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SEVERAL NEW INTEGRAL INEQUALITIES VIA K -RIEMANN–LIOUVILLE FRACTIONAL INTEGRALS OPERATORS

Abstract. The main objective of this paper is to establish several new integral inequalities including k -Riemann–Liouville fractional integrals for convex, s -Godunova–Levin convex functions, quasi-convex, η -quasi-convex. In order to obtain our results, we have used classical inequalities as Hölder inequality, Power mean inequality and Weighted Hölder inequality. We also give some applications.

Key words: η -quasi-convex, s -Godunova–Levin type, k -Riemann–Liouville fractional integral, Hölder inequality, weighted Hölder inequality, power mean inequality

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1. Introduction. In the recent years, theory of inequalities has been attracting many researchers due to its applications in our daily life and within the mathematics. Theory of convex functions has great importance in various fields of pure and applied sciences. It is known that theory of convex functions is closely related to theory of inequalities. It has now become a trending aspect of mathematical research to generalize classical known results via the fractional integral operator.

Fractional calculus may be described as an extension of the concept of derivative operator from integer order n to arbitrary order a . Fractional derivative has been considered as the inverse of a fractional integral. Considerable work has been done in the recent years on fractional integrals because of their applications in many fields of science and technology. The fractional integrals are a powerful tool in applied mathematics, used to solve many problems from different fields of science and engineering.

The Riemann–Liouville integral is named after Bernhard Riemann and Joseph Liouville, the latter was the first to consider the possibility of fractional calculus in 1832. The concept of k -Riemann–Liouville fractional

integral is an important extension of Riemann–Liouville fractional integrals. We want to stress here that for $k \neq 1$, the properties of k -Riemann–Liouville fractional integrals are quite dissimilar from those of general Riemann–Liouville fractional integrals. For this, the k -Riemann–Liouville fractional integrals have aroused interest of many researchers. Our concern is the k -Riemann–Liouville fractional integral operator. We invite the interested reader to see [5], [17], [13], [16], [11] and references therein.

The following definitions are well-known:

Definition 1. A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$.

Many important inequalities are established for the class of convex functions; one of the most important is called Hermite–Hadamard’s inequality (or Hadamard’s inequality). This double inequality is stated as follows:

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and let $a, b \in I$, with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.$$

This inequality is in the reversed direction if f is concave.

Definition 2. [9] The function $f : [a, b] \rightarrow \mathbb{R}$ is said to be η -convex if the inequality

$$f(tx + (1-t)y) \leq f(y) + t\eta(f(x), f(y))$$

holds for all $x, y \in [a, b]$ and $t \in [0, 1]$, and η is defined by

$$\eta : f([a, b]) \times f([a, b]) \rightarrow \mathbb{R}.$$

If we set $\eta(x, y) = x - y$ in the definition above, we can directly obtain the classical definition of a convex function.

Definition 3. [9] Let a real function f be defined on some non-empty interval I of real numbers. The function f is said to be quasi-convex on I if following inequality holds:

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\} \quad (QC)$$

Also, f is called η -quasi-convex if

$$f(tx + (1-t)y) \leq \max\{f(y), f(y) + \eta(f(x), f(y))\} \quad (\eta - QC)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Clearly, any convex function is a QC function, though not every QC function is convex.

For example the function $f : R^+ \rightarrow R$, $f(x) = \ln(x)$, $x \in R^+$ is quasi-convex but it is not convex.

Definition 4. [8] we say that the function $f : C \subset X \rightarrow [0, \infty)$ is of s -Godunova–Levin type, with $s \in [0, 1]$, if the inequality

$$f(tx + (1-t)y) \leq t^{-s}f(x) + (1-t)^{-s}f(y) \quad (s - GL)$$

holds for all $t \in (0, 1)$ and $x, y \in C$.

Now we give a necessary definition of fractional calculus theory, which is used throughout this paper.

Definition 5. [14] Let $f \in L_1[a, b]$. The k -Riemann–Liouville integrals ${}_k J_{a+}^\alpha f$ and ${}_k J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$${}_k J_{a+}^\alpha f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt, \quad x > a, k > 0$$

and

$${}_k J_{b-}^\alpha f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t) dt, \quad x < b, k > 0$$

respectively, where $\Gamma_k(\cdot)$ stands for the k -Gamma function.

For a positive real number k and a complex number α with $\text{Re}(\alpha) > 0$, the k -Gamma function is given by the integral [7]:

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt \quad \text{with} \quad \alpha\Gamma_k(\alpha) = \Gamma_k(\alpha + k).$$

Remark. If $k = 1$, then k -Riemann–Liouville fractional integrals reduce to the classical Riemann–Liouville fractional integral. And, if $\alpha = 1$ and $k = 1$, the fractional integral reduces to the classical integral.

For some recent results connected with fractional integral inequalities, see [1], [3], [6], [12].

The main aim of this paper is to establish three new integral identities and to prove, using these equalities, some new integral inequalities for η -QC and s -GL convex functions via the k -Riemann–Liouville fractional integral operators.

2. Main Results. In order to move further, we need the following useful lemma.

Lemma 1. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I° , where $a, b \in I$ with $t \in [0, 1]$. If $f' \in L[a, b]$, then for all $a \leq x < y \leq b$ and $k, \alpha > 0$, we have:*

$$\frac{f(y)}{y-x} - \frac{\Gamma_k(\alpha+k)}{(y-x)^{\frac{\alpha}{k}+1}} {}_k J_{y^-}^\alpha f(x) = \int_0^1 (1-t)^{\frac{\alpha}{k}} f'(tx+(1-t)y) dt.$$

Proof. Integrate by parts:

$$\begin{aligned} \int_0^1 (1-t)^{\frac{\alpha}{k}} f'(tx+(1-t)y) dt &= \\ &= \frac{f(y)}{y-x} - \frac{\frac{\alpha}{k}}{y-x} \int_0^1 (1-t)^{\frac{\alpha}{k}-1} f(tx+(1-t)y) dt. \end{aligned}$$

