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RATIONAL APPROXIMATIONS OF LIPSCHITZ FUNCTIONS FROM THE HARDY CLASS ON THE LINE

Abstract. We study a rate of uniform approximations on the real line of summable Lipschitz functions f having a summable Hilbert transform Hf by normalized logarithmic derivatives of rational functions. Inequalities between different metrics of the logarithmic derivatives of algebraic polynomials on the line are also considered.

Key words: *logarithmic derivative of a rational function, simple partial fraction, Hilbert transform, uniform approximation, inequality between different metrics*

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1. Main result. Denote the Hilbert transform of a function f by Hf,

$$Hf(x) := \frac{1}{\pi} \lim_{\varepsilon\to+0} \int\limits_{|t-x|>\varepsilon} \frac{f(t)}{t-x}\,dt, \quad -\infty < x < +\infty.$$

In this paper, by using the results of [10] and certain properties of the Hilbert transform, we obtain upper bounds for uniform approximations on $\mathbb{R}$ of a sufficiently wide subclass of real-valued functions f by rational fractions R_n, $n = 2, 3, \ldots$, of the special form

$$R_n(x) = \frac{C}{n}\left(\frac{P'(x)}{P(x)} - \frac{Q'(x)}{Q(x)}\right), \quad C = C(f) > 0, \tag{1}$$

where P, Q are real polynomials of degree $n-1$. Here the expression in brackets is a difference of the so-called *simple partial fractions*

$$\sum_{k=1}^{n-1} \frac{1}{x - z_k}. \tag{2}$$

Setting $r_{n-1} = P/Q$, we represent the fraction (1) in the form of the normalized logarithmic derivative of a rational function of degree $n-1$:

$$R_n(x) = \frac{C}{n} \cdot \frac{r'_{n-1}(x)}{r_{n-1}(x)}, \quad \deg R_n = 2n-2.$$

Further we denote $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{R})}$ $(1 \leqslant p \leqslant \infty)$ and $L := L^1(\mathbb{R})$. We write $f \in \mathrm{Lip}_\alpha\{A; E\}$, if a function f is defined on a set $E \subseteq \mathbb{R}$ and there are constants $\alpha \in (0,1]$ and $A > 0$, such that

$$|f(x_1) - f(x_2)| \leqslant A|x_1 - x_2|^\alpha$$

for any two points $x_1, x_2 \in E$. By $C_0(\mathbb{R})$ we denote the class of continuous on $\mathbb{R}$ functions having the zero limit as $x \to \pm\infty$.

Theorem 1. *Let $n = 2, 3, \ldots$. Let a real-valued function f belong to $C_0(\mathbb{R})$ and let the following conditions hold:*

1) *$f \in \mathrm{Lip}_\alpha\{A; \mathbb{R}\}$ with some $\alpha \in (0,1)$, $A > 0$,*
2) *$f \in L$, $Hf \in L$.*

Then there are real polynomials Q_1, Q_2 of degree $n-1$, such that

$$\left| f(x) - \frac{\|Hf\|_1}{4\pi n} \frac{r'(x)}{r(x)} \right| < \frac{c(f)}{n^{\alpha/(1+\alpha)}}, \quad -\infty < x < +\infty,$$

where $r(x) = Q_1(x)/Q_2(x)$; $c(f) > 0$ is a constant depending only on f.

Theorem 1 is proved in Section 3; we show that the constant

$$c(f) = 4A(\alpha\pi)^{-1}\mathrm{B}\big(\tfrac{\alpha}{2}, \tfrac{1-\alpha}{2}\big) + 2\alpha^{-1}\|Hf\|_1$$

is suitable (here $\mathrm{B}(x,y)$ is Euler's beta-function). Our proof uses the well-known implication: $f \in \mathrm{Lip}_\alpha\{A; \mathbb{R}\}$ $(\alpha \in (0,1)) \Rightarrow Hf \in \mathrm{Lip}_\alpha\{\tilde{A}; \mathbb{R}\}$. Here the restriction $\alpha < 1$ is essential; nevertheless, if the assumptions of Theorem 1 hold for $\alpha = 1$ and, additionally, $Hf \in \mathrm{Lip}_1\{\tilde{A}; \mathbb{R}\}$, then the estimate given in Theorem 1 holds with the bound

