) is given in Section 2, examples are given in Section 3, and the conclusion is in Section 4.

2. Local convergence. Let us define real parameters and functions needed for our analysis. Assume that there exists a continuous increasing function ω_0 that maps the interval $S := [0, \infty)$ into itself and such that the equation

$$\omega_0(s) - 1 = 0,$$

has the least positive solution r_0. Define the real functions g_1 and h_1 on $(0, r_0)$ as

$$g_1(s) = \frac{\int_0^1 \omega((1-\tau)s)d\tau + \frac{1}{3}\int_0^1 \omega_1(\tau s)d\tau}{1-\omega_0(s)}$$

and

$$h_1(s) = g_1(s) - 1,$$

where ω, ω_1 are continuous increasing functions on $S_0 := [0, r_0)$.

Assume that the equation

$$h_1(s) = 0.$$

has the least solution in $(0, r_0)$ denoted by R_1. Assume that the equation

$$p(s) - 1 = 0$$

has the least solution in $(0, r_0)$ denoted r_p, where

$$p(s) = \frac{1}{2}(\omega_0(s) + \omega_0(g_1(s)s)).$$

Define the functions g_2 and h_2 on $[0, r_p)$ as

$$g_2(s) = g_0(s) + \frac{q(s)\int_0^1 \omega_1(\tau s)d\tau}{1-\omega_0(s)}$$

and

$$h_2(s) = g_2(s) - 1,$$

where

$$g_0(s) = \frac{\int_0^1 \omega((1-\tau)s)d\tau}{1-\omega_0(s)}$$

and q is a continuous increasing real function on $[0, r_p)$.

Assume that the equation

$$h_2(s) = 0$$

has the least solution in $(0, r_p)$ denoted R_2.

Assume the the equation

$$\omega_0(g_2(s)s) - 1 = 0 \tag{2}$$

has the least solution in $(0, r_p)$ denoted r_1. Define the functions g_3 and h_3 on $[0, r_1)$ as

$$g_3(s) = \Bigg[g_0(g_2(s)s) + \Bigg(2\frac{\omega_0(g_1(s)s) + \omega_0(s)}{1 - \omega_0(s)} + \\ + \frac{\omega_0(s) + \omega_0(g_1(s)s) + 2\omega_0(g_2(s)s)}{1 - \omega_0(g_2(s)s)} \Bigg) \frac{\int_0^1 \omega_1(\tau g_2(s)s)d\tau}{2(1 - p(s))} \Bigg] g_2(s)$$

and

$$h_3(s) = g_3(s) - 1.$$

Assume that the equation

$$h_3(s) = 0$$

has the least solution in $(0, r_1)$ denoted R_3. Let the radius of convergence R be as

$$R = \min\{R_m\},\ m = 1, 2, 3, \dots \tag{3}$$

Thus, for all $s \in [0, R)$:

$$0 \leqslant \omega_0(s) < 1, \tag{4}$$

$$0 \leqslant \omega_0(g_2(s)s) < 1, \tag{5}$$

$$0 \leqslant p(s) < 1, \tag{6}$$

$$0 \leqslant g_m(s) < 1. \tag{7}$$

The following definitions are used: $U(x, a) = \{y \in B_1 : \|x - y\| < a\}$ and let $\bar{U}(x,a)$ be its closure for $a > 0$. Let us use the notation $e_n = \|x_n - x_*\|$, for all $n = 0, 1, 2, \dots$

The following assumptions $(\mathcal{A})$ are used:

$(\mathcal{A}_1)$ $F : \Omega \longrightarrow Y$ has a simple solution x_* and the inverse of $F'(x_*)$ exists.

$(\mathcal{A}_2)$ There exists a continuous increasing function ω_0 on S, such that for all $x \in \Omega$

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \leqslant \omega_0(\|x - x_*\|).$$

Set $\Omega_0 = \Omega \cap U(x_*, r_0)$.

($\mathcal{A}_3$) There exists a continuous increasing functions ω on S_0, such that for each $x, y \in \Omega_0$

$$\|F'(x_*)^{-1}(F'(y) - F'(x))\| \leqslant \omega(\|y - x\|),$$

$$\|F'(x_*)^{-1}F'(x)\| \leqslant \omega_1(\|x - x_*\|).$$

($\mathcal{A}_4$) There exists a continuous increasing real function q defined on $(0, r_p)$, such that for all $x \in \Omega_0$

$$\|I - A((F'(x) + F'(y))^{-1}F'(x))\| \leqslant q(\|x - x_*\|),$$

where $y = x - \frac{2}{3}F'(x)^{-1}F(x)$.

($\mathcal{A}_5$) $\bar{U}(x_*, R) \subset \Omega$, and r_0, r_1, R_1 and R_2 exist, where R is defined by (2).