Change the variable $u = tx + (1-t)y$ to these integrals, and get

$$\begin{aligned} \frac{1}{y-x} f(y) - \frac{\frac{\alpha}{k}}{(y-x)^{\frac{\alpha}{k}+1}} \int_x^y (u-x)^{\frac{\alpha}{k}-1} f(u) du &= \\ &= \frac{1}{y-x} f(y) - \frac{\Gamma_k(\alpha+k)}{(y-x)^{\frac{\alpha}{k}+1}} {}_k J_{y^-}^\alpha f(x). \end{aligned}$$

This completes the proof. \square

Theorem 1. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$, $I \subset [0, \infty)$, be a differentiable function on I° , such that $f' \in L[a, b]$, where $a, b \in I$, with $a \leq x < y \leq b$. If f' is η -QC on $[x, y]$ for $t \in [0, 1]$, then we have*

$$\begin{aligned} \frac{1}{y-x}f(y) - \frac{\Gamma_k(\alpha+k)}{(y-x)^{\frac{\alpha}{k}+1}} {}_k J_{y^-}^\alpha f(x) &\leq \\ &\leq \left(\frac{k}{\alpha+k}\right) \max \{f'(y), f'(y) + \eta(f'(x), f'(y))\} \end{aligned}$$

for all $k, \alpha > 0$.

Proof. Since $f'(tx + (1-t)y) \leq \max \{f'(y), f'(y) + \eta(f'(x), f'(y))\}$ for $t \in [0, 1]$, we obtain, from Lemma 1:

$$\begin{aligned} \frac{1}{y-x}f(y) - \frac{\Gamma_k(\alpha+k)}{(y-x)^{\frac{\alpha}{k}+1}} {}_k J_{y^-}^\alpha f(x) &= \int_0^1 (1-t)^{\frac{\alpha}{k}} f'(tx + (1-t)y) dt \leq \\ &\leq \max \{f'(y), f'(y) + \eta(f'(x), f'(y))\} \int_0^1 (1-t)^{\frac{\alpha}{k}} dt = \\ &= \left(\frac{k}{\alpha+k}\right) \max \{f'(y), f'(y) + \eta(f'(x), f'(y))\}. \end{aligned}$$

This completes the proof. \square

Corollary 1. If we set $\eta(x, y) = x - y$ in Theorem 1, we obtain

$$\frac{1}{y-x}f(y) - \frac{\Gamma_k(\alpha+k)}{(y-x)^{\frac{\alpha}{k}+1}} {}_k J_{y^-}^\alpha f(x) \leq \left(\frac{k}{\alpha+k}\right) \max \{f'(y), f'(x)\}.$$

Corollary 2. For $x = a$ and $y = b$, we obtain

$$\begin{aligned} \frac{1}{b-a}f(b) - \frac{\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}+1}} {}_k J_{b^-}^\alpha f(a) &\leq \left(\frac{k}{\alpha+k}\right) \max \{f'(b), f'(a)\} \leq \\ &\leq \left(\frac{k}{\alpha+k}\right) \|f'\|_\infty, \end{aligned}$$

where $\|f'\|_\infty = \sup |f'(x)|$. Moreover, if we choose $k = \alpha = 1$, we obtain

$$f(b) - \frac{1}{b-a} \int_a^b f(x) dx \leq \left(\frac{b-a}{2}\right) \max \{f'(b), f'(a)\}. \tag{1}$$

Theorem 2. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$, $I \subset [0, \infty)$, be a differentiable function on I° , such that $f' \in L[a, b]$, where $a, b \in I$, $a \leq x < y \leq b$. If $|f'|^q$ is η -QC on $[x, y]$ for $t \in [0, 1]$, $q > 1$, $p = \frac{q}{q-1}$, then we have

$$\left| \frac{1}{y-x} f(y) - \frac{\Gamma_k(\alpha+k)}{(y-x)^{\frac{\alpha}{k}+1}} {}_k J_{y^-}^\alpha f(x) \right| \leq \left(\frac{k}{\alpha p + k} \right)^{\frac{1}{p}} \max \{ |f'(y)|^q, |f'(y)|^q + \eta(|f'(x)|^q, |f'(y)|^q) \}^{\frac{1}{q}}. \quad (2)$$

for all $k, \alpha > 0$.

Proof. First of all, we know that

$$|f'(tx + (1-t)y)|^q \leq [\max \{ |f'(y)|^q, |f'(y)|^q + \eta(|f'(x)|^q, |f'(y)|^q) \}].$$

Using the well known Hölder inequality, properties of the absolute value, and using Lemma 1, we obtain

$$\begin{aligned} \left| \frac{1}{y-x} f(y) - \frac{\Gamma_k(\alpha+k)}{(y-x)^{\frac{\alpha}{k}+1}} {}_k J_{y^-}^\alpha f(x) \right| &= \left| \int_0^1 (1-t)^{\frac{\alpha}{k}} f'(tx + (1-t)y) dt \right| \leq \\ &\leq \int_0^1 |(1-t)^{\frac{\alpha}{k}}| |f'(tx + (1-t)y)| dt \leq \\ &\leq \left(\int_0^1 (1-t)^{\frac{\alpha p}{k}} dt \right)^{\frac{1}{p}} (|f'(tx + (1-t)y)|^q dt)^{\frac{1}{q}} = \\ &= \left(\frac{k}{\alpha p + k} \right)^{\frac{1}{p}} [\max \{ |f'(y)|^q, |f'(y)|^q + \eta(|f'(x)|^q, |f'(y)|^q) \}]^{\frac{1}{q}}, \end{aligned}$$

which completes the desired inequality. \square

Corollary 1. Set $\eta(x, y) = x - y$ to obtain

$$\left| \frac{1}{y-x} f(y) - \frac{\Gamma_k(\alpha+k)}{(y-x)^{\frac{\alpha}{k}+1}} {}_k J_{y^-}^\alpha f(x) \right| \leq \left(\frac{k}{\alpha p + k} \right)^{\frac{1}{p}} [\max \{ |f'(y)|^q, |f'(x)|^q \}]^{\frac{1}{q}}.$$