$$c(f)/\sqrt{n}, \quad c(f) := 4\tilde{A} + 2\|Hf\|_1.$$

For a small α, the order of approximation $O(n^{-\alpha/(1+\alpha)}) \approx O(n^{-\alpha})$, established by the theorem, cannot be essentially improved in the following sense: if approximations of a function f by the class of all rational functions of degree $2n-2$ have order $O(n^{-\alpha_0-\varepsilon})$ with $\alpha_0 \in (\alpha, 1)$ (while $\varepsilon > 0$ is arbitrarily small) for all $n = 2, 3, \dots$, then, by the Gonchar converse theorem [6], f satisfies the Lipschitz condition of degree α_0 almost everywhere on $\mathbb{R}$ (in contrast to the condition $f \in \mathrm{Lip}_\alpha\{A; \mathbb{R}\}$, where $\alpha < \alpha_0$).

Borodin and Kosukhin [2] have proved that any function $f \in C_0(\mathbb{R})$ can be approximated uniformly on $\mathbb{R}$ by sums of the form (2) with poles z_k outside any given strip $|\mathrm{Im}\, z| <$ const. In our construction, all poles of the approximating functions R_n (i.e., the zeros of the polynomials Q_1, Q_2) lie on the two lines $\mathrm{Im}\, z = \pm n^{-1/(1+\alpha)}$, so that $|\mathrm{Im}\, z_k| \to 0$ as $n \to \infty$.

Some estimates of uniform approximations on $\mathbb{R}$ of certain functions f by differences of simple partial fractions were obtained in [9]. For example, an order of such approximations is $O(n^{-1})$ if a function f has the form

$$f(x) = \frac{x}{(1+x^2)^2} F\left(\frac{1-x^2}{1+x^2}\right), \quad -\infty < x < +\infty,$$

with some function $F(t) \in \mathrm{Lip}_1\{A; [-1, 1]\}$. Concerning the uniform approximation rate by simple partial fractions themselves (not by their differences) on the whole real axis recall the result by Danchenko [4]: for any function f of the form

$$f(x) = f_a(x) = -\frac{1}{x-a}, \quad a \in \mathbb{C} \setminus \mathbb{R},$$

and sufficiently large $n \geqslant n_0(a)$ there is a complex polynomial P of degree n, such that

$$\left| f_a(x) - P'(x)/P(x) \right| < C \cdot \ln\ln n / \ln n, \quad -\infty < x < +\infty,$$

where $C > 0$ is a constant depending only on a (the order of approximation cannot be improved). At the end of Section 3, we discuss the rate of the uniform approximation by *normalized* simple partial fractions.

2. Some remarks on the assumptions of Theorem 1. The class of functions f, such that $f \in L$ and $Hf \in L$, is called [12, p. 165] the *Hardy class* $H_1(-\infty,\infty)$. Thus, the second condition of Theorem 1 can be written as follows: $f \in H_1(-\infty, \infty)$. For example, the class $H_1(-\infty, \infty)$ contains the derivative R' of any bounded on $\mathbb{R}$ rational function R, because of the Rusak inequality

$$\|R'\|_1 + \|H(R')\|_1 \leqslant 4\pi n \|R\|_\infty, \quad n = \deg R$$

(see [12, p. 165]). Further nontrivial examples of functions $f \in H_1(-\infty, \infty)$ can be found in the paper by Kober [8].

Protasov [13] described the class $V_p = V_p(\mathbb{R})$ of functions $f \in L^p(\mathbb{R})$, $p \in (1, \infty)$, that can be approximated in $L^p(\mathbb{R})$ by sums of the form

$$\sum_{k=1}^{N} \frac{p_k}{x - z_k}, \quad p_k \geqslant 0. \tag{3}$$

In particular, [13, Corollary 1], if a function f belongs to $L^p(\mathbb{R})$ and is real-valued, then $f \in V_p$ if and only if $Hf(x) \geqslant 0$ for almost all $x \in \mathbb{R}$.

Let us show that a nonzero function f, satisfying the conditions of Theorem 1, cannot be approximated by sums (3) in $L^p(\mathbb{R})$.

