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ESTIMATES FOR SOBOLEV-ORTHONORMAL FUNCTIONS AND GENERATED BY LAGUERRE FUNCTIONS

Abstract. In this paper, we consider the system of functions $\lambda_{r,n}^{\alpha}(x)$ $(n = 0, 1, \ldots)$, $\alpha > -1$, $r \in \mathbb{N}$, orthonormal with respect to a Sobolev-type inner product and generated by the system of Laguerre functions. Using asymptotic formulas for the Laguerre polynomials, we obtain estimates for functions $\lambda_{r,n}^{\alpha}(x)$, $x \in [0, \infty)$.

Key words: *Laguerre functions, Sobolev-type inner product, Sobolev-orthonormal functions*

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1. Introduction. Recently, the theory of polynomials orthogonal with respect to a Sobolev-type inner product has been intensively developed [2], [4], [5], [10], [11], [14], [16], [17]. In particular, this is due to the fact that Fourier series by Sobolev orthogonal polynomials (Fourier-Sobolev series) is a useful tool for solving initial-value problems for ordinary differential equations [17]. Note that the most representative results in this theory are associated with the following inner product:

$$\langle f, g\rangle_S = \sum_{\nu=0}^{r-1} f^{(\nu)}(0)g^{(\nu)}(0) + \int_a^b f^{(r)}(x)g^{(r)}(x)w(x)dx, \qquad (1)$$

where $w(x)$ is the weight function. One of the methods for constructing systems of functions orthogonal with respect to the inner product (1) was developed in the works of Sharapudinov I. I. [16], [17]. A system of functions

$$\lambda_{r,n}^{\alpha}(x) = \frac{x^n}{n!}, \quad n = 0, 1, \ldots, r-1,$$

$$\lambda_{r,r+n}^{\alpha}(x) = \frac{1}{(r-1)!}\int\limits_0^x (x-t)^{r-1}\mathcal{L}_n^{\alpha}(t)dt, \quad n = 0, 1, \ldots, \tag{2}$$

orthonormal on $[0, \infty)$ with respect to the inner product

$$\langle f, g\rangle_S = \sum_{\nu=0}^{r-1} f^{(\nu)}(0)g^{(\nu)}(0) + \int\limits_0^{\infty} f^{(r)}(x)g^{(r)}(x)dx$$

and generated by the system of Laguerre functions $\mathcal{L}_n^{\alpha}(x) = \frac{e^{-x/2}x^{\alpha/2}}{\sqrt{h_n^{\alpha}}}L_n^{\alpha}(x)$ was introduced in [7] using this method. Here $h_n^{\alpha} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}$, and by $L_n^{\alpha}(x)$ we denote the Laguerre polynomial of degree n. In the same paper, asymptotic properties were investigated and estimates were obtained for $\lambda_{1,1+n}^{\alpha}(x)$, $x \in [0, \omega]$:

$$\lambda_{1,1+n}^{\alpha}(x) = \begin{cases} O\left(\frac{1}{\nu}\right), & 0 \leqslant x \leqslant \frac{1}{\nu}, \\ O\left(\frac{1}{\nu^{3/4}}\right), & \frac{1}{\nu} < x \leqslant \omega, \end{cases}$$

where ω is a fixed positive real number, $\nu = 4n + 2\alpha + 2$. In this paper, we obtain estimates for $\lambda_{r,r+n}^{\alpha}(x)$, $x \in [0, \infty)$. The following theorem holds:

Theorem 1. *Let $\alpha > -1$, $r \in \mathbb{N}$. Then the following estimates hold:*

$$\lambda_{r,r+n}^{\alpha}(x) = \begin{cases} O\left(\nu^{\frac{\alpha}{2}}x^{r+\frac{\alpha}{2}}\right), & 0 \leqslant x \leqslant \frac{1}{\nu}, \\ O\left(\frac{x^{r-\frac{3}{4}}}{(\nu-x)^{\frac{3}{4}}}\right), & \frac{1}{\nu} < x \leqslant \nu - \nu^{1/3}, \\ O\left(\nu^{r-1}\right), & \nu - \nu^{1/3} < x. \end{cases}$$

Remark. *Estimates for the function $\lambda_{1,1+n}^{0}(x)$ were obtained in [8].*

To prove this theorem, we need some properties of Laguerre polynomials given in the next section.

2. Some properties of Laguerre polynomials. Let α be a real number. Then the following relations hold for Laguerre polynomials:

- Rodrigues' formula [18]:

$$L_n^{\alpha}(x) = \frac{1}{n!}x^{-\alpha}e^{x}\frac{d^n}{dx^n}(x^{n+\alpha}e^{-x}).$$

- Orthogonality relations [18]:

$$\int_0^\infty L_n^\alpha(x) L_m^\alpha(x)\rho(x)dx = \delta_{nm} h_n^\alpha, \quad \alpha > -1,$$

where $\rho(x) = e^{-x}x^\alpha$, δ_{nm} is the Kronecker delta.