($\mathcal{A}_6$) There exists $R_* \geqslant R$, such that

$$\int_0^1 \omega_0(\tau R_*)d\tau < 1.$$

Set $\Omega_1 = \Omega \cap \bar{U}(x_*, R_*)$.

Under these assumptions, we present the ball convergence for (1).

Theorem 1. *Suppose that $x_0 \in U(x_*, R) - \{x_*\}$ under the conditions ($\mathcal{A}$). Then the following assertions hold:*

$$\{x_n\} \in U(x_*, R),$$

$$\lim_{n \longrightarrow \infty} x_n = x_*, \tag{8}$$

$$\|y_n - x_*\| \leqslant g_1(e_n)e_n \leqslant e_n < R, \tag{9}$$

$$\|z_n - x_*\| \leqslant g_2(e_n)e_n \leqslant e_n, \tag{10}$$

$$\|x_{n+1} - x_*\| \leqslant g_3(e_n)e_n \leqslant e_n, \tag{11}$$

and x_* is unique in the set Ω_1 as a solution of equation $F(x) = 0$.

Proof. Let us choose $x \in U(x_*, R) - \{x_*\}$. Then, due to (3), (4), ($\mathcal{A}_1$) and ($\mathcal{A}_2$), we get

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \leqslant \omega_0(\|x - x_*\|) < \omega_0(R) \leqslant 1,$$

leading to $F'(x)$ being invertible,

$$\|F'(x)^{-1}F'(x_*)\| \leqslant \frac{1}{1-\omega_0(\|x-x_*\|)} \tag{12}$$

by the Banach perturbation lemma [8], and to the existence of y_0 by the method (1). Further, in view of

$$F(x) = F(x) - F(x_*) = \int_0^1 F'(x+\tau(x-x_*))d\tau(x-x_*),$$

$(\mathcal{A}_1)$ and $(\mathcal{A}_3)$, we have

$$\|F'(x_*)^{-1}F'(x)\| \leqslant \int_0^1 \omega_1(\tau\|x-x_*\|)d\tau\|x-x_*\|. \tag{13}$$

Then it follows from (3), (7) (for $m=1$), $(\mathcal{A}_3)$, (12) (for $x=x_0$) and (13) (for $x=x_0$)

$$\begin{aligned}
\|y_0-x_*\| &= \left\|x_0-x_*-F'(x_0)^{-1}F(x_0)+\frac{1}{3}F'(x_0)^{-1}F(x_0)\right\| \leqslant \\
&\leqslant \|F'(x_0)^{-1}F'(x_*)\|\left\|\int_0^1 F'(x_*)^{-1}(F'(x_*+\tau(x_0-x_*))-F'(x_0))d\tau(x_0-x_*)\right\|+ \\
&\quad + \frac{1}{3}\|F'(x_0)^{-1}F(x_*)\|\|F'(x_*)^{-1}F'(x_0)\| \leqslant \\
&\leqslant \frac{\int_0^1 \omega((1-\tau)\|x_0-x_*\|)d\tau\|x_0-x_*\|}{1-\omega_0(\|x_0-x_*\|)}+ \\
&\quad + \frac{\frac{1}{3}\int_0^1 \omega_1(\tau\|x_0-x_*\|)d\tau\|x_0-x_*\|}{1-\omega_0(\|x_0-x_*\|)} \leqslant \\
&\leqslant g_1(\|x_0-x_*\|)\|x_0-x_*\| \leqslant \|x_0-x_*\| < R,
\end{aligned} \tag{14}$$

leading to the estimate (9) for $n=0$ and $y_0 \in U(x_*,R)$.

Next, we need to show that $F'(x_0)+F'(y_0)$ is an invertible operator. Indeed, using (3), (6) and (12), we have

$$\begin{aligned}
&\|(2F'(x_*))^{-1}(F'(x_0)+F'(y_0)-2F'(x_*))\| \leqslant \\
&\leqslant \frac{1}{2}(\|F'(x_*)^{-1}(F'(x_0)-F'(x_*))\| + \|F'(x_*)^{-1}(F'(y_0)-F'(x_*))\|) \leqslant \\
&\leqslant \frac{1}{2}(\omega_0(\|x_0-x_*\|)+\omega_0(\|y_0-x_*\|)) \leqslant \\
&\leqslant \frac{1}{2}(\omega_0(\|x_0-x_*\|)+\omega_0(g_1(\|x_0-x_*\|)\|x_0-x_*\|)) \leqslant p(R) < 1,
\end{aligned}$$

so

$$\|(F'(x_0)+F'(y_0))^{-1}F'(x_*)\| \leqslant \frac{1}{2(1-p(\|x_0-x_*\|))} \tag{15}$$