Proposition 1. *Let a real-valued function f belong to $L^p(\mathbb{R})$, $p \in (1, \infty)$. Then $f \in V_p \cap H_1(-\infty, \infty)$ if and only if $f(x) = 0$ a.e.*

Proof. The sufficient condition is obvious. To prove the necessary condition, we first recall the result of Kober [8, Theorem 1]:

$$f \in L, \quad Hf \in L \quad \Rightarrow \quad \int_{-\infty}^{\infty} f(x)dx = 0. \tag{4}$$

On the other hand (Hille and Tamarkin, see [8, Lemma 2]), we have[1]

$$f \in L, \quad Hf \in L \quad \Rightarrow \quad HHf = -f \quad \text{a.e.} \tag{5}$$

Thus, if $f \in H_1(-\infty, \infty)$, then $\tilde{f} := Hf \in L$ and, by (5), $H\tilde{f} = -f \in L$; by applying (4) to the function $\tilde{f}$, we get

$$f \in H_1(-\infty, \infty) \quad \Rightarrow \quad \int_{-\infty}^{\infty} Hf(x)dx = 0. \tag{6}$$

But if $f \in V_p$, then $Hf(x) \geqslant 0$ a.e. [13]. Hence, for any function $f \in V_p \cap H_1(-\infty, \infty)$ we have $Hf(x) = 0$ a.e. Therefore, $f(x) = 0$ a.e. by $f = -HHf$, see (5). □

Let us formulate another simple observation concerning the class V_p.

Proposition 2. *Let an even real-valued function f belong to $L^p(\mathbb{R})$, $p \in (1, \infty)$. Then $f \in V_p$ if and only if $f(x) = 0$ a.e.*

[1] Of course, we also have $HHf = -f$ (a.e.) due to $f \in L^p(\mathbb{R})$, $p > 1$ [7, p. 148].

Proof. Indeed, the Hilbert transform Hf of an even function f is odd (see [7, p. 146]). But if an odd function $Hf(x)$ is non-negative (a.e.), then $Hf(x) = 0$ (a.e.). Finally, we use, again, the formula $f = -HHf$, which is correct due to $f \in L^p(\mathbb{R})$. □

The results of Danchenko [4] yield that the functions $f_a(x)$ (see Section 1) cannot be approximated by simple partial fractions in $L^p(\mathbb{R})$ with finite p. In particular, this remark is also true for the real-valued function

$$g(x) := -\frac{2x}{x^2+1} \equiv -\frac{1}{x+i} - \frac{1}{x-i}.$$

At the end of Section 4, we establish that the normalized logarithmic derivatives $Q'(x)/(nQ(x))$ of real polynomials $Q(x)$ rapidly converge to $g(x)$ on the line in $L^p(\mathbb{R})$ with any $1 < p \leqslant \infty$. Note that $g \in V_p$ for all $1 < p < \infty$ by the theorem of Protasov, because

$$Hg(x) = \frac{i}{x+i} + \frac{-i}{x-i} = \frac{2}{x^2+1} \geqslant 0$$

(see [7, p. 104] for explicit values of $H((x+i\alpha)^{-1})$ with nonzero α).

Although the class V_p is narrow, Protasov has showed [13, Remark 1] that any function $f \in L^p(\mathbb{R})$, $p \in (1, \infty)$, can be approximated in $L^p(\mathbb{R})$ by differences of sums of the form (3). Obviously, the normalized logarithmic derivatives of rational functions, see (1), belong to the space of such differences.

3. Proof of Theorem 1. Put $\tilde{f} = Hf$. Since $f \in \mathrm{Lip}_\alpha\{A; \mathbb{R}\}$ with $\alpha \in (0,1)$, it follows by the theorem of Aleksandrov [1] that

$$\tilde{f} \in \mathrm{Lip}_\alpha\{\tilde{A}; \mathbb{R}\}, \quad \tilde{A} = A\pi^{-1}\mathrm{B}\big(\tfrac{\alpha}{2}, \tfrac{1-\alpha}{2}\big).$$