- Equalities for derivatives [18]:

$$\frac{d^r}{dx^r} L_{n+r}^{\alpha-r} = (-1)^r L_n^\alpha(x), \tag{3}$$

$$L_n^{-r}(x) = \frac{(-x)^r}{n(n-1)\cdots(n-r+1)} L_{n-r}^r(x). \tag{4}$$

- Equality [15, p.623, formula 6]:

$$\sum_{k=0}^\infty \frac{k! L_k^\alpha(x) L_k^\alpha(y)}{(\alpha+1)_k(k+1)} = \frac{\alpha}{(xy)^\alpha} e^{x+y}\gamma(\alpha,x)\Gamma(\alpha,y), \quad 0 < x \leqslant y, \tag{5}$$

where $\gamma(\alpha,x)$ and $\Gamma(\alpha, y)$ are incomplete gamma functions defined by

$$\gamma(\alpha,x) = \int_0^x t^{\alpha-1}e^{-t}dt, \quad \Gamma(\alpha,x) = \int_x^\infty t^{\alpha-1}e^{-t}dt.$$

- Asymptotic formulas [3], [12]:
  i) Let $\alpha > -1$, $0 \leqslant x \leqslant b\nu$, $0 < b < 1$, $n > n_0$; then

$$L_n^\alpha(x) = \frac{\Gamma(n+\alpha+1)2^{\alpha-\frac{1}{2}} e^{\frac{x}{2}} \psi^{\frac{1}{2}}}{n!\, \nu^{\frac{\alpha}{2}-\frac{1}{2}} x^{\frac{\alpha}{2}+\frac{1}{2}} (\psi')^{\frac{1}{2}}} \left[J_\alpha(\nu\psi) + O\Big(\frac{x^{\frac{1}{2}}}{\nu^{\frac{3}{2}}} \tilde{J}_\alpha(\nu\psi)\Big)\right]. \tag{6}$$

  ii) Let $\alpha > -1$, $a\nu \leqslant x$, $a > 0$, $n > n_0$; then

$$L_n^\alpha(x) = \frac{(-1)^n \pi^{\frac{1}{2}} 2^{\frac{5}{6}} N^{N+\frac{1}{6}} e^{\frac{x}{2}}}{n!\, x^{\frac{\alpha}{2}+\frac{1}{2}} e^N (-\phi')^{\frac{1}{2}}} \left[Ai(-\nu^{\frac{2}{3}}\phi) + O\Big(\frac{\widetilde{Ai}(-\nu^{\frac{2}{3}}\phi)}{x}\Big)\right]. \tag{7}$$

In (6) and (7) above $N = n + (\alpha+1)/2$, $\nu = 4n + 2\alpha + 2$, $t = x/\nu$,

$$\psi = \psi(t) = \frac{1}{2}\left(\sqrt{t-t^2} + \arcsin(\sqrt{t})\right), \ 0 \leqslant t < 1,$$

$J_\alpha(x)$ is the Bessel function of the first kind, for which the following asymptotic formula holds [18, p.15, formula 1.71.7]:

$$J_\alpha(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O\left(x^{-\frac{3}{2}}\right),\ x \to +\infty, \tag{8}$$

$$\tilde{J}_\alpha(u) = \begin{cases} u^\alpha,\ 0 < u \leqslant 1, \\ u^{-1/2},\ 1 < u. \end{cases}$$

$$\phi = \phi(t) = \begin{cases} \left[\frac{3}{4}(\arccos\sqrt{t} - \sqrt{t(1-t)})\right]^{2/3}, & 0 < t \leqslant 1, \\ -\left[\frac{3}{4}(\sqrt{t(t-1)} - \operatorname{arcosh}\sqrt{t})\right]^{2/3}, & 1 < t, \end{cases} \tag{9}$$

$$\operatorname{arcosh}(t) = \ln(t + \sqrt{t^2 - 1}),$$

$Ai(u)$ and $Bi(u)$ are Airy functions; if $u > 0$ the following estimates hold [9, pp. 508–511]:

$$|Ai(-u)| = O\left(u^{-1/4}\right), \tag{10}$$

$$|Bi(-u)| = O\left(u^{-1/4}\right), \tag{11}$$

$$|Ai(u)| = O\Big(u^{-1/4} \exp\Big(-\frac{2}{3}u^{3/2}\Big)\Big), \tag{12}$$

$$\widetilde{Ai}(u) = \begin{cases} Ai(u), & u \geqslant 0, \\ (|Ai(u)|^2 + |Bi(u)|^2)^{1/2}, & u \leqslant 0. \end{cases} \tag{13}$$

Also, note the estimates for Laguerre functions $\mathcal{L}_n^\alpha(x)$, which were obtained in [1], [13]:

$$\mathcal{L}_n^\alpha(x) = \begin{cases} O\left(x^{\frac{\alpha}{2}}\nu^{\frac{\alpha}{2}}\right), & 0 \leqslant x \leqslant \frac{1}{\nu}, \\ O\left(\nu^{-\frac{1}{4}}x^{-\frac{1}{4}}\right), & \frac{1}{\nu} < x \leqslant \frac{\nu}{2}, \\ O\left(\nu^{-\frac{1}{4}}(\nu^{\frac{1}{3}} - |x-\nu|)^{-\frac{1}{4}}\right), & \frac{\nu}{2} < x \leqslant \frac{3\nu}{2}, \\ O\left(e^{-\frac{x}{4}}\right), & \frac{3\nu}{2} < x. \end{cases} \tag{14}$$

In (14) for $n = 0$ and $-1 < \alpha < 0$ we will assume $\nu = 2$.