Corollary 2. Choose $x = a$, $y = b$ and $\alpha = k = 1$ to obtain

$$\left| f(b) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{(p+1)^{\frac{1}{p}}} [\max \{ |f'(b)|^q, |f'(a)|^q \}]^{\frac{1}{q}}. \quad (3)$$

Lemma 2. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I° , where $a, b \in I$ with $t \in [0, 1]$. If $f' \in L[a, b]$, then we have, for all $a \leq x < y \leq b$ and $k, \alpha > 0$:

$$\begin{aligned} & \frac{1}{x-y} [f(x) - f(y)] + \frac{\Gamma_k(\alpha + k)}{(y-x)^{\frac{\alpha}{k}+1}} [{}_k J_{x^+}^\alpha f(y) - {}_k J_{y^-}^\alpha f(x)] = \\ & = \int_0^1 t^{\frac{\alpha}{k}} f'(tx + (1-t)y) dt + \int_0^1 (1-t)^{\frac{\alpha}{k}} f'(tx + (1-t)y) dt. \end{aligned}$$

Proof. Integrating by parts, we can write the integrals above as follows:

$$\begin{aligned} \int_0^1 t^{\frac{\alpha}{k}} f'(tx + (1-t)y) dt &= \frac{1}{x-y} f(x) + \frac{\Gamma_k(\alpha + k)}{(y-x)^{\frac{\alpha}{k}+1}} {}_k J_{x^+}^\alpha f(y), \\ \int_0^1 (1-t)^{\frac{\alpha}{k}} f'(tx + (1-t)y) dt &= \\ &= -\frac{1}{x-y} f(y) - \frac{\frac{\alpha}{k}}{(y-x)^{\frac{\alpha}{k}+1}} \int_x^y (u-x)^{\frac{\alpha}{k}-1} f(u) du = \\ &= -\frac{1}{x-y} f(y) - \frac{\Gamma_k(\alpha + k)}{(y-x)^{\frac{\alpha}{k}+1}} {}_k J_{y^-}^\alpha f(x). \end{aligned}$$

Adding the above integral equalities, we get the desired inequality. \square

Theorem 3. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I° , where $a, b \in I$ with $t \in [0, 1]$. If $f' \in L[a, b]$ and $|f'|$ is a s -GL type function, then

$$\begin{aligned} & \left| \frac{1}{x-y} [f(x) - f(y)] + \frac{\Gamma_k(\alpha + k)}{(y-x)} [{}_k J_{x^+}^\alpha f(y) - {}_k J_{y^-}^\alpha f(x)] \right| \leq \\ & \leq (|f'(x)| + |f'(y)|) \left[\beta \left(\frac{\alpha}{k} + 1, 1-s \right) + \frac{k}{\alpha - ks + k} \right], \end{aligned}$$

for all $a \leq x < y \leq b$ and $k, \alpha > 0, s \in [0, 1]$; $\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, x > 1, y > 0$ is the Euler Beta function.

Proof. From Lemma 2 and the properties of the absolute value:

$$\begin{aligned} & \left| \int_0^1 t^{\frac{\alpha}{k}} f'(tx + (1-t)y) dt + \int_0^1 (1-t)^{\frac{\alpha}{k}} f'(tx + (1-t)y) dt \right| \leq \\ & \leq \int_0^1 t^{\frac{\alpha}{k}} |f'(tx + (1-t)y)| dt + \int_0^1 (1-t)^{\frac{\alpha}{k}} |f'(tx + (1-t)y)| dt. \end{aligned}$$

Since $|f'|$ is a s -Godunova–Levin type function, we get, applying integration by parts to every integral, respectively:

$$\begin{aligned} & \int_0^1 t^{\frac{\alpha}{k}} |f'(tx + (1-t)y)| dt \leq |f'(x)| \int_0^1 t^{\frac{\alpha}{k}-s} dt + |f'(y)| \int_0^1 t^{\frac{\alpha}{k}} (1-t)^{-s} dt = \\ & = \left[\frac{k}{\alpha - ks + k} |f'(x)| + |f'(y)| \beta \left(\frac{\alpha}{k} + 1, 1 - s \right) \right] \end{aligned}$$

and

$$\int_0^1 (1-t)^{\frac{\alpha}{k}} |f'(tx + (1-t)y)| dt = |f'(x)| \beta \left(1 - s, \frac{\alpha}{k} + 1 \right) + \frac{k |f'(y)|}{\alpha - ks + k}.$$

Finally, since $\beta(x, y) = \beta(y, x)$, we have

$$\begin{aligned} & \left| \frac{1}{x-y} [f(x) - f(y)] + \frac{\Gamma_k(\alpha + k)}{(y-x)} [{}_k J_{x^+}^\alpha f(y) - {}_k J_{y^-}^\alpha f(x)] \right| \\ & \leq (|f'(x)| + |f'(y)|) \left[\beta \left(\frac{\alpha}{k} + 1, 1 - s \right) + \frac{k}{\alpha - ks + k} \right]. \end{aligned}$$

This completes the proof. \square

Lemma 3. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function on I° , where $a, b \in I$ with $t \in [0, 1]$. If $f'' \in L[a, b]$, then for all $a < b$ and $\frac{\alpha}{k} - 1 > 0$, we have

$$\begin{aligned} & \frac{k 2^{\frac{\alpha}{k}-2}}{\alpha(b-a)^{\frac{\alpha}{k}-1}} \Gamma_k(\alpha + k) \left[{}_k J_{\frac{a+b}{2}^+}^{\alpha-1} f(b) + {}_k J_{\frac{a+b}{2}^-}^{\alpha-1} f(a) \right] - f\left(\frac{a+b}{2}\right) = \\ & = \frac{k(b-a)^2}{\alpha 2^{2-\frac{\alpha}{k}}} \left[\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}} f''(ta + (1-t)b) dt + \int_{\frac{1}{2}}^1 (1-t)^{\frac{\alpha}{k}} f''(ta + (1-t)b) dt \right]. \end{aligned}$$