Let us write the real-valued function $\tilde{f}$ in the form $\tilde{f} = \tilde{f}_1 - \tilde{f}_2$,

$$\tilde{f}_1(x) := \max\{\tilde{f}(x); 0\} \geqslant 0, \quad \tilde{f}_2(x) := \max\{-\tilde{f}(x); 0\} \geqslant 0.$$

Both functions $\tilde{f}_k$ also belong to the class $\mathrm{Lip}_\alpha\{\tilde{A}; \mathbb{R}\}$: for example, the identity $\tilde{f}_1(x) = (\tilde{f}(x) + |\tilde{f}(x)|)/2$ and the triangle inequality yield

$$\big|\tilde{f}_1(x_1) - \tilde{f}_1(x_2)\big| \leqslant \tfrac{1}{2}\big|\tilde{f}(x_1) - \tilde{f}(x_2)\big| + \tfrac{1}{2}\big||\tilde{f}(x_1)| - |\tilde{f}(x_2)|\big| \leqslant \big|\tilde{f}(x_1) - \tilde{f}(x_2)\big|.$$

By the assumptions of the theorem, $\tilde{f} \in L$. Hence, $\tilde{f}_1, \tilde{f}_2 \in L$ and

$$\|\tilde{f}\|_1 = \int_{-\infty}^{\infty} \tilde{f}_1(x)dx + \int_{-\infty}^{\infty} \tilde{f}_2(x)dx = \|\tilde{f}_1\|_1 + \|\tilde{f}_2\|_1.$$

From this, we get $\|\tilde{f}_1\|_1 = \|\tilde{f}_2\|_1 = \frac{1}{2}\|\tilde{f}\|_1$ using the formula (6).

Further, we can assume $\|\tilde{f}\|_1 > 0$. Both functions

$$F_k(x) := \tilde{f}_k(x)/\|\tilde{f}_k\|_1 = 2\tilde{f}_k(x)/\|\tilde{f}\|_1, \quad k = 1; 2,$$

are non-negative and

$$\|F_k\|_1 = 1, \quad F_k \in \mathrm{Lip}_\alpha\{A^*; \mathbb{R}\} \quad (A^* := 2\tilde{A}/\|\tilde{f}\|_1).$$

By [10, Theorem 3], there are real polynomials Q_1, Q_2 of degree $n-1$, such that

$$\left| HF_k(x) + \frac{1}{2\pi n}\frac{Q_k'(x)}{Q_k(x)} \right| < \frac{2A^* + 2}{\alpha n^{\alpha/(1+\alpha)}}, \quad -\infty < x < \infty, \quad k = 1; 2.$$

Namely (see [10, Lemma 2]), we can take

$$Q_k(x) = \prod_{j=1}^{n-1} \left((x_{k,j} - x)^2 + n^{-2/(1+\alpha)} \right), \quad k = 1; 2,$$

where the points $x_{k,0} = -\infty < x_{k,1} < \ldots < x_{k,n-1} < x_{k,n} = \infty$ are defined by

$$\int\limits_{x_{k,j}}^{x_{k,j+1}} F_k(x)dx = \frac{1}{n}, \quad j = 0, \ldots, n-1.$$

Hence,

$$\left| HF_1(x) - HF_2(x) + \frac{1}{2\pi n}\frac{r'(x)}{r(x)} \right| < \frac{4A^* + 4}{\alpha n^{\alpha/(1+\alpha)}}, \quad -\infty < x < +\infty,$$

where $r(x) := Q_1(x)/Q_2(x)$ and

$$HF_1(x) - HF_2(x) = 2\|\tilde{f}\|_1^{-1} H(\tilde{f}_1(x) - \tilde{f}_2(x)) = -2\|Hf\|_1^{-1} f(x)$$

by (5). Theorem 1 is proved. □

By using very similar arguments, we easily obtain the following assertion, which complements the theorem in the case when $f \notin L$ and $f \in L^p(\mathbb{R})$, $p > 1$.