3. The proof of Theorem 1. First, we obtain an asymptotic representation for $\lambda^{\alpha}_{r,r+n}(x)$, $0 < x \leqslant b\nu$ in terms of the Bessel function $J_\alpha(x)$. To this end, we use formula (6) and write

$$\lambda^{\alpha}_{r,r+n}(x) = \frac{2^{\alpha-\frac{1}{2}}}{\nu^{\frac{\alpha}{2}-\frac{1}{2}}}\frac{\sqrt{h^{\alpha}_{n}}}{(r-1)!}\int\limits_0^x \frac{(x-t)^{r-1}\psi^{\frac{1}{2}}}{t^{\frac{1}{2}}(\psi')^{\frac{1}{2}}}\Big[J_\alpha(\nu\psi)+O\Big(\frac{t^{\frac{1}{2}}}{\nu^{\frac{3}{2}}}\tilde{J}_\alpha(\nu\psi)\Big)\Big] =$$

$$= \frac{2^{\alpha-\frac{1}{2}}}{\nu^{\frac{\alpha}{2}-\frac{1}{2}}}\frac{\sqrt{h^{\alpha}_{n}}}{(r-1)!}\int\limits_0^x \frac{(x-t)^{r-1}\psi^{\frac{1}{2}}}{t^{\frac{1}{2}}(\psi')^{\frac{1}{2}}}J_\alpha(\nu\psi)dt+$$

$$+O\Big(\frac{1}{\nu}\Big)\int\limits_0^x \frac{(x-t)^{r-1}\psi^{\frac{1}{2}}}{(\psi')^{\frac{1}{2}}}\tilde{J}_\alpha(\nu\psi)dt.$$

Further, we note that $\psi(\tau) = \tau^{1/2}(1+O(\tau))$, $\tau = \frac{t}{\nu}$ and

$$\nu\psi = \nu\sqrt{\tau}(1+O(\tau)) = \sqrt{\nu t} + O\Big(\frac{t^{3/2}}{\nu^{1/2}}\Big).$$

Then

$$\nu\psi \leqslant 1 \quad \text{if} \quad 0 < t \leqslant \frac{1}{\nu},$$

$$\nu\psi > 1 \quad \text{if} \quad t > \frac{1}{\nu}.$$

Using the definition of the function $\tilde{J}_\alpha(u)$, we find:

$$O\Big(\frac{1}{\nu}\Big)\int\limits_0^x \frac{(x-t)^{r-1}\psi^{\frac{1}{2}}}{(\psi')^{\frac{1}{2}}}\tilde{J}_\alpha(\nu\psi)dt = O\left(\nu^{\alpha-1}\right)\int\limits_0^x \frac{(x-t)^{r-1}}{(\psi')^{\frac{1}{2}}}\psi^{\alpha+\frac{1}{2}} =$$

$$= O\left(\nu^{\alpha-1}\right)\int\limits_0^x \frac{(x-t)^{r-1}t^{\frac{1}{4}}}{(\nu-t)^{\frac{1}{4}}}\Big(\frac{t}{\nu}\Big)^{\frac{\alpha}{2}+\frac{1}{4}}dt =$$

$$= O\Big(\nu^{\frac{\alpha}{2}-\frac{5}{4}}\Big)\frac{x^{\frac{\alpha}{2}+\frac{1}{2}}x^r}{(\nu-x)^{\frac{1}{4}}} = O\left(\nu^{\frac{\alpha}{2}-\frac{3}{2}}x^{r+\frac{\alpha}{2}+\frac{1}{2}}\right),\ 0 < x \leqslant \frac{1}{\nu},$$

$$O\Big(\frac{1}{\nu}\Big)\int\limits_0^x \frac{(x-t)^{r-1}\psi^{\frac{1}{2}}}{(\psi')^{\frac{1}{2}}}\tilde{J}_\alpha(\nu\psi)dt = O\Big(\frac{1}{\nu^{r+2}}\Big)+$$

$$+O\Big(\frac{1}{\nu^{\frac{3}{2}}}\Big)\int\limits_{1/\nu}^{x}\frac{(x-t)^{r-1}t^{\frac{1}{4}}}{(\nu-t)^{\frac{1}{4}}}dt=O\Big(\frac{x^{r+\frac{1}{4}}}{\nu^{\frac{3}{2}}(\nu-x)^{\frac{1}{4}}}\Big),\quad \frac{1}{\nu}<x<b\nu.$$

So, we have

$$\lambda_{r,r+n}^{\alpha}(x)=\frac{2^{\alpha-\frac{1}{2}}}{\nu^{\frac{\alpha}{2}-\frac{1}{2}}}\frac{\sqrt{h_n^\alpha}}{(r-1)!}\int\limits_{0}^{x}\frac{(x-t)^{r-1}\psi^{\frac{1}{2}}}{t^{\frac{1}{2}}(\psi')^{\frac{1}{2}}}J_\alpha(\nu\psi)dt+R_\nu^{\alpha,r}(x), \tag{15}$$

where

$$R_\nu^{\alpha,r}(x)=\begin{cases}O\left(\nu^{\frac{\alpha}{2}-\frac{3}{2}}x^{r+\frac{\alpha}{2}+\frac{1}{2}}\right), & 0<x\leqslant\frac{1}{\nu},\\ O\Big(\frac{x^{r+\frac{1}{4}}}{\nu^{\frac{3}{2}}(\nu-x)^{\frac{1}{4}}}\Big), & \frac{1}{\nu}<x<b\nu.\end{cases} \tag{16}$$