Proof. Integrating by parts several times, we obtain the following equalities:

$$\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}} f''(ta + (1-t)b) dt = \frac{f'\left(\frac{a+b}{2}\right)}{(a-b) 2^{\frac{\alpha}{k}}} - \frac{\frac{\alpha}{k}}{(a-b)^2 2^{\frac{\alpha}{k}-1}} f\left(\frac{a+b}{2}\right) + \frac{\frac{\alpha}{k} \left(\frac{\alpha}{k} - 1\right)}{(a-b)^2} \int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}-2} f(ta + (1-t)b) dt,$$

$$\int_{\frac{1}{2}}^1 (1-t)^{\frac{\alpha}{k}} f''(ta + (1-t)b) dt = -\frac{f'\left(\frac{a+b}{2}\right)}{(a-b) 2^{\frac{\alpha}{k}}} - \frac{\frac{\alpha}{k}}{(a-b)^2 2^{\frac{\alpha}{k}-1}} f\left(\frac{a+b}{2}\right) + \frac{\frac{\alpha}{k} \left(\frac{\alpha}{k} - 1\right)}{(a-b)^2} \int_{\frac{1}{2}}^1 (1-t)^{\frac{\alpha}{k}-2} f(ta + (1-t)b) dt.$$

Now, changing the variable $u = ta + (1-t)b$ in every integral, we have

$$\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}-2} f(ta + (1-t)b) dt = \frac{1}{(b-a)^{\frac{\alpha}{k}-1}} \int_{\frac{a+b}{2}}^b (b-u)^{\frac{\alpha}{k}-2} f(u) du = \frac{k\Gamma\left(\frac{\alpha}{k} - 1\right)}{(b-a)^{\frac{\alpha}{k}-1}} {}_k J_{\frac{a+b}{2}}^{\alpha-1} f(b),$$

$$\int_{\frac{1}{2}}^1 (1-t)^{\frac{\alpha}{k}-2} f(ta + (1-t)b) dt = \frac{1}{(b-a)^{\frac{\alpha}{k}-1}} \int_a^{\frac{a+b}{2}} (u-a)^{\frac{\alpha}{k}-2} f(u) du = \frac{k\Gamma\left(\frac{\alpha}{k} - 1\right)}{(b-a)^{\frac{\alpha}{k}-1}} {}_k J_{\frac{a+b}{2}}^{\alpha-1} f(a).$$

By adding these inequalities and multiplying by $\frac{\alpha}{k(a-b)^2 2^{\frac{\alpha}{k}-1}}$, we get the required inequality. \square

Remark 1. Taking $k = 1$ in Lemma 3, we obtain equality (2) from [2] for $m = 1$.

Theorem 4. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I° , such that $f'' \in L[a, b]$, where $a, b \in I$, $a < b$ with $t \in [0, 1]$. If $|f''|^q$ is η -QC on $[a, b] \subset I$ and $q \geq 1$, then the following inequality for fractional integrals holds:

$$\begin{aligned} & \left| \frac{k2^{\frac{\alpha}{k}-2}}{\alpha(b-a)^{\frac{\alpha}{k}-1}} \Gamma_k(\alpha+k) \left[{}_k J_{\frac{a+b}{2}^+}^{\alpha-1} f(b) + {}_k J_{\frac{a+b}{2}^-}^{\alpha-1} f(a) \right] - f\left(\frac{a+b}{2}\right) \right| \leq \\ & \leq \left(\frac{b-a}{2}\right)^2 \frac{k^2}{\alpha(\alpha+k)} \left[\max(|f''(b)|^q, |f''(b)|^q + \eta(|f''(a)|^q, |f''(b)|^q)) \right]^{\frac{1}{q}}, \end{aligned}$$

where $\frac{\alpha}{k} - 1 > 0$.

Proof. Using Lemma 3, the power-mean inequality, and properties of absolute value, we can write

$$\begin{aligned} & \left| \frac{k2^{\frac{\alpha}{k}-2}}{\alpha(b-a)^{\frac{\alpha}{k}-1}} \Gamma_k(\alpha+k) \left[{}_k J_{\frac{a+b}{2}^+}^{\alpha-1} f(b) + {}_k J_{\frac{a+b}{2}^-}^{\alpha-1} f(a) \right] - f\left(\frac{a+b}{2}\right) \right| \leq \\ & \leq \frac{k(b-a)^2}{\alpha 2^{2-\frac{\alpha}{k}}} \left[\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}} + \int_{\frac{1}{2}}^1 (1-t)^{\frac{\alpha}{k}} \right] |f''(ta + (1-t)b)| dt \leq \\ & \leq \frac{k(b-a)^2}{\alpha 2^{2-\frac{\alpha}{k}}} \left(\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}} dt \right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}} |f''(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} + \\ & + \left(\int_{\frac{1}{2}}^1 (1-t)^{\frac{\alpha}{k}} dt \right)^{1-\frac{1}{q}} \left(\int_{\frac{1}{2}}^1 (1-t)^{\frac{\alpha}{k}} |f''(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} = \\ & = \left(\frac{b-a}{2}\right)^2 \frac{k^2}{\alpha(\alpha+k)} \left[\max(|f''(b)|^q, |f''(b)|^q + \eta(|f''(a)|^q, |f''(b)|^q)) \right]^{\frac{1}{q}}. \end{aligned}$$

And it can be easily checked that $\int_0^{\frac{1}{2}} t^{\frac{\alpha}{k}} dt = \int_{\frac{1}{2}}^1 (1-t)^{\frac{\alpha}{k}} dt = \frac{k}{2^{\frac{\alpha}{k}+1}(\alpha+k)}$;

this completes the proof. \square

Corollary 1. Set $\eta(a, b) = a - b$ in Theorem 4 to obtain

$$\begin{aligned} & \left| \frac{k2^{\frac{\alpha}{k}-2}}{\alpha(b-a)^{\frac{\alpha}{k}-1}} \Gamma_k(\alpha+k) \left[{}_k J_{\frac{a+b}{2}^+}^{\alpha-1} f(b) + {}_k J_{\frac{a+b}{2}^-}^{\alpha-1} f(a) \right] - f\left(\frac{a+b}{2}\right) \right| \leq \\ & \leq \left(\frac{b-a}{2}\right)^2 \frac{k^2}{\alpha(\alpha+k)} \left[\max(|f''(b)|^q, |f''(a)|^q) \right]^{\frac{1}{q}}. \quad (4) \end{aligned}$$

Corollary 2. Let $a, b \in \mathbb{R}^+$ and $a < b$. Choose $\alpha = 1$ and $k = 1$ in (4) to obtain an estimate for Jensen’s inequality:

$$\left| \frac{f(b) + f(a)}{2} - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{8} \left[\max(|f''(b)|^q, |f''(a)|^q) \right]^{\frac{1}{q}}.$$

Proof. Indeed, for $\alpha = 1$ and $k = 1$ we can rewrite the left-hand side of (4):

$$\begin{aligned} & \left| \frac{k2^{\frac{\alpha}{k}-2}}{\alpha(b-a)^{\frac{\alpha}{k}-1}} \Gamma_k(\alpha+k) \left[{}_k J_{\frac{a+b}{2}^+}^{\alpha-1} f(b) + {}_k J_{\frac{a+b}{2}^-}^{\alpha-1} f(a) \right] - f\left(\frac{a+b}{2}\right) \right| = \\ & = \left| \frac{2^{-1}}{1} \Gamma_1(2) \left[J_{\frac{a+b}{2}^+}^0 f(b) + J_{\frac{a+b}{2}^-}^0 f(a) \right] - f\left(\frac{a+b}{2}\right) \right| = \\ & = \left| \frac{f(b) + f(a)}{2} - f\left(\frac{a+b}{2}\right) \right|, \end{aligned}$$

as well as the right-hand side

$$\begin{aligned} & \left(\frac{b-a}{2}\right)^2 \frac{k^2}{\alpha(\alpha+k)} \left[\max(|f''(b)|^q, |f''(a)|^q) \right]^{\frac{1}{q}} = \\ & = \frac{(b-a)^2}{8} \left[\max(|f''(b)|^q, |f''(a)|^q) \right]^{\frac{1}{q}}. \end{aligned}$$

This inequality is of the same order as the inequality obtained in [4] (see Corollary 2.2). □

Similarly, we can prove the following corollary:

Corollary 3. Let $a, b \in \mathbb{R}^+$ and $a < b$. Choose $\alpha = 2$ and $k = 1$ in (4) to obtain

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{24} \left[\max(|f''(b)|^q, |f''(a)|^q) \right]^{\frac{1}{q}}.$$

For $q = 1$, this inequality is of the same order as the inequality obtained in [15] (see Proposition 1) and in [2] inequality (11).

Theorem 5. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$, $I \subset [0, \infty)$, be a differentiable function on I° , such that $f, g \in L[a, b]$ and $a, b \in I$, $0 \leq a < b$. $|f|^p$ and $|g|^q$ are η -QC on $[a, b]$ for $t \in [0, 1]$, $q > 1$. Then, for all $x \in [a, b]$, $\frac{\alpha}{k} + 1 > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, the following inequality holds:

$$\begin{aligned} \frac{1}{b-a} \left| \int_a^b f(x)g(x)h(x)dx \right| &\leq \\ &\leq M^{\frac{1}{p}} N^{\frac{1}{q}} \left[\frac{\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}+1}} [{}_k J_{a^+}^\alpha f(b) + {}_k J_{b^-}^\alpha f(a)] - \frac{1}{b-a} \int_a^b f(x)dx \right], \end{aligned}$$

where

$$\begin{aligned} M &= \max \{|f(b)|^p, |f(b)|^p + \eta(|f(a)|^p, |f(b)|^p)\}, \\ N &= \max \{|g(b)|^q, |g(b)|^q + \eta(|g(a)|^q, |g(b)|^q)\}, \end{aligned}$$

$h(ta + (1-t)b) = [(1-t)^{\frac{\alpha}{k}} + (t^{\frac{\alpha}{k}} - 1)] f(ta + (1-t)b) \geq 0$ for all $t \in [0, 1]$ and $\frac{\alpha}{k} \in [0, 1]$.

Proof. We will use the weighted Hölder inequality:

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x)g(x)h(x)dx &= \\ &= \int_0^1 f(ta + (1-t)b)g(ta + (1-t)b)h(ta + (1-t)b)dt, \\ \frac{1}{b-a} \left| \int_a^b f(x)g(x)h(x)dx \right| &\leq \left(\int_0^1 |f(ta + (1-t)b)|^p h(ta + (1-t)b)dt \right)^{\frac{1}{p}} \times \\ &\quad \times \left(\int_0^1 |g(ta + (1-t)b)|^q h(ta + (1-t)b)dt \right)^{\frac{1}{q}} \leq \\ &\leq \left(\int_0^1 h(ta + (1-t)b)dt \right)^{\frac{1}{p}} \left(\int_0^1 h(ta + (1-t)b)dt \right)^{\frac{1}{q}} \times \\ &\quad \times [\max \{|f(b)|^p, |f(b)|^p + \eta(|f(a)|^p, |f(b)|^p)\}]^{\frac{1}{p}} \times \end{aligned}$$

$$\begin{aligned} & \times [\max \{|g(b)|^q, |g(b)|^q + \eta(|g(a)|^q, |g(b)|^q)\}]^{\frac{1}{q}} = \\ & = \left(\int_0^1 h(ta + (1-t)b) dt \right)^{\frac{1}{p} + \frac{1}{q}} \times M^{\frac{1}{p}} N^{\frac{1}{q}} = \\ & = M^{\frac{1}{p}} N^{\frac{1}{q}} \left(\int_0^1 [(1-t)^{\frac{\alpha}{k}} + (t^{\frac{\alpha}{k}} - 1)] f(ta + (1-t)b) dt \right) = \\ & = M^{\frac{1}{p}} N^{\frac{1}{q}} \left[\frac{\Gamma_k(\alpha + k)}{(b-a)^{\frac{\alpha}{k} + 1}} [{}_k J_{b^-}^\alpha f(a) + {}_k J_{a^+}^\alpha f(b)] - \frac{1}{b-a} \int_a^b f(x) dx \right]; \end{aligned}$$

this completes the proof. \square

Corollary 1. Set $\eta(a, b) = a - b$ to obtain

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x)g(x)h(x)dx \right| \leq \\ & \leq \left[\frac{\Gamma_k(\alpha + k)}{(b-a)^{\frac{\alpha}{k} + 1}} [{}_k J_{a^+}^\alpha f(b) + {}_k J_{b^-}^\alpha f(a)] - \frac{1}{b-a} \int_a^b f(x)dx \right] \times \\ & \quad \times [\max \{|f(b)|^p, |f(a)|^p\}]^{\frac{1}{p}} [\max \{|g(b)|^q, |g(a)|^q\}]^{\frac{1}{q}}. \end{aligned}$$

Further in this article, the following notation is used:

$$U = \frac{\frac{\alpha}{k} + 1}{b-a} [{}_k J_{b^-}^\alpha f(b) + {}_k J_{b^-}^\alpha f(a)] - [{}_k J_{b^-}^{\alpha-1} f(b) + {}_k J_{b^-}^{\alpha-1} f(a)],$$

$$V = |f''(ta + (1-t)b)| \leq [\max \{|f''(b)|, |f''(b)| + \eta(|f''(a)|, |f''(b)|)\}].$$

$$W = |f''(tb + (1-t)a)| \leq [\max \{|f''(a)|, |f''(a)| + \eta(|f''(b)|, |f''(a)|)\}].$$

Lemma 4. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° . If $f'' \in L[a, b]$, $a, b \in I$, then the following equality holds for all $\frac{\alpha}{k} > 1$:

$$\frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b-a)^{\frac{\alpha}{k} - 1}} U = \frac{(b-a)^2}{2} (I_1 + I_2), \tag{5}$$

where

$$I_1 = \int_0^1 (1-t)t^{\frac{\alpha}{k}} f''((1-t)a + tb) dt,$$

$$I_2 = \int_0^1 (1-t)t^{\frac{\alpha}{k}} f''((ta + (1-t)b)dt).$$

Proof. Integrating by parts twice, transform the first integral:

$$\begin{aligned} I_1 &= \int_0^1 (1-t)t^{\frac{\alpha}{k}} f''((1-t)a + tb)dt = \\ &= \frac{f(b)}{(b-a)^2} + \frac{\frac{\alpha}{k}(\frac{\alpha}{k} - 1)}{(b-a)^2} \int_0^1 t^{\frac{\alpha}{k}-2} f((1-t)a + tb)dt - \\ &\quad - \frac{\frac{\alpha}{k}(\frac{\alpha}{k} + 1)}{(b-a)^2} \int_0^1 t^{\frac{\alpha}{k}-1} f((1-t)a + tb)dt. \end{aligned}$$

Make the $(1-t)a + tb = x$ transformation in both obtained integrals and add them, then write:

$$\begin{aligned} I_1 &= \frac{f(b)}{(b-a)^2} + \frac{\frac{\alpha}{k}(\frac{\alpha}{k} - 1)}{(b-a)^{\frac{\alpha}{k}+1}} \int_a^b (x-a)^{\frac{\alpha}{k}-2} f(x)dx - \\ &\quad - \frac{\frac{\alpha}{k}(\frac{\alpha}{k} + 1)}{(b-a)^{\frac{\alpha}{k}+2}} \int_a^b (x-a)^{\frac{\alpha}{k}-1} f(x)dx \end{aligned}$$

or

$$I_1 = \frac{f(b)}{(b-a)^2} + \frac{\Gamma_k(\alpha + k)}{(b-a)^{\frac{\alpha}{k}+1}} {}_k J_{b^-}^{\alpha-1} f(a) - \frac{(\frac{\alpha}{k} + 1)\Gamma_k(\alpha + k)}{(b-a)^{\frac{\alpha}{k}+2}} {}_k J_{b^-}^{\alpha} f(a).$$

Similarly, for the other integral

$$I_2 = \frac{f(a)}{(b-a)^2} + \frac{\Gamma_k(\alpha + k)}{(b-a)^{\frac{\alpha}{k}+1}} {}_k J_{b^-}^{\alpha-1} f(b) - \frac{(\frac{\alpha}{k} + 1)\Gamma_k(\alpha + k)}{(b-a)^{\frac{\alpha}{k}+2}} {}_k J_{b^-}^{\alpha} f(b).$$

Get, by summing these equalites and then grouping the summands:

$$\begin{aligned} I_1 + I_2 &= \frac{[f(a) + f(b)]}{(b-a)^2} - \frac{(\frac{\alpha}{k} + 1)\Gamma_k(\alpha + k)}{(b-a)^{\frac{\alpha}{k}+2}} [{}_k J_{b^-}^{\alpha} f(b) + {}_k J_{b^-}^{\alpha} f(a)] + \\ &\quad + \frac{\Gamma_k(\alpha + k)}{(b-a)^{\frac{\alpha}{k}+1}} [{}_k J_{b^-}^{\alpha-1} f(b) + {}_k J_{b^-}^{\alpha-1} f(a)]. \quad (6) \end{aligned}$$

Obtain (5) by multiplying both sides of equality (6) by the expression $\frac{(b-a)^2}{2}$. The proof is completed. \square

Remark 2. For $k = 1$ in Lemma 4, we obtain equality (20) in [2].

Theorem 6. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}, I \subset [0, \infty)$, be a differentiable function on I° , such that $f'' \in L[a, b]$, where $a, b \in I, a \leq x < y \leq b$. If $|f''|$ is η -QC on $[x, y]$ for $t \in [0, 1]$, then we have, for all $k, \alpha > 0$:

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b-a)^{\frac{\alpha}{k}-1}} \cdot U \right| \leq \frac{(b-a)^2}{2(\frac{\alpha}{k} + 1)(\frac{\alpha}{k} + 2)} \times [V + W].$$

Proof. Using properties of the absolute value and Lemma 4, we obtain

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b-a)^{\frac{\alpha}{k}-1}} \cdot U \right| \leq \\ & \leq \frac{(b-a)^2}{2} \int_0^1 (1-t)t^{\frac{\alpha}{k}} [|f''((1-t)a + tb)| + |f''(ta + (1-t)b)|] dt \leq \\ & \leq \frac{(b-a)^2}{2(\frac{\alpha}{k} + 1)(\frac{\alpha}{k} + 2)} [\max \{ |f''(b)|, |f''(b)| + \eta(|f''(a)|, |f''(b)|) \}] + \\ & + \frac{(b-a)^2}{2(\frac{\alpha}{k} + 1)(\frac{\alpha}{k} + 2)} [\max \{ |f''(a)|, |f''(a)| + \eta(|f''(b)|, |f''(a)|) \}]. \end{aligned}$$

The proof is completed. \square

Corollary 1. Set $\eta(a, b) = a - b$ in Theorem 6 to obtain

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b-a)^{\frac{\alpha}{k}-1}} \cdot U \right| \leq \frac{(b-a)^2 \max \{ |f''(b)|, |f''(a)| \}}{(\frac{\alpha}{k} + 1)(\frac{\alpha}{k} + 2)}. \quad (7)$$

3. New Hölder and improved-iscan inequalities.

Theorem 7. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}, I \subset [0, \infty)$, be a differentiable function on I° , such that $f'' \in L[a, b]$, where $a, b \in I, a \leq x < y \leq b$. If $|f''|^q$ is η -QC on $[x, y]$ for $t \in [0, 1], q > 1, p = \frac{q}{q-1}$, then we have, for all $k, \alpha > 0$:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b-a)^{\frac{\alpha}{k}-1}} \cdot U \right| \leq \\ & \leq \frac{(b-a)^2}{2^{1+\frac{1}{q}}} \left[\left(\beta \left(3, \frac{\alpha}{k}p + 1 \right) \right)^{\frac{1}{p}} + \left(\beta \left(2, \frac{\alpha}{k}p + 2 \right) \right)^{\frac{1}{p}} \right] \left(W^{\frac{1}{q}} + V^{\frac{1}{q}} \right). \end{aligned}$$

Proof. Using the Hölder-Iskan inequality (see [10]), properties of the absolute value, and Lemma 4, we obtain

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b-a)^{\frac{\alpha}{k}-1}} \cdot U \right| = \\ & = \frac{(b-a)^2}{2} \left| \int_0^1 (1-t)t^{\frac{\alpha}{k}} [f''((1-t)a + tb) + f''(ta + (1-t)b)] dt \right| \leq \\ & \leq \frac{(b-a)^2}{2} \left[\left\{ \left(\int_0^1 (1-t)^2 t^{\frac{\alpha}{k}p} dt \right)^{\frac{1}{p}} \left(\int_0^1 (1-t) |f''((1-t)a + tb)|^q dt \right)^{\frac{1}{q}} + \right. \right. \\ & \quad \left. \left. + \left(\int_0^1 (1-t)t^{\frac{\alpha}{k}p+1} dt \right)^{\frac{1}{p}} \left(\int_0^1 t |f''((1-t)a + tb)|^q dt \right)^{\frac{1}{q}} \right\} + \right. \\ & \quad \left. + \left\{ \left(\int_0^1 (1-t)^2 t^{\frac{\alpha}{k}p} dt \right)^{\frac{1}{p}} \left(\int_0^1 (1-t) |f''((1-t)b + ta)|^q dt \right)^{\frac{1}{q}} + \right. \right. \\ & \quad \left. \left. + \left(\int_0^1 (1-t)t^{\frac{\alpha}{k}p+1} dt \right)^{\frac{1}{p}} \left(\int_0^1 t |f''((1-t)b + ta)|^q dt \right)^{\frac{1}{q}} \right\} \right]. \end{aligned}$$

Given the fact that

$$\int_0^1 (1-t)^2 t^{\frac{\alpha}{k}p} dt = \beta \left(3, \frac{\alpha}{k}p + 1 \right) \text{ and } \int_0^1 (1-t)t^{\frac{\alpha}{k}p+1} dt = \beta \left(2, \frac{\alpha}{k}p + 2 \right),$$

we can write

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b-a)^{\frac{\alpha}{k}-1}} \cdot U \right| \leq \\ & \leq \frac{(b-a)^2}{2^{1+\frac{1}{q}}} \left[\left(\beta \left(3, \frac{\alpha}{k}p + 1 \right) \right)^{\frac{1}{p}} + \left(\beta \left(2, \frac{\alpha}{k}p + 2 \right) \right)^{\frac{1}{p}} \right] \left(W^{\frac{1}{q}} + V^{\frac{1}{q}} \right). \end{aligned}$$

The proof is completed. \square

Theorem 8. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$, $I \subset [0, \infty)$, be a differentiable function on I° , such that $f'' \in L[a, b]$, where $a, b \in I$, $a \leq x < y \leq b$. If $|f''|^q$ is η -QC on $[x, y]$ for $t \in [0, 1]$, $q > 1$, $p = \frac{q}{q-1}$, then we have, for all $k, \alpha > 0$:

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b-a)^{\frac{\alpha}{k}-1}} \cdot U \right| \leq \frac{(b-a)^2}{2} \cdot S \cdot \left(W^{\frac{1}{q}} + V^{\frac{1}{q}} \right),$$

where $S = \frac{k^2}{(\alpha + k)(\alpha + 2k)}$.

Proof. Using the improved power-mean integral inequality, properties of the absolute value, and Lemma 4, we obtain

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b - a)^{\frac{\alpha}{k} - 1}} \cdot U \right| \leq \\ & \leq \frac{(b - a)^2}{2} \left[\left\{ \left(\int_0^1 (1 - t)^2 t^{\frac{\alpha}{k}} dt \right)^{1 - \frac{1}{q}} \left(\int_0^1 (1 - t)^2 t^{\frac{\alpha}{k}} |f''((1 - t)a + tb)|^q dt \right)^{\frac{1}{q}} + \right. \right. \\ & \quad \left. \left. + \left(\int_0^1 (1 - t)t^{\frac{\alpha}{k} + 1} dt \right)^{1 - \frac{1}{q}} \left(\int_0^1 (1 - t)t^{\frac{\alpha}{k} + 1} |f''((1 - t)a + tb)|^q dt \right)^{\frac{1}{q}} \right\} + \right. \\ & \quad \left. + \left\{ \left(\int_0^1 (1 - t)^2 t^{\frac{\alpha}{k}} dt \right)^{1 - \frac{1}{q}} \left(\int_0^1 (1 - t)^2 t^{\frac{\alpha}{k}} |f''((1 - t)b + ta)|^q dt \right)^{\frac{1}{q}} + \right. \right. \\ & \quad \left. \left. + \left(\int_0^1 (1 - t)t^{\frac{\alpha}{k} + 1} dt \right)^{1 - \frac{1}{q}} \left(\int_0^1 (1 - t)t^{\frac{\alpha}{k} + 1} |f''((1 - t)b + ta)|^q dt \right)^{\frac{1}{q}} \right\} \right]. \end{aligned}$$

Or

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b - a)^{\frac{\alpha}{k} - 1}} \cdot U \right| \leq \\ & \leq \frac{(b - a)^2}{2} \left[\int_0^1 (1 - t)^2 t^{\frac{\alpha}{k}} dt + \int_0^1 (1 - t)t^{\frac{\alpha}{k} + 1} dt \right] \left(W^{\frac{1}{q}} + V^{\frac{1}{q}} \right) \leq \\ & \leq \frac{(b - a)^2 \left(W^{\frac{1}{q}} + V^{\frac{1}{q}} \right)}{2} \int_0^1 (1 - t)t^{\frac{\alpha}{k}} [(1 - t) + t] dt \leq \\ & \leq \frac{(b - a)^2}{2} \frac{k^2}{(\alpha + k)(\alpha + 2k)} \left(W^{\frac{1}{q}} + V^{\frac{1}{q}} \right). \end{aligned}$$

The proof is completed. \square

4. Applications to special means. Now we consider some means for arbitrary real numbers α, β .

1) *Arithmetic mean* : $A(\alpha, \beta) = \frac{\alpha + \beta}{2}$,

- 2) *Geometric mean* : $G(\alpha, \beta) = \sqrt{\alpha\beta}$, $\alpha\beta \geq 0$
- 3) *Logarithmic mean*: $L(\alpha, \beta) = \frac{\alpha-\beta}{\ln|\alpha|-\ln|\beta|}$, $|\alpha| \neq |\beta|$, $\alpha, \beta \neq 0$,
- 4) *Generalized log-mean*: $L_n(\alpha, \beta) = \left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)} \right]^{\frac{1}{n}}$, $n \in \mathbb{Z} \setminus \{-1, 0\}$,
 $\alpha, \beta > 0$.
- 5) *The Identric Mean*: $I(\alpha, \beta) = \frac{1}{e} \left[\frac{\beta^\beta}{\alpha^\alpha} \right]^{\frac{1}{\beta-\alpha}}$, $\alpha \neq \beta$ and $\alpha, \beta > 0$.

Now, using the obtained results, we give some applications to special means of real numbers.

Proposition 1. *Let $a, b \in \mathbb{R}^+$ and $a < b$; then we have*

$$I(a, b) \leq be^{\frac{a-b}{2b}}. \quad (8)$$

Proof. The assertion follows from Corollary 2 applied to the QC mapping $\alpha = k = 1$ and $f(x) = -\ln(x)$, where $x \in \mathbb{R}^+$. \square

Proposition 2. *Let $a, b \in \mathbb{R}^+$, $a < b$. Then we have, for all $q \geq 1$:*

$$\left| \ln \frac{I(a, b)}{b} \right| \leq \frac{b-a}{2a}. \quad (9)$$

Proof. The assertion follows from Corollary 2 applied to the QC mapping $\alpha = k = 1$ and $f(x) = -\ln(x)$, where $x \in \mathbb{R}^+$. \square

Remark 3. *Obviously, the upper bound (8) for $f(x) = -\ln(x)$ is better than (9).*

Proposition 3. *Let $a, b \in [e, \infty)$ and $a < b$. Then we have*

$$\left| \frac{A(b \ln a, a \ln b)}{G^2(a, b)} - \frac{A(\ln a, \ln b)}{L(b, a)} \right| \leq \frac{(b-a)^2}{12} \cdot \frac{2 \ln b - 3}{b^3}.$$

Proof. The assertion follows from (7) applied to the QC mapping $f(x) = \frac{\ln x}{x}$, $x \in [e, \infty)$, $\alpha = 2$ and $k = 1$ (indeed, since the function $f(x)$ is on the interval $[e, \infty)$ decreases, it is QC). \square

Proposition 4. *Let $a, b \in [\frac{1}{e}, \infty)$ and $a < b$. Then we have*

$$\left| A(a \ln a, b \ln b) - \left[A(a, b) \ln b + \frac{a^2}{2} L^{-1}(b, a) - \frac{1}{2} A(a, b) \right] \right| \leq \frac{(b-a)^2}{12a}.$$

Proof. The assertion follows from (7) applied to the QC mapping $f(x) = x \ln x$, $x \in [\frac{1}{e}, \infty)$, $\alpha = 2$ and $k = 1$ (indeed, since the function $f(x)$ is on the interval $[e, \infty)$ and decreases, it is QC). \square

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