and z_0 is well-defined. Using (3), (8) (for $i=2$), ($\mathcal{A}_4$), (12) (for $x=x_0$), (13)–(14), and the method (1), we obtain

$$\begin{aligned}
\|z_0-x_*\| &= \|(x_0-x_*-F'(x_0)^{-1}F(x_0)) + (I-A(V_0))F'(x_0)^{-1}F(x_0)\| \leqslant \\
&\leqslant \|x_0-x_*-F(x_0)^{-1}F(x_0)\| + \\
&+ \|I-A(V_0)\|\|F'(x_0)^{-1}F'(x_*)\|\|F'(x_*)^{-1}F(x_0)\| \leqslant \\
&\leqslant [g_0(\|x_0-x_*\|) + \\
&+ \frac{q(\|x_0-x_*\|)\int_0^1 \omega_1(\tau\|x_0-x_*\|)d\tau}{1-\omega_0(\|x_0-x_*\|)}]\|x_0-x_*\| = \\
&= g_2(\|x_0-x_*\|)\|x_0-x_*\| \leqslant \|x_0-x_*\|,
\end{aligned} \tag{16}$$

leading to the verification of (10), $z_0 \in U(x_*, R)$, and so x_1 is well defined. We need an estimate

$$\begin{aligned}
&-2[F'(x_0)^{-1}-3B_0^{-1}] = -2F'(x_0)^{-1}(B_0-3F'(x_0))B_0^{-1} = \\
&= -2F'(x_0)^{-1}(F'(x_0)+F'(y_0)-3F'(x_0))B_0^{-1} = \\
&= -2F'(x_0)^{-1}(F'(y_0)-F'(x_0))+2B_0^{-1}.
\end{aligned} \tag{17}$$

Then, by (5), (8) (for $m=3$), (12) (for $x=z_0$), (13) (for $x=z_0$), (15)–(17), we get

$$\begin{aligned}
\|x_1-x_*\| &= \|(z_0-x_*-F'(z_0)^{-1}F(z_0)) + [2F'(x_0)^{-1}(F'(y_0)-F'(x_0))B_0^{-1} + \\
&+ (F'(z_0)^{-1}-2B_0^{-1})]F(z_0)\| \leqslant \\
&\leqslant \|z_0-x_*-F'(z_0)^{-1}F(z_0)\| + \|2F'(x_0)^{-1}(F'(y_0)-F'(x_0)) + \\
&+ F'(z_0)^{-1}((F'(x_0)-F'(z_0)) +
\end{aligned}$$

$$
\begin{aligned}
&+ (F'(y_0) - F'(z_0))\| \|B_0^{-1}F'(x_*)\| \|F'(x_*)^{-1}F(z_0)\| \leqslant \\
&\leqslant \Bigg[g_0(\|z_0 - x_*\|) + \Big(2\frac{\omega_0(\|y_0 - x_*\|) + \omega_0(\|x_0 - x_*\|)}{1 - \omega_0(\|x_0 - x_*\|)} + \\
&+ (\omega_0(\|x_0 - x_*\|) + \omega_0(\|y_0 - x_*\|) + 2\omega_0(\|z_0 - x_*\|) \Big) \times \\
&\times \frac{\int_0^1 \omega_1(\tau\|z_0 - x_*\|)d\tau}{2(1 - p(\|x_0 - x_*\|))} \Bigg] \|z_0 - x_*\| = \\
&= g_3(\|x_0 - x_*\|)\|x_0 - x_*\| \leqslant \|x_0 - x_*\|,
\end{aligned}
$$

leading to the completion of the induction for (9)–(11) for $n = 0$ and $x_1 \in U(x_*, R)$. By supposing they hold for all $m = 0, 1, \ldots, n-1$, we complete the induction for (9)–(11) by replacing x_0, y_0, z_0, x_1 by x_m, y_m, z_m, x_{m+1} in the previous calculations. In view of the estimate

$$\|x_{m+1} - x_*\| \leqslant b\|x_m - x_*\| < R,$$

where $b = g_3(\|x_0 - x_*\|) \in [0, 1)$, we obtain $x_{m+1} \in U(x_*, R)$ and $\lim_{m\to\infty} x_m = x_*$, showing (9). Let $u \in \Omega_1$ with $F(u) = 0$. Define

$$C = \int_0^1 F'(u + \tau(x_* - u))d\tau.$$

Then, by $(\mathcal{A}_2)$ and $(\mathcal{A}_5)$, we get

$$\|F'(x_*)^{-1}(C - F'(x_*))\| \leqslant \int_0^1 \omega_0((1-\tau)\|x_* - u\|)d\tau \leqslant \int_0^1 \omega_0(\tau R_*)d\tau < 1,$$

so $x_* = u$; this follows from invertability of C and the identity $0 = F(x_*) - F(u) = C(x_* - u)$. □

Remark 1.