Proposition 3. *Let $p \in (1,\infty)$, $n = 2, 3, \ldots$ Let a real-valued function f belong to $C_0(\mathbb{R}) \cap L^p(\mathbb{R})$. If the function Hf is nonnegative, $Hf \in L$*

and $Hf \in \mathrm{Lip}_\alpha\{\tilde{A};\mathbb{R}\}$ *with some* $\alpha \in (0,1]$, $\tilde{A} > 0$, *then there is a real polynomial* Q *of degree* $n-1$, *such that*

$$\left| f(x) - \frac{\|Hf\|_1}{2\pi n}\frac{Q'(x)}{Q(x)} \right| < 2 \cdot \frac{\tilde{A} + \|Hf\|_1}{\alpha n^{\alpha/(1+\alpha)}}, \quad -\infty < x < +\infty.$$

Note that any function f, satisfying the conditions of Proposition 3, belongs to V_p. If, moreover, $f \not\equiv 0$, then $f \notin L$ (sf. Proposition 1).

Proof. Assume that $d := \|Hf\|_1 > 0$ and set $F(x) = Hf(x)/d$. The function F is nonnegative and $\|F\|_1 = 1$, $F \in \mathrm{Lip}_\alpha\{\tilde{A}/d;\mathbb{R}\}$. By [10, Theorem 3], there is a real polynomial Q of degree $n-1$, such that

$$\left| HF(x) + \frac{1}{2\pi n}\frac{Q'(x)}{Q(x)} \right| < \frac{2\tilde{A} + 2d}{\alpha n^{\alpha/(1+\alpha)} d}, \quad -\infty < x < +\infty.$$

By $f \in L^p(\mathbb{R})$, we have $f = -HHf$. Hence, $HF(x) \equiv -f(x)/d$, and the assertion follows. The case $d = 0$ is trivial ($f \equiv 0$). □

4. Inequalities between different metrics for simple partial fractions. Nikol'skii inequalities for simple partial fractions

$$\rho_n(z) = \sum_{k=1}^{n} \frac{1}{z - z_k}$$

were studied by many authors (see, for example, [5], [3] and references therein). Let us recall one result of the paper by Chunaev and Danchenko [3], stated as Theorem 4.5: for any $z_1, \ldots, z_n \in \mathbb{C}\backslash\mathbb{R}$ and $1 < p < q \leqslant \infty$,

$$\|\rho_n\|_q^{q'} \leqslant 2^{q'-p'} \left(\frac{\kappa_p}{\pi}\right)^{p'q'(\frac{1}{p}-\frac{1}{q})} (1 + c_p)^{p'} \|\rho_n\|_p^{p'}, \tag{7}$$

where κ_p is a unique natural number, which belongs to $[\frac{p}{2}, \frac{p}{2}+1)$,

$$\frac{1}{q} + \frac{1}{q'} = 1 = \frac{1}{p} + \frac{1}{p'}, \quad c_p = \begin{cases} \tan\frac{\pi}{2p}, & 1 < p \leqslant 2, \\ \cot\frac{\pi}{2p}, & 2 \leqslant p < \infty, \end{cases}$$

$c_p \geqslant 1$ is the norm of the Hilbert transform in $L^p(\mathbb{R})$. A similar inequality with a bigger constant was first obtained by Danchenko and Dodonov in the paper [5], where the authors raised the problem of finding a better upper bound for the ratio $\|\rho_n\|_q^{q'}/\|\rho_n\|_p^{p'}$.

Thus, our goal is to improve the constant factor in the estimate (7). If all z_k are non-real, then $\rho_n(x)$ is bounded on the real line:

$$M := \|\rho_n\|_\infty < \infty.$$

Putting $q = \infty$ in (7), we get

$$M \leqslant 2^{1-p'} \left(\frac{\kappa_p}{\pi}\right)^{p'/p} (1 + c_p)^{p'} \|\rho_n\|_p^{p'}, \quad 1 < p < \infty. \tag{8}$$

But

$$1 - p' = p'\left(\frac{1}{p'} - 1\right) = -\frac{p'}{p},$$

therefore, (8) can be written in the form

$$M \leqslant \left(\frac{\kappa_p}{2\pi}\right)^{p'/p} (1 + c_p)^{p'} \|\rho_n\|_p^{p'}, \quad 1 < p < \infty. \tag{9}$$

Now, let $q < \infty$. Since $|\rho_n(x)| \leqslant M$ at points $x \in \mathbb{R}$, we see that

$$\int_{-a}^{a} |\rho_n(x)|^q dx = \int