Now we estimate the function $\lambda_{r,r+n}^{\alpha}(x)$ for $0<x\leqslant b\nu$. Let $0<x\leqslant\frac{1}{\nu}$. Then from (2) and (14) it follows that

$$\lambda_{r,r+n}^{\alpha}(x)=O\left(\nu^{\frac{\alpha}{2}}\right)\int\limits_{0}^{x}(x-t)^{r-1}t^{\frac{\alpha}{2}}dt=O\left(\nu^{\frac{\alpha}{2}}x^{r+\frac{\alpha}{2}}\right). \tag{17}$$

Consider the case $\frac{1}{\nu}<x\leqslant b\nu$, $0<b<1$. From (15)–(17) and (8) we obtain

$$\lambda_{r,r+n}^{\alpha}(x)=O\Big(\frac{1}{\nu^r}\Big)+\frac{2^{\alpha-\frac{1}{2}}}{\nu^{\frac{\alpha}{2}-\frac{1}{2}}}\frac{\sqrt{h_n^\alpha}}{(r-1)!}\times$$

$$\times\int\limits_{1/\nu}^{x}\frac{(x-t)^{r-1}\psi^{\frac{1}{2}}}{t^{\frac{1}{2}}(\psi')^{\frac{1}{2}}}\Big[\sqrt{\frac{2}{\pi\nu\psi}}\cos\left(\nu\psi-\frac{\alpha\pi}{2}-\frac{\pi}{4}\right)+O\Big(\frac{1}{(\nu\psi)^{\frac{3}{2}}}\Big)\Big]dt+$$

$$+R_\nu^{\alpha,r}(x)=I_1+I_2+O\Big(\frac{x^{r+\frac{1}{4}}}{\nu^{\frac{3}{2}}(\nu-x)^{\frac{1}{4}}}\Big), \tag{18}$$

where

$$I_2=O\Big(\frac{1}{\nu}\Big)\int\limits_{1/\nu}^{x}\frac{(x-t)^{r-1}}{\sqrt{t}\sqrt{\psi'}\psi}dt=O\Big(\frac{1}{\nu^{\frac{1}{2}}}\Big)\int\limits_{1/\nu}^{x}\frac{(x-t)^{r-1}}{t^{\frac{3}{4}}(\nu-t)^{\frac{1}{4}}}dt=$$

$$=O\Big(\frac{x^{r-\frac{3}{4}}}{\nu^{\frac{1}{2}}(\nu-x)^{\frac{1}{4}}}\Big), \tag{19}$$

$$I_1 = \frac{2^\alpha \sqrt{h_n^\alpha}}{\sqrt{\pi}\nu^{\frac{\alpha}{2}}(r-1)!} \int\limits_{1/\nu}^{x} \frac{(x-t)^{r-1}\cos\left(\nu\psi - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right)}{\sqrt{t\psi'}} dt =$$

$$= \frac{2^{\alpha+\frac{3}{2}} \sqrt{h_n^\alpha}}{\sqrt{\pi}\nu^{\frac{\alpha}{2}}(r-1)!} \int\limits_{1/\nu}^{x} \frac{(x-t)^{r-1} t^{\frac{1}{4}}}{(\nu-t)^{\frac{3}{4}}} \psi'\left(\frac{t}{\nu}\right) \cos\left(\nu\psi - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) dt.$$

Using the second mean-value theorem [6, p.600], we have:

$$I_1 = O(1)\Big| \frac{(1/\nu)^{1/4}}{(\nu - 1/\nu)^{3/4}} \int\limits_{1/\nu}^{\xi} (x-t)^{r-1} \psi'\left(\frac{t}{\nu}\right) \cos\left(\nu\psi - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) dt +$$

$$+ \frac{x^{1/4}}{(\nu - x)^{3/4}} \int\limits_{\xi}^{x} (x-t)^{r-1} \psi'\left(\frac{t}{\nu}\right) \cos\left(\nu\psi - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) dt \Big| =$$

$$= O(1)\Big| \frac{(1/\nu)^{1/4}}{(\nu - 1/\nu)^{3/4}} \Big((x - 1/\nu)^{r-1} \int\limits_{1/\nu}^{y} \psi'\left(\frac{t}{\nu}\right) \cos\left(\nu\psi - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) dt +$$

$$+ (x-\xi)^{r-1} \int\limits_{y}^{\xi} \psi'\left(\frac{t}{\nu}\right) \cos\left(\nu\psi - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) dt \Big) +$$

$$+ \frac{x^{1/4}}{(\nu - x)^{3/4}} (x-\xi)^{r-1} \int\limits_{\xi}^{\tau} \psi'\left(\frac{t}{\nu}\right) \cos\left(\nu\psi - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) dt \Big| =$$

$$= O\Big(\frac{\nu^{\frac{1}{2}} x^{r-1}}{(\nu^2 - 1)^{\frac{3}{4}}}\Big) + O\Big(\frac{x^{r-\frac{3}{4}}}{(\nu - x)^{\frac{3}{4}}}\Big) =$$

$$= O\Big(\frac{x^{r-\frac{3}{4}}}{(\nu - x)^{\frac{3}{4}}}\Big), \quad \frac{1}{\nu} < y \leqslant \xi \leqslant \tau \leqslant x. \tag{20}$$

From (18) and estimates (19), (20) we deduce

$$\lambda_{r,r+n}^\alpha(x) = O\Big(\frac{x^{r-\frac{3}{4}}}{(\nu - x)^{\frac{3}{4}}}\Big), \quad \frac{1}{\nu} < x \leqslant b\nu,\ 0 < b < 1. \tag{21}$$

Now let $\nu-\nu^{\frac{1}{3}}<x<\infty$. From (17) and (21) we get

$$\lambda_{r,r+n}^{\alpha}(x)=\frac{1}{(r-1)!}\Big(\int\limits_{0}^{1/\nu}+\int\limits_{1/\nu}^{\nu-\nu^{\frac{1}{3}}}+\int\limits_{\nu-\nu^{\frac{1}{3}}}^{x}\Big)(x-t)^{r-1}\mathcal{L}_n^{\alpha}(t)dt=$$

$$=O\Big(\frac{1}{\nu^r}\Big)+O\left(\nu^{r-1}\right)+\frac{1}{(r-1)!\sqrt{h_n^{\alpha}}}\int\limits_{\nu-\nu^{\frac{1}{3}}}^{x}(x-t)^{r-1}t^{\frac{\alpha}{2}}e^{-\frac{t}{2}}L_n^{\alpha}(t)dt.$$

We use formula (7):

$$\lambda_{r,r+n}^{\alpha}(x)=O\left(\nu^{r-1}\right)+\frac{(-1)^n\pi^{\frac{1}{2}}2^{\frac{5}{6}}N^{N+\frac{1}{6}}}{(r-1)!\,n!\,e^N\sqrt{h_n^{\alpha}}}\times$$

$$\times\int\limits_{\nu-\nu^{\frac{1}{3}}}^{x}\frac{(x-t)^{r-1}}{t^{\frac{1}{2}}(-\phi')^{\frac{1}{2}}}\Big[Ai(-\nu^{\frac{2}{3}}\phi)+O\Big(\frac{\widetilde{Ai}(-\nu^{\frac{2}{3}}\phi)}{t}\Big)\Big]dt=O\Big(\nu^{r-1}\Big)+$$

$$+\frac{(-1)^n\pi^{\frac{1}{2}}2^{\frac{5}{6}}N^{N+\frac{1}{6}}}{(r-1)!\,n!\,e^N\sqrt{h_n^{\alpha}}}\int\limits_{\nu-\nu^{\frac{1}{3}}}^{x}\frac{(x-t)^{r-1}}{t^{\frac{1}{2}}(-\phi')^{\frac{1}{2}}}Ai(-\nu^{\frac{2}{3}}\phi)dt+V_{\nu}^{\alpha,r}(x), \tag{22}$$

where

$$V_{\nu}^{\alpha,r}(x)=\frac{(-1)^n\pi^{\frac{1}{2}}2^{\frac{5}{6}}N^{N+\frac{1}{6}}}{(r-1)!\,n!\,e^N\sqrt{h_n^{\alpha}}}\int\limits_{\nu-\nu^{\frac{1}{3}}}^{x}\frac{(x-t)^{r-1}}{t^{\frac{1}{2}}(-\phi')^{\frac{1}{2}}}O\Big(\frac{\widetilde{Ai}(-\nu^{\frac{2}{3}}\phi)}{t}\Big)dt. \tag{23}$$

For $\nu-\nu^{1/3}<x\leqslant\nu$ from (9)–(11) and (13) it follows that

$$\frac{Ai(-\nu^{2/3}\phi)}{\sqrt{-\phi'}}=\frac{O(\nu^{-1/6}\phi^{-1/4})}{\sqrt{-\phi'}}=\Big(\frac{t}{\nu-t}\Big)^{1/4}O\Big(\frac{1}{\nu^{1/6}}\Big), \tag{24}$$

$$O\Big(\frac{\widetilde{Ai}(-\nu^{2/3}\phi)}{t\sqrt{-\phi'}}\Big)=\frac{O(\nu^{-1/6}\phi^{-1/4})}{t\sqrt{-\phi'}}=\frac{1}{t}\Big(\frac{t}{\nu-t}\Big)^{1/4}O\Big(\frac{1}{\nu^{1/6}}\Big). \tag{25}$$

Further, from the Stirling formula $n!=n^ne^{-n}\sqrt{2\pi n}(1+O(1/n))$ we have

$$\frac{\pi^{\frac{1}{2}}2^{\frac{5}{6}}N^{N+\frac{1}{6}}}{n!e^N\sqrt{h_n^{\alpha}}}=O(n^{1/6}). \tag{26}$$

From (23), (25) and (26) we get

$$V_\nu^{\alpha,r}(x) = O(1)\int\limits_{\nu-\nu^{\frac{1}{3}}}^{x} \frac{(x-t)^{r-1}}{t^{\frac{5}{4}}(\nu-t)^{\frac{1}{4}}}dt =$$

$$= O\Big(\frac{(x-\nu+\nu^{\frac{1}{3}})^{r-1}(\nu^{\frac{1}{4}}-(\nu-x)^{\frac{3}{4}})}{(\nu-\nu^{\frac{1}{3}})^{\frac{5}{4}}}\Big) = O\Big(\nu^{\frac{r-4}{3}}\Big). \quad (27)$$

For $\nu < x < \infty$, from the definition of the function $\phi = \phi(\tau)$, $\tau = \frac{t}{\nu}$ it follows that

$$-\phi' = \frac{\sqrt{\tau-1}}{2\sqrt{\tau}\sqrt{-\phi}}$$

and

$$-\phi = (\tau-1)\left[\frac{3}{4}\sum_{k=0}^{\infty}\frac{(-1)^k(1/2)_k}{(k+3/2)k!}(\tau-1)^k\right]^{2/3}.$$

This series is of the Leibniz type, therefore,

$$-\phi \geqslant (\tau-1)\left[\frac{3}{4}\left(\frac{2}{3}-\frac{1}{5}(\tau-1)\right)\right]^{2/3} = (\tau-1)\left[\frac{1}{20}(13-3\tau)\right]^{2/3}.$$

Then from (12) and (13) we obtain

$$\frac{Ai(-\nu^{2/3}\phi)}{\sqrt{-\phi'}} = \frac{O\Big(\nu^{-1/6}(-\phi)^{-1/4}\exp(-\frac{2}{3}\nu(-\phi)^{3/2})\Big)}{\sqrt{-\phi'}} =$$

$$= O\Big(\frac{1}{\nu^{1/6}}\Big)\frac{\tau^{1/4}(-\phi)^{1/4}}{(-\phi)^{1/4}(\tau-1)^{1/4}\exp(\frac{2}{3}\nu(-\phi)^{3/2})} \leqslant$$

$$\leqslant O\Big(\frac{1}{\nu^{1/6}}\Big)\frac{t^{1/4}}{(t-\nu)^{1/4}\exp\left(\frac{1}{30}(\frac{t-\nu}{\nu})^{3/2}(13\nu-3t)\right)}, \quad (28)$$

$$O\Big(\frac{\widetilde{Ai}(-\nu^{2/3}\phi)}{t\sqrt{-\phi'}}\Big) = O\Big(\frac{Ai(-\nu^{2/3}\phi)}{t\sqrt{-\phi'}}\Big) \leqslant$$

$$\leqslant O\Big(\frac{1}{\nu^{1/6}}\Big)\frac{1}{t^{3/4}(t-\nu)^{1/4}\exp\left(\frac{1}{30}(\frac{t-\nu}{\nu})^{3/2}(13\nu-3t)\right)}. \quad (29)$$

Taking into account estimates (27) and (29), we deduce

$$V_\nu^{\alpha,r}(x) = O\left(\nu^{\frac{r-4}{3}}\right) + \\ + O(1)\int\limits_\nu^x \frac{(x-t)^{r-1}}{t^{5/4}(t-\nu)^{1/4}} \frac{1}{\exp\left(\frac{1}{30}(\frac{t-\nu}{\nu})^{3/2}(13\nu-3t)\right)}dt \leqslant$$

$$\leqslant O\left(\nu^{\frac{r-4}{3}}\right) + O\left(\frac{1}{\nu^{5/4}}\right)\int\limits_\nu^x \frac{(x-t)^{r-1}}{(t-\nu)^{1/4}\exp\left(\frac{1}{3}\frac{(t-\nu)^{3/2}}{\nu^{1/2}}\right)}dt =$$

$$= O\left(\nu^{\frac{r-4}{3}}\right) + O\left(\frac{1}{\nu^{5/4}}\right)\int\limits_0^{(x-\nu)^{1/4}} \frac{(x-\nu-y^4)^{r-1}y^2}{\exp\left(\frac{y^6}{3\nu^{1/2}}\right)}dy. \tag{30}$$

For $\nu \leqslant x \leqslant \nu + \nu^{1/3}$, we have

$$\int\limits_0^{(x-\nu)^{1/4}} \frac{(x-\nu-y^4)^{r-1}y^2}{\exp\left(\frac{y^6}{3\nu^{1/2}}\right)}dy \leqslant \\ \leqslant (x-\nu)^{r-1}(x-\nu)^{1/2}(x-\nu)^{1/4} = (x-\nu)^{r-\frac{1}{4}} \leqslant \nu^{\frac{r}{3}-\frac{1}{12}}. \tag{31}$$

When $\nu + \nu^{1/3} < x \leqslant \frac{3\nu}{2}$, we get

$$\int\limits_0^{(x-\nu)^{1/4}} \frac{(x-\nu-y^4)^{r-1}y^2}{\exp\left(\frac{y^6}{3\nu^{1/2}}\right)}dy \leqslant \\ \leqslant \nu^{\frac{r}{3}-\frac{1}{12}} + (x-\nu-\nu^{1/3})^{r-1}\int\limits_{\nu^{1/12}}^{(x-\nu)^{1/4}} \frac{y^2dy}{\exp\left(\frac{y^6}{3\nu^{1/2}}\right)} \leqslant \\ \leqslant \nu^{\frac{r}{3}-\frac{1}{12}} + \nu^{r-1}\nu^{\frac{1}{6}}\nu^{\frac{1}{12}}\int\limits_{1/3}^{\frac{(x-\nu)^{3/2}}{3\nu^{1/2}}} \frac{t^{\frac{1}{3}}}{t^{\frac{5}{6}}}e^{-t}dt = \\ = \nu^{\frac{r}{3}-\frac{1}{12}} + \nu^{r-\frac{3}{4}}\int\limits_{1/3}^{\frac{(x-\nu)^{3/2}}{3\nu^{1/2}}} \frac{1}{\sqrt{t}}e^{-t}dt = O\left(\nu^{r-\frac{3}{4}}\right). \tag{32}$$

Let $x > \frac{3\nu}{2}$. In this case, we can write

$$\int\limits_{0}^{(x-\nu)^{1/4}} \frac{(x-\nu-y^4)^{r-1}y^2}{\exp\left(\frac{y^6}{3\nu^{1/2}}\right)}dy = O\big(\nu^{r-\frac{3}{4}}\big) + \int\limits_{(\nu/2)^{1/4}}^{(x-\nu)^{1/4}} \frac{(x-\nu-y^4)^{r-1}y^2}{\exp\left(\frac{y^6}{3\nu^{1/2}}\right)}dy \leqslant$$

$$\leqslant O\big(\nu^{r-\frac{3}{4}}\big) + \left(x-\frac{3\nu}{2}\right)^{r-1} \int\limits_{\frac{\nu}{6\sqrt{2}}}^{\frac{(x-\nu)^{3/2}}{3\nu^{1/2}}} \frac{1}{\sqrt{t}}e^{-t}dt \leqslant$$

$$\leqslant O\big(\nu^{r-\frac{3}{4}}\big) + \left(x-\frac{3\nu}{2}\right)^{r-1} \frac{\sqrt{6\sqrt{2}}}{\sqrt{\nu}}e^{-\frac{\nu}{6\sqrt{2}}} = O\big(\nu^{r-\frac{3}{4}}\big). \tag{33}$$

Therefore, from (27) and (30)–(33) we have

$$V_\nu^{\alpha,r}(x) = \begin{cases} O\big(\nu^{\frac{r-4}{3}}\big), & \nu-\nu^{1/3} < x \leqslant \nu+\nu^{1/3}, \\ O\big(\nu^{r-2}\big), & x > \nu+\nu^{1/3}. \end{cases} \tag{34}$$

So, from relation (22) and estimates (34) we obtain the following asymptotic representation for $\lambda_{r,r+n}^{\alpha}(x)$, $\nu-\nu^{\frac{1}{3}} < x < \infty$:

$$\lambda_{r,r+n}^{\alpha}(x) = \frac{(-1)^n \pi^{\frac{1}{2}} 2^{\frac{5}{6}} N^{N+\frac{1}{6}}}{(r-1)!\, n!\, e^N \sqrt{h_n^\alpha}} \int\limits_{\nu-\nu^{\frac{1}{3}}}^{x} \frac{(x-t)^{r-1}}{t^{\frac{1}{2}}(-\phi')^{\frac{1}{2}}} Ai(-\nu^{\frac{2}{3}}\phi)dt + O\left(\nu^{r-1}\right).$$

In turn, from (24), (26), (28) and (31)–(34) we deduce the estimate

$$\lambda_{r,r+n}^{\alpha}(x) = O\left(\nu^{r-1}\right), \quad \nu-\nu^{\frac{1}{3}} < x < \infty.$$

Theorem 1 is proved.

Note another important property of functions $\lambda_{r,r+n}^{\alpha}(x)$ for $\alpha = 0$ and $r = 1$. As noted in the introduction, Fourier-Sobolev series are a convenient tool for solving initial-value problems for ordinary differential equations. In [17], an iterative method for solving the Cauchy problem for ODEs was developed. It was shown that if the system of functions $\{\varphi_{1,n}(x)\}_{n=0}^{\infty}$ orthonormal with respect to the inner product (1) for $r = 1$ satisfies the condition of the form

$$\kappa_\varphi = \int\limits_a^b \sum_{n=1}^{\infty} (\varphi_{1,n}(x))^2 w(x)dx < \infty,$$

then the iterative process converges. When $\alpha = 0$, $r = 1$, we get

$$\kappa_\varphi = \int\limits_0^\infty e^{-2ax} \sum_{n=0}^\infty (\lambda^0_{1,1+n}(x))^2 dx.$$

The following statement holds:

Theorem 2. *Let $\frac{1}{2} < a \in \mathbb{R}$. Then the following estimate holds:*

$$\int\limits_0^\infty e^{-2ax} \sum_{n=0}^\infty (\lambda^0_{1,1+n}(x))^2 dx \leqslant \frac{1}{(2a-1)2a}.$$

Proof. We use the second mean-value theorem [6, p. 600]:

$$\int\limits_0^x e^{-t/2} L^0_n(t)dt = \int\limits_0^\xi L^0_n(t)dt + e^{-x/2} \int\limits_\xi^x L^0_n(t)dt, \quad 0 \leqslant \xi \leqslant x.$$

Then

$$(\lambda^0_{1,1+n}(x))^2 = \left((1-e^{-x/2}) \int\limits_0^\xi L^0_n(t)dt + e^{-x/2} \int\limits_0^x L^0_n(t)dt\right)^2 =$$

$$= (1-e^{-x/2})^2 \Big(\int\limits_0^\xi L^0_n(t)dt\Big)^2 + e^{-x} \Big(\int\limits_0^x L^0_n(t)dt\Big)^2 +$$

$$+ 2(1-e^{-x/2})e^{-x/2} \int\limits_0^\xi L^0_n(t)dt \int\limits_0^x L^0_n(t)dt.$$

From (3) and (4), we have

$$\int\limits_0^y L^0_n(\tau)d\tau = \frac{y}{n+1} L^1_n(y).$$

Hence,

$$(\lambda^0_{1,1+n}(x))^2 = (1-e^{-x/2})^2 \left(\frac{\xi}{n+1} L^1_n(\xi)\right)^2 + e^{-x} \Big(\frac{x}{n+1} L^1_n(x)\Big)^2 +$$

$$+\,2(1-e^{-x/2})e^{-x/2}\frac{\xi x}{(n+1)^2}L_n^1(\xi)L_n^1(x), \quad 0\leqslant\xi\leqslant x.$$

Further, from the formula (5) (for $\alpha = 1$) it follows that

$$\sum_{n=0}^{\infty}\frac{\xi x}{(n+1)^2}L_n^1(\xi)L_n^1(x) = e^{\xi+x}\gamma(1,\xi)\Gamma(1,x) =$$
$$= e^{\xi+x}\int_0^{\xi}e^{-t}dt\int_x^{\infty}e^{-t}dt = e^{\xi+x}(1-e^{-\xi})e^{-x} = (e^{\xi}-1), \quad 0\leqslant\xi\leqslant x.$$

Then we have

$$\sum_{n=0}^{\infty}(\lambda^0_{1,1+n}(x))^2 = (1-e^{-\frac{x}{2}})^2(e^{\xi}-1) + e^{-x}(e^x-1)+$$
$$+\,2(1-e^{-\frac{x}{2}})e^{-\frac{x}{2}}(e^{\xi}-1)\leqslant e^x-1.$$

Thus,

$$\int_0^{\infty}e^{-2ax}\sum_{n=0}^{\infty}(\lambda^0_{1,1+n}(x))^2dx\leqslant\int_0^{\infty}e^{-2ax}(e^x-1)dx = \frac{1}{(2a-1)2a}.$$

□

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References

[1] Askey R., Wainger S. *Mean convergence of expansions in Laguerre and Hermite series.* Amer. J. Math., 1965, vol. 87, pp. 698 – 708.
DOI: https://doi.org/10.2307/2373069

[2] Ciaurri Ó., Mínguez-Ceniceros J. *Fourier series of Jacobi - Sobolev polynomials.* Integral Transforms Spec. Funct., 2019, vol. 30, no. 4, pp. 334 – 346.
DOI: https://doi.org/10.1080/10652469.2018.1560279

[3] Erdélyi A. *Asymptotic forms for Laguerre polynomials.* J. Indian Math. Soc., 1960, vol. 24, pp. 235 – 250.

[4] Everitt W. N., Kwon K. H., Littlejohn L. L., Wellman R. *On the spectral analysis of the Laguerre polynomials $\{L_n^{-k}(x)\}$ for positive integers k.* In *Spectral theory and computational methods of Sturm-Liouville problems (Knoxville, TN, 1996)*, 251 – 283, Lecture Notes in Pure and Appl. Math., 191, Dekker, New York, 1997.

[5] Everitt W. N., Littlejohn L. L., Wellman R. *The Sobolev orthogonality and spectral analysis of the Laguerre polynomials $\{L_n^{-k}\}$ for positive integers k.* J. Comput. Appl. Math., 2004, vol. 171, no. 1-2, pp. 199 – 234.
DOI: https://doi.org/10.1016/j.cam.2004.01.017

[6] Fikhtengolts G. M. *Course of Differential and Integral Calculus.* Moscow: Fizmatlit, 2001, vol. 2. (in Russian)

[7] Gadzhimirzaev R. M. *Sobolev-orthonormal system of functions generated by the system of Laguerre functions.* Probl. Anal. Issues Anal., 2019, vol. 8 (26). no. 1, pp. 32 – 46.
DOI: https://doi.org/10.15393/j3.art.2019.5150

[8] Gadzhimirzaev R. M. *Integral estimates for Laguerre polynomials with exponential weight function.* Russian Math. (Iz. VUZ), 2020, vol. 64, no. 4, pp. 12 – 20. DOI: https://doi.org/10.3103/S1066369X20040027

[9] Jeffreys H, Jeffreys B. S. *Methods of mathematical physics.* Cambridge Univ. Press, Cambridge, 1956. Third edition.

[10] Kwon K. H., Littlejohn L. L. *Sobolev orthogonal polynomials and second-order differential equations.* Rocky Mountain J. Math. 1998, vol. 28, no. 2, pp. 547 – 594. DOI: https://doi.org/10.1216/rmjm/1181071786

[11] Marcellán F., Xu Y. *On Sobolev orthogonal polynomials.* Expo. Math., 2015, vol. 33, no. 3, pp. 308 – 352.
DOI: https://doi.org/10.1016/j.exmath.2014.10.002

[12] Muckenhoupt B. *Asymptotic forms for Laguerre polynomials.* Proc. Amer. Math. Soc., 1960, vol. 24, no. 2, pp. 288 – 292.
DOI: https://doi.org/10.2307/2036349

[13] Muckenhoupt B. *Mean convergence of Hermite and Laguerre series. II.* Trans. Amer. Math. Soc., 1970, vol. 147, no. 2, pp. 433 – 460.
DOI: https://doi.org/10.2307/1995205

[14] Pérez T. E., Piñar M. A. *On Sobolev orthogonality for the generalized Laguerre polynomials.* J. Approx. Theory, 1996, vol. 86, no. 3, pp. 278 – 285.
DOI: https://doi.org/10.1006/jath.1996.0069

[15] Prudnikov A. P., Brychkov Yu. A., Marichev O. I. *Integrals and Series. Vol 2. Special functions. Additional chapters.* M.: Fizmatlit, 2003. Second edition. (in Russian)

[16] Sharapudinov I. I. *Sobolev-orthogonal systems of functions associated with an orthogonal system.* Izv. Math., 2018, vol. 82, no. 1, pp. 212 – 244. DOI: https://doi.org/10.1070/IM8536

[17] Sharapudinov I. I. *Sobolev-orthogonal systems of functions and some of their applications.* Russian Math. Surveys, 2019, vol. 74, no. 4. pp. 659 – 733. DOI: https://doi.org/10.1070/RM9846

[18] Szegö G. *Orthogonal Polynomials.* Amer. Math. Soc. Colloq. Publ, 23, Amer. Math. Soc., Providence R. I. 1975. Fourth Edition.

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