(a) *We can find the convergence order by resorting to the computational order of convergence (COC) defined by*

$$\xi = \ln\left(\frac{\|x_{n+1} - x_*\|}{\|x_n - x_*\|}\right) / \ln\left(\frac{\|x_n - x_*\|}{\|x_{n-1} - x_*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way, we obtain in practice the order of convergence without resorting to the computation of higher-order derivatives appearing in the method or in the sufficient convergence criteria usually appearing in the Taylor expansions for the proofs of those results.

(b) *Let us consider specializations of method (1).*
Case I: $A(V) = I$*; then, clearly,* $q(s) = 0$.
Case II: $A(V) = 2V$. *Then, from the estimate*

$$\begin{aligned} I - A(V) &= I - 2(F'(x) + F'(y))^{-1}F'(x) = \\ &= (F'(x) + F'(y))^{-1}(F'(x) + F'(y) - 2F'(x)) = \\ &= (F'(x) + F'(y))^{-1}(F'(y) - F'(x)), \end{aligned}$$

so, by the proof of Theorem 1, we can choose

$$q(s) = \frac{\omega_0(g_1(s)s) + \omega_0(s)}{2(1 - p(s))}.$$

3. Numerical Examples.

Example 3.1 Let us consider a system of differential equations governing the motion of an object and given by

$$F_1'(x) = e^x, \;\; F_2'(y) = (e-1)y + 1, \;\; F_3'(z) = 1$$

with the initial conditions $F_1(0) = F_2(0) = F_3(0) = 0$. Let $F = (F_1, F_2, F_3)$. Let $B_1 = B_2 = \mathbb{R}^3, \Omega = \bar{U}(0,1)$, $x_* = (0,0,0)^T$. Define the function F on Ω for $w = (x,y,z)^T$ by

$$F(w) = \left(e^x - 1, \frac{e-1}{2}y^2 + y, z\right)^T.$$

The Fréchet-derivative is

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the $(\mathcal{A})$ conditions, we get $\omega_0(s) = (e-1)s$, $\omega(s) = e^{\frac{1}{e-1}}s$, $\omega_1(s) = e^{\frac{1}{e-1}}$. The radii are given in Table 1.

Example 3.2 Let $B_1 = B_2 = C[0,1]$, the space of continuous functions defined on $[0,1]$, be equipped with the max norm. Let $\Omega = \overline{U}(0,1)$. Define the function F on Ω by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta.$$

Radius	Case-I	Case-II
R_1	0.1544069513571540708252	0.1544069513571540708252
R_2	0.3826919122323857447298	0.1492777526031611734502
R_3	0.2952384459889182410918	0.0798310047170279202255
R	0.1544069513571540708252	0.0798310047170279202255

Table 1: example 3.1.

We have

$$F'(\varphi(\xi))(x) = \xi(x) - 15 \int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in \Omega.$$

Then we get $x^* = 0$, so $\omega_0(s) = 7.5s$, $\omega(s) = 15s$, and $\omega_1(s) = 2$. The radii are given in Table 2.

Radius	Case-I	Case-II
R_1	0.02222222222222222222222	0.02222222222222222222222
R_2	0.06666666666666666666666	0.028225943743472654834381
R_3	0.05665528918936485469615	0.13333333333333333333333
R	0.02222222222222222222222	0.02222222222222222222222

Table 2: example 3.2.

Example 3.3 Returning back to the motivational example at the introduction of this study, we have $\omega_0(s) = \omega(s) = 96.6629073s$ and $\omega_1(s) = 2$. The parameters for the method (1) are given in Table 3.

Radius	Case-I	Case-II
R_1	0.002298939980488484951387	0.002298939980488484951387
R_2	0.006896819941465455287843	0.002471912165730189275131
R_3	0.005208424091120038290636	0.01034522991219942976426
R	0.002298939980488484951387	0.002298939980488484951387

Table 3: example 3.3.

4. Conclusions. Different techniques are used to develop iterative methods. Moreover, different set of criteria, usually based on the seventh derivative, are needed in the ball convergence of the sixth-order methods. Then these methods are compared using numerical examples. But we do not know: if the results of those comparisons remain true if the examples change; the largest radii of convergence; error estimates on $\|x_n - x_*\|$; and uniqueness results that are computable. We address these concerns using only the first derivative and a common set of criteria. Numerical experiments are used to test the convergence criteria and further validate the theoretical results. Our technique can be used to make comparisons between other methods of the same order.

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