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S. S. Volosivets

MODIFIED MODULUS OF SMOOTHNESS AND APPROXIMATION IN WEIGHTED LORENTZ SPACES BY BOREL AND EULER MEANS

Abstract. Using one-sided Steklov means, we introduce a new modulus of smoothness in weighted Lorentz spaces. The direct and inverse approximation theorem for this modulus of smoothness are proved. Also, we estimate the rate of approximation by the Borel and Euler means in weighted Lorentz spaces.

Key words: *weighted Lorentz spaces, direct and inverse approximation theorems, Borel means, Euler means*

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1. Introduction. Let f be a 2π-periodic continuous function ($f \in C_{2\pi}$), T_n be the space of trigonometric polynomilas of degree at most n, $n \in \mathbb{Z}_+ = \{0, 1, \ldots\}$, $\|f\|_\infty = \max_{x\in[0,2\pi]} |f(x)|$. Let us consider the best approximation $E_n(f)_\infty = \inf\{\|f - t_n\|_\infty : t_n \in T_n\}$, $n \in \mathbb{Z}_+$, and the modulus of continuity $\omega(f, \delta) = \sup_{0\leqslant h\leqslant\delta} \|f(\cdot+h) - f(\cdot)\|_\infty$. Then the classical Jackson theorem states that

$$E_n(f)_\infty \leqslant C\omega(f, (n + 1)^{-1}), \quad n \in \mathbb{Z}_+,$$

while the inverse Salem-Stechkin inequality gives

$$\omega(f, 1/n) \leqslant Cn^{-1} \sum_{k=0}^{n-1} E_k(f)_\infty, \quad n \in \mathbb{N} = \{1, 2, \ldots\}$$

(see [6, Ch. 7]). For a 2π-periodic locally integrable function f, we can consider two variants of Steklov means:

$$s_h(f)(x) = h^{-1} \int\limits_x^{x+h} f(u)\, du, \quad s_h^{(2)}(f)(x) = (2h)^{-1} \int\limits_{x-h}^{x+h} f(u)\, du.$$

Israfilov, Kokilashvili, and Samko [9] introduced a modulus of smoothness in a weighted Lebesgue space with variable exponent $L_w^{p(\cdot)}$ of order $r \in \mathbb{N}$

$$\Omega_r^*(f,\delta)_{L_w^p} = \sup_{0 \leqslant h_i \leqslant \delta} \left\| \prod_{i=1}^{r} (I - s_{h_i}^{(2)})(f) \right\|_{L_w^p},$$

where I is the identical operator, and obtained a Jackson-type estimate $E_n(f)_{L_w^p} \leqslant C\Omega_r^*(f, 1/n)_{L_w^p}$, $n \in \mathbb{N}$, and the inverse result. For another modulus of smoothness, direct and inverse approximation theorems were obtained by Ky [13]. Many mathematicians, such as Akgün, Guven, Israfilov, Kokilashvili, Yildirir, studied the approximation by trigonometric polynomials in various weighted spaces. We note only the papers [17], [10] studying the moduli of smoothness defined with help of $s_h(f)$ and the papers connected with Lorentz spaces: [12], [20], [21], [1], [2].

2. Definitions. A Lebesgue measurable 2π-periodic function $w : [0, 2\pi] \to [0, \infty)$ is called a weight function if $w^{-1}(\{0\})$ has measure zero. If $w(E) = \int_E w(x)\,dx$ for a measurable subset E of $[0, 2\pi]$, then

$$f_w^*(t) = \inf\{\lambda \geqslant 0 : w(\{x \in [0, 2\pi] : |f(x)| > \lambda\}) \leqslant t\}.$$

Let $1 < p, q < \infty$, w be a weight. A measurable function f on $[0, 2\pi]$ belongs to the weighted Lorentz space $L_w^{p,q}$, if

$$\|f\|_{p,q,w} = \left(\int_0^{2\pi} (f^{**}(t))^q t^{q/p-1}\,dt \right)^{1/q} < \infty, \quad f^{**}(t) = t^{-1} \int_0^t f_w^*(u)\,du.$$

The classical Lorentz spaces were introduced by Lorentz (see [15]). If $p = q$, then L_w^{pq} coincides with the weighted Lebesgue space L_w^p.

A weight w belongs to the Muckenhoupt class $A_p(\mathbb{T})$, $1 < p < \infty$, if

$$|w|_{A_p} = \sup \left(|I|^{-1} \int_I w(x)\,dx \right) \left(|I|^{-1} \int_I w^{1-p'}(x)\,dx \right)^{p-1} < \infty,$$

where $p' = p/(p-1)$ and the supremum is taken with respect to all intervals $I \subset \mathbb{R}$ whose length $|I|$ does not exceed 2π (see [16]).

If $w \in A_p(\mathbb{T})$, $1 < p,q < \infty$, then the Hardy-Littlewood maximal operator is bounded in $L_w^{p,q}$ (see [5]). As a consequence, the operators s_h

and $s_h^{(2)}$ are uniformly bounded in $L_w^{p,q}$. Now, we define, for $r \in \mathbb{N}$, the following modulus of smoothness:

$$\Omega_r(f,\delta)_{p,q,w} = \sup_{0\leqslant h_i\leqslant\delta,\, i=1,\dots,r} \left\| \prod_{i=1}^{r}(I - s_{h_i})(f) \right\|_{p,q,w} \tag{1}$$

It is clear that $\Omega_r(f,\delta)_{p,q,w}$ is finite for $w \in A_p(\mathbb{T})$ and $f \in L_w^{p,q}$. In [12] and [1], the authors consider another modulus of smoothness $\Omega_r^*(f,\delta)_{p,q,w}$, where operators s_{h_i} in (1) are substituted by $s_{h_i}^{(2)}$. By definition, $E_n(f)_{p,q,w} = \inf_{t_n\in T_n} \|f - t_n\|_{p,q,w}$, $n \in \mathbb{Z}_+$. For $r \in \mathbb{N}$, $1 < p, q < \infty$, and $w \in A_p(\mathbb{T})$ we denote by $W_{p,q,w}^r$ the collection of all absolutely continuous on each period functions f ($f \in AC_{2\pi}$), such that $f', \dots, f^{(r-1)} \in AC_{2\pi}$ and $f^{(r)} \in L_w^{p,q}$.

For a function $f \in L_w^{p,q}$ and $r \in \mathbb{N}$, we define the Peetre's K-functiomal by

$$K(f,t,L_w^{p,q},W_{p,q,w}^r) = \inf_{g\in W_{p,q,w}^r} \{\|f-g\|_{p,q,w} + t\|g^{(r)}\|_{p,q,w}\}.$$

If $1 < p, q < \infty$, $w \in A_p(\mathbb{T})$, then $L_w^{p,q} \subset L_{2\pi}^1 := L_{w_0}^{1,1}$, $w_0(x) \equiv 1$, (see the proof of Proposition 3.3 in [12]) and $f \in L_w^{p,q}$ has the Fourier series

$$a_0(f)/2 + \sum_{k=1}^{\infty}(a_k(f)\cos kx + b_k(f)\sin kx) =: \sum_{k=0}^{\infty} A_k(f)(x).$$

Let us consider partial sums $S_n(f)(x) = \sum_{k=0}^{n} A_k(f)(x)$, the Borel means

$$B_r(f)(x) = e^{-r}\sum_{k=0}^{\infty} r^k S_k(t)/k!, \quad r > 0,$$

and the Euler means

$$e_n^t(f)(x) = (1+t)^{-n}\sum_{k=0}^{n}\binom{n}{k} t^{n-k} S_k(f)(x), \quad t > 0, \quad n \in \mathbb{N}.$$

More about these means can be found in the monograph by Hardy [7]. It is well known that for $f \in L_{2\pi}^1$, the following limit

$$\widetilde{f}(x) = (2\pi)^{-1} \lim_{t\to 0+0} \int_t^{\pi} (f(x-u) - f(x+u))\,\mathrm{ctg}(u/2)\,du$$

exists a. e. on $\mathbb{R}$ (see [3, Ch. VIII, § 7]). The function $\widetilde{f}(x)$ is called the conjugate function to f. If $\widetilde{f} \in L^1_{2\pi}$, then its Fourier series has the form $\sum\limits_{k=1}^{\infty}(a_k(f)\sin kx - b_k(f)\cos kx)$.

3. Auxiliary propositions.

Lemma 1. *Let $1 < p, q < \infty$, $w \in A_p(\mathbb{T})$. Then the conjugation operator is bounded in $L^{p,q}_w$ and the inequalities*

$$\|S_n(f)\|_{p,q,w} \leqslant C_1\|f\|_{p,q,w}, \quad \|f - S_n(f)\|_{p.q,w} \leqslant (C_1+1)E_n(f)_{p,q,w}, \quad (2)$$

hold for $n \in \mathbb{Z}_+$ and $f \in L^{p,q}_w$.

The statement concerning conjugation operator can be found in [11, Ch. 6, Theorem 6.6.2], while the inequalities (2) can be proved as in [3, Ch. VIII, § 20].

Lemma 2 is stated in [20] for arbitrary $r > 0$ with a reference to the method of Ky [14]. We give another proof for $r \in \mathbb{N}$.

Lemma 2. *Let $1 < p, q < \infty$, $w \in A_p(\mathbb{T})$, $t_n \in T_n$, $n \in \mathbb{N}$, $r \in \mathbb{N}$. Then*

$$\|t_n^{(r)}\|_{p,q,w} \leqslant Cn^r\|t_n\|_{p,q,w} \quad (3)$$

holds.

Proof. It is sufficient to prove (3) in the case $r = 1$. Note that for $t_n(x) = \sum_{k=0}^{n}(c_k\cos kx + d_k\sin kx)$ we have

$$t_n'(x) = -\sum_{k=1}^{n} k(S_k(\widetilde{t_n})(x) - S_{k-1}(\widetilde{t_n})(x)) = \sum_{k=1}^{n-1} S_k(\widetilde{t_n})(x) - n\widetilde{t_n}(x).$$

Since the operators S_k are uniformly bounded in $L^{p,q}_w$ and $\|\widetilde{t_n}\|_{p,q,w} \leqslant C_1\|t_n\|_{p,q,w}$, we obtain

$$\|t_n'\|_{p,q,w} \leqslant C_2(n-1)\|\widetilde{t_n}\|_{p,q,w} + n\|\widetilde{t_n}\|_{p,q,w} \leqslant C_1(C_2+1)n\|t_n\|_{p,q,w}.$$

□

Lemma 3 is proved in [1] also for $r > 0$.

Lemma 3. *Let $1 < p, q < \infty$, $w \in A_p(\mathbb{T})$, $r \in \mathbb{N}$, $f \in W^r_{p,q,w}$. Then*

$$E_n(f)_{p,q,w} \leqslant C(n+1)^{-r}E_n(f^{(r)})_{p,q,w} \leqslant C(n+1)^{-r}\|f^{(r)}\|_{p,q,w}, \quad n \in \mathbb{Z}_+.$$

Lemma 4 is proved in [12, Proposition 3.2].

Lemma 4. *Let $1 < p, q < \infty$, $w \in A_p(\mathbb{T})$ and $\varphi(x, y)$ is a measurable 2π-periodic in each variable function. Then*

$$\left\| \int_0^{2\pi} \varphi(x, \cdot)\, dx \right\|_{p,q,w} \leqslant \int_0^{2\pi} \|\varphi(x, \cdot)\|_{p,q,w}\, dx.$$

Lemma 5. *Let $1 < p, q < \infty$, $w \in A_p(\mathbb{T})$, $r, k \in \mathbb{N}$, $f \in W^r_{p,q,w}$. Then*

$$\Omega_{r+k}(f, \delta)_{p,q,w} \leqslant C\delta^k \Omega_r(f^{(k)}, \delta)_{p,q,w}, \quad \delta \in [0, 2\pi],$$

$$\Omega_k(f, \delta)_{p,q,w} \leqslant C\|f^{(k)}\|_{p,q,w}\delta^k, \quad \delta \in [0, 2\pi].$$

Proof. It is sufficient to prove the first inequality of the Lemma for $k = 1$. Let $0 \leqslant h_i \leqslant \delta$, $1 \leqslant i \leqslant r + 1$, $f \in W^1_{p,q,w}$ and $g(x) = \prod_{i=2}^{r+1}(I - s_{h_i})(f)(x)$. Then we have

$$\prod_{i=1}^{r+1}(I - s_{h_i})(f)(x) = -h_1^{-1} \int_0^{h_1} \int_0^{t} g'(x + s)\, ds\, dt. \tag{4}$$

By Lemma 4 and the uniform boundedness of s_h in $L^{p,q}_w$, we obtain

$$\left\| \prod_{i=1}^{r+1}(I - s_{h_i})(f) \right\|_{p,q,w} \leqslant C_1 h_1^{-1} \int_0^{h_1} t \left\| t^{-1} \int_0^{t} g'(\cdot + s)\, ds \right\|_{p,q,w} dt \leqslant$$

$$\leqslant C_2 h_1^{-1} \|g'\|_{p,q,w} \int_0^{h_1} t\, dt \leqslant 2^{-1} C_2 h_1 \|g'\|_{p,q,w} \leqslant 2^{-1} C_2 \delta \|g'\|_{p,q,w}. \tag{5}$$

It is clear that $g' = \prod_{i=2}^{r+1}(I - s_{h_i})(f')$ a. e. on $\mathbb{R}$. Taking the supremum in the left-hand side of (5) with respect to $h_i \in [0, \delta]$, $1 \leqslant i \leqslant r + 1$, we find that $\Omega_{r+1}(f,\delta)_{p,q,w} \leqslant C_3 \delta \Omega_r(f', \delta)_{p,q,w}$. If we use the equality $(s_h - I)(f)(x) = h^{-1} \int_0^h \int_0^t f'(x + s)\, ds\, dt$ instead of (4) similarly to (5), we obtain the second inequality of Lemma 5 in the case $k = 1$. The general case easily follows from this one. □

Lemma 6 is proved in [19].

Lemma 6. *Let $l \in \mathbb{N}$, $q > 0$. Then there exists $C = C(l, q)$ independent of n, such that*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{q^{n-k}}{(k+1)^l} \leqslant c\frac{(q+1)^n}{(n+1)^l}, \quad n \in \mathbb{N}.$$

Lemma 7 was established by Iofina [8].

Lemma 7. *Let $\gamma_n(t) = e^{-t}\sum\limits_{k=0}^{n} t^k/k!$, $t \geqslant 1$. Then $\gamma_{[t]}(t) \geqslant C > 0$, where $[t]$ is the integer part of t.*

4. Direct and inverse approximation theorems. As usually, $A(t) \asymp B(t)$, $t \in T$, means that there exist $C_1, C_2 > 0$, such that $C_1A(t) \leqslant B(t) \leqslant C_2A(t)$, $t \in T$.

Theorem 1. *Let $1 < p, q < \infty$, $w \in A_p(\mathbb{T})$, $r \in \mathbb{N}$, $f \in W^r_{p,q,w}$. Then*

$$\Omega_r(f, t)_{p,q,w} \asymp K(f, t^r, L^{p,q}_w, W^r_{p,q,w}), \quad t \in [0, 2\pi].$$

Proof. By the uniform boundedness of the Steklov operators s_h in $L^{p,q}_w$ and Lemma 5 for $g \in W^r_{p(\cdot),2\pi}$, we have

$$\Omega_r(f, t)_{p,q,w} \leqslant \Omega_r(f - g, t)_{p,q,w} + \Omega_r(g, t)_{p,q,w} \leqslant \\ \leqslant C_1\|f - g\|_{p,q,w} + C_2t^r\|g^{(r)}\|_{p,q,w}. \quad (6)$$

Taking the infimum in the right-hand side of (6) over $g \in W^r_{p,q,w}$, we obtain

$$\Omega_r(f,t)_{p,q,w} \leqslant \max(C_1,C_2)K(f, t^r, L^{p,q}_w, W^r_{p,q,w}).$$

For the converse inequality, we use the operator

$$\Theta_z(f)(x) = \frac{2}{z^2}\int\limits_0^z\int\limits_0^t f(x + s)\,ds\,dt.$$

In [18], it is proved that

$$(\Theta^k_z(f))^{(k)}(x) = \frac{2^k}{z^k}[s_z - I]^k(f)(x). \quad (7)$$

Similar to the proof of Lemma 5, by Lemma 4 we have

$$\|\Theta_z(f)\|_{p,q,w} \leqslant \frac{2}{z^2}\int_0^z t\|t^{-1}\int_0^t f(x+s)\,ds\|_{p,q,w}\,dt \leqslant$$
$$\leqslant C_1\|f\|_{p,q,w}\frac{2}{z^2}\int_0^z t\,dt = C_1\|f\|_{p,q,w}. \quad (8)$$

For the operators $A_z^{[r]} = I-(I-\Theta_z^r)^r$ and $U_j = \Theta_z^{r(r-j-1)}$ by (7) we find that

$$\|(A_z^{[r]}(f))^{(r)}\|_{p,q,w} \leqslant \sum_{j=0}^{r-1}\binom{r}{j}\|(\Theta_z^{r(r-j)}(f))^{(r)}\|_{p,q,w} =$$
$$= \sum_{j=0}^{r-1}\binom{r}{j}\|(\Theta_z^r(U_j(f)))^{(r)}\|_{p,q,w} = \sum_{j=0}^{r-1}\binom{r}{j}\frac{2^r}{z^r}\|(I-s_z)^r[U_j(f)]\|_{p,q,w}.$$

Since s_h and Θ_z commute, by (8) we obtain

$$\|(A_z^{[r]}(f))^{(r)}\|_{p,q,w} \leqslant 2^r\sum_{j=0}^{r-1}\binom{r}{j}z^{-r}\|U_j[(I-s_z)^r(f)]\|_{p,q,w} \leqslant$$
$$\leqslant C_2 z^{-r}\|(I-s_z)^r(f)\|_{p,q,w} \leqslant C_2 z^{-r}\Omega_r(f,z)_{p,q,w}. \quad (9)$$

Using (8) and Lemma 4, we have for $g \in L_{2\pi}^{p(\cdot)}$:

$$\|(I-\Theta_z^r)(g)\|_{p,q,w} \leqslant C_3 \sup_{0\leqslant t\leqslant z}\|(I-s_z)(g)\|_{p,q,w}. \quad (10)$$

(see similar arguments in [18, (4.6)]). Using the equality $I-A_z^{[r]} = (I-\Theta_z^r)^r$ and applying (10) r times, we obtain

$$\|f - A_z^{[r]}(f)\|_{p,q,w} \leqslant C_3^r\Omega_r(f,z)_{p,q,w}. \quad (11)$$

From (9) and (11) we deduce

$$\|f - A_z^{[r]}(f)\|_{p,q,w} + z^r\|(A_z^{[r]}(f))^{(r)}\|_{p,q,w} \leqslant C_4\Omega_r(f,z)_{p,q,w},$$

where $A_z^{[r]}(f) \in W_{p(\cdot),2\pi}^r$. Thus, $K(f, z^r, L_w^{p,q}, W_{p,q,w}^r) \leqslant C_4\Omega_r(f,z)_{p(\cdot)}$, and the proof of Theorem 1 is complete. □

Now we compare our modulus of smoothness with $\Omega^*_r(f,\delta)_{p,q,w}$ used in [12] and [1].

Corollary 1. *Let* $1<p,q<\infty$, $w\in A_p(\mathbb{T})$, $r\in\mathbb{N}$. *Then*

$$\Omega_{2r}(f,\delta)_{p,q,w}\asymp\Omega^*_r(f,\delta)_{p,q,w},\quad \delta\in[0,2\pi]. \tag{12}$$

Proof. In [1] it is proved that, under conditions of the Corollary,

$$\Omega^*_r(f,\delta)_{p,q,w}\asymp K(f,\delta^r,L^{p,q}_w,W^{2r}_{p,q,w}),\quad \delta\in[0,2\pi].$$

Combining this result with Theorem 1, we obtain (12). □

Theorem 2. *Let* $1<p,q<\infty$, $w\in A_p(\mathbb{T})$, $r\in\mathbb{N}$. *Then*

$$E_n(f)_{p,q,w}\leqslant C\Omega_r(f,(n+1)^{-1})_{p,q,w},\quad n\in\mathbb{Z}_+.$$

Proof. For $n\in\mathbb{Z}_+$ we choose a function $g\in W^r_{p,q,w}$, such that

$$\|f-g\|)_{p,q,w}+(n+1)^{-r}\|g^{(r)}\|_{p,q,w}\leqslant 2K(f,(n+1)^{-r},L^{p,q}_w,W^r_{p,q,w}).$$

By Lemma 3 and Theorem 1, we have, for $n\in\mathbb{Z}_+$:

$$\begin{aligned}E_n(f)_{p,q,w}&\leqslant E_n(f-g)_{p,q,w}+E_n(g)_{p,q,w}\leqslant\\&\leqslant C_1\left(\|f-g\|_{p,q,w}+\|g^{(r)}\|_{p,q,w}(n+1)^{-r}\right)\leqslant\\&\leqslant 2C_1K(f,(n+1)^{-r},L^{p,q}_w,W^r_{p,q,w})\leqslant C_2\Omega_r(f,(n+1)^{-1})_{p,q,w}.\end{aligned}$$

□

Theorem 3. *Let* $1<p,q<\infty$, $w\in A_p(\mathbb{T})$, $r\in\mathbb{N}$. *Then*

$$\Omega_r(f,n^{-1})_{p,q,w}\leqslant Cn^{-r}\sum_{k=1}^{n}k^{r-1}E_{k-1}(f)_{p,q,w},\quad n\in\mathbb{N}. \tag{13}$$

Proof. Let $t_k\in T_k$ be the polynomial of the best approximation for $f\in L^{p,q}_w$, $k\in\mathbb{Z}_+$. Using Lemma 5 and Lemma 2, we obtain

$$\Omega_r(f,n^{-1})_{p,q,w}\leqslant\Omega_r(f-t_{2^m},n^{-1})_{p,q,w}+\Omega_r(t_{2^m},n^{-1})_{p,q,w}\leqslant$$

$$\leqslant C_1(\|f-t_{2^m}\|_{p,q,w}+n^{-r}\|t^{(r)}_{2^m}\|_{p,q,w})\leqslant$$

$$\leqslant C_2\left[E_{2^m}(f)_{p,q,w}+n^{-r}\left(\|t^{(r)}_1-t^{(r)}_0\|_{p,q,w}+\sum_{i=0}^{m-1}\|t^{(r)}_{2^{i+1}}-t^{(r)}_{2^i}\|_{p,q,w}\right)\right]\leqslant$$

$$\leqslant C_3\left[E_{2^m}(f)_{p,q,w}+n^{-r}\left(\|t_1-t_0\|_{p,q,w}+\sum_{i=0}^{m-1}2^{ir}\|t_{2^{i+1}}-t_{2^i}\|_{p,q,w}\right)\right]\leqslant$$

$$\leqslant C_4\left[E_{2^m}(f)_{p,q,w}+n^{-r}\left(E_0(f)_{p,q,w}+\sum_{i=0}^{m-1}2^{ir}E_{2^i}(f)_{p,q,w}\right)\right]\leqslant$$

$$\leqslant C_5\sum_{k=1}^{2^m}k^{r-1}E_{k-1}(f)_{p,q,w},\quad n\in\mathbb{N}.$$

If $n\in\mathbb{N}$ is fixed and $2^m\leqslant n<2^{m+1}$, $m\in\mathbb{Z}_+$, then (13) easily follows from the last inequality. □

If ω is increasing and continuous on $[0;2\pi]$, $\omega(0)=0$, then $\omega\in\Phi$. A function $\omega\in\Phi$ belongs to the Bary-Stechkin class B_α, $\alpha>0$, if $\sum_{k=1}^{n}k^{\alpha-1}\omega(k^{-1})=O(n^\alpha\omega(n^{-1}))$, $n\in\mathbb{N}$ (see [4]).

Corollary 1. *Let $1<p,q<\infty$, $w\in A_p(\mathbb{T})$, $r\in\mathbb{N}$ and $\omega\in B_r$. Then, the conditions $E_n(f)_{p,q,w}=O(\omega((n+1)^{-1}))$, $n\in\mathbb{Z}_+$, and $\Omega_r(f,\delta)_{p,q,w}=O(\omega(\delta))$, $\delta\in[0,2\pi]$, are equivalent.*

Remark 1. *The converse inequality from Theorem 3 has the same form as the classical converse approximation theorem (see [6, Ch. 7, Theorem 3.1]), while (see Corollary 1)*

$$\Omega_r^*(f,n^{-1})_{p,q,w}\leqslant Cn^{-2r}\sum_{k=1}^{n}k^{2r-1}E_{k-1}(f)_{p,q,w},\quad n\in\mathbb{N}.$$

5. Approximation by the Borel and Euler means.

Theorem 4. *Let $1<p,q<\infty$, $w\in A_p(\mathbb{T})$, $r\in\mathbb{N}$, $t>0$ and $f\in L_w^{p,q}$. Then*

$$\|f-e_n^t(f)\|_{p,q,w}\leqslant C\Omega_r(f,n^{-1})_{p,q,w},\quad n\in\mathbb{N},\tag{14}$$

$$\|f-e_n^t(f)\|_{p,q,w}\leqslant Cn^{-r}\sum_{k=1}^{n}k^{r-1}E_{k-1}(f)_{p,q,w},\quad n\in\mathbb{N}.\tag{15}$$

Proof. By the definition of the Euler means, Lemma 1, and Theorem 2, we obtain

$$\|f-e_n^t(f)\|_{p,q,w}=(1+t)^{-n}\left\|\sum_{j=0}^{n}\binom{n}{j}t^{n-j}(f-S_j(f))\right\|_{p,q,w}\leqslant$$

$$\leqslant \frac{C_1}{(1+t)^n}\sum_{j=0}^{n}\binom{n}{j}t^{n-j}E_j(f)_{p,q,w} \leqslant$$

$$\leqslant \frac{C_2}{(1+t)^n}\sum_{j=0}^{n}\binom{n}{j}t^{n-j}\Omega_r(f,(j+1)^{-1})_{p,q,w}.$$

Due to Theorem 1, we have the property

$$\Omega_r(f,\lambda\delta)_{p,q,w} \leqslant C(\lambda+1)^r\Omega_r(f,\delta)_{p,q,w}, \quad \lambda>0.$$

By this property and Lemma 6, we find that

$$\|f-e_n^t(f)\|_{p,q,w} \leqslant \frac{C_2}{(1+t)^n}\sum_{j=0}^{n}\binom{n}{j}t^{n-j}\left(\frac{n+1}{j+1}+1\right)^r\Omega_r(f,\frac{1}{n+1})_{p,q,w} \leqslant$$

$$\leqslant \frac{C_3}{(1+t)^n}(n+1)^r\Omega_r(f,\frac{1}{n+1})_{p,q,w}\sum_{j=0}^{n}\binom{n}{j}\frac{t^{n-j}}{(j+1)^r} \leqslant C_4\Omega_r(f,\frac{1}{n+1})_{p,q,w}$$

and (14) is proved. The inequality (15) follows from (14) and Theorem 3. □

Theorem 5. *Let $1<p,q<\infty$, $w\in A_p(\mathbb{T})$, $r\in\mathbb{N}$, $t\geqslant 1$ and $f\in L_w^{p,q}$. Then*

$$\|f-B_t(f)\|_{p,q,w} \leqslant C\sum_{k=0}^{[t]}\frac{t^k}{k!}e^{-t}E_k(f)_{p,q,w},$$

where $[t]$ is the integer part of t.

Proof. Let $\tau_n\in T_n$ be such that $\|f-\tau_n\|_{p,q,w}=E_n(f)_{p,q,w}$. Then, by Lemma 1,

$$\|f-B_t(f)\|_{p,q,w} \leqslant \|B_t(\tau_n)-B_t(f)\|_{p,q,w}+\|B_t(\tau_n)-\tau_n\|_{p,q,w}+$$

$$+\|\tau_n-f\|_{p,q,w} \leqslant e^{-t}\sum_{k=0}^{\infty}\frac{t^k}{k!}\|S_k(f-\tau_n)\|_{p,q,w}+\|B_t(\tau_n)-\tau_n\|_{p,q,w}+$$

$$+\|\tau_n-f\|_{p,q,w} \leqslant C_1E_n(f)_{p,q,w}+\|B_t(\tau_n)-\tau_n\|_{p,q,w}.$$

Now,

$$\|B_t(\tau_n)-\tau_n\|_{p,q,w}=\left\|\sum_{k=0}^{n-1}\frac{t^ke^{-t}}{k!}(S_k(\tau_n)-\tau_n)\right\|_{p,q,w} \leqslant$$

$$\leqslant C_2 \sum_{k=0}^{n-1} \frac{t^k e^{-t}}{k!} E_k(\tau_n)_{p,q,w}.$$

By the definition of τ_n,

$$E_k(\tau_n)_{p,q,w} \leqslant E_k(f-\tau_n)_{p,q,w} + E_k(f)_{p,q,w} \leqslant 2E_k(f)_{p,q,w}.$$

Using the previous inequaities and Lemma 7 and taking $n = [t]$, we obtain

$$\|f - B_t(f)\|_{p,q,w} \leqslant C_1 E_{[t]}(f)_{p,q,w} + 2C_2 \sum_{k=0}^{[t]-1} \frac{t^k e^{-t}}{k!} E_k(f)_{p,q,w} \leqslant$$

$$\leqslant C_3 \sum_{k=0}^{[t]} \frac{t^k e^{-t}}{k!} E_k(f)_{p,q,w}.$$

□

Let us show that the estimate of Theorem 5 gives a clear result on some subclasses of $L_w^{p,q}$.

Corollary 1. *Let $1 < p, q < \infty$, $w \in A_p(\mathbb{T})$, $r \in \mathbb{N}$, $t \geqslant 1$ and $f \in L_w^{p,q}$. If $E_n(f)_{p,q,w} = O((n+1)^{-\alpha})$, $n \in \mathbb{Z}_+$, or $\Omega(f,\delta)_{p,q,w} = O(\delta^\alpha)$, $\delta \in [0, 2\pi]$, then*

$$\|f - B_t(f)\|_{p,q,w} \leqslant Ct^{-\alpha}, \quad t \geqslant 1.$$

Proof. Under conditions of Corollary 1, we have, by Theorem 5 and Theorem 2:

$$\|f - B_t(f)\|_{p,q,w} \leqslant C_1 \sum_{k=0}^{[t]} \frac{t^k e^{-t}}{k!(k+1)^\alpha} = C_1 (t+1)^{-\alpha} \sum_{k=0}^{[t]} \frac{t^k e^{-t}}{k!} \left(\frac{t+1}{k+1}\right)^\alpha.$$

For $m = [\alpha] + 1$ and $0 \leqslant k \leqslant [t]$, we have

$$\left(\frac{t+1}{k+1}\right)^\alpha \leqslant \left(\frac{t+1}{k+1}\right)^m \leqslant \frac{2^m t^m}{(k+1)^m}$$

and

$$\|f - B_t(f)\|_{p,q,w} \leqslant C_2 (t+1)^{-\alpha} e^{-t} \sum_{k=0}^{[t]} \frac{t^{k+m}}{k!(k+1)^m} \leqslant$$

$$\leqslant \frac{C_2}{(t+1)^\alpha e^t}\sum_{k=0}^{\infty}\frac{t^{k+m}}{(k+m)!}\frac{(k+1)\dots(k+m)}{(k+1)^m}\leqslant$$
$$\leqslant \frac{C_2 m!}{(t+1)^\alpha e^t}\sum_{k=0}^{\infty}\frac{t^{k+m}}{(k+m)!}\leqslant\frac{C_3}{(t+1)^\alpha}.$$

□

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Saratov State University
83 Astrakhanskaya St., Saratov 410012, Russia
E-mail: VolosivetsSS@mail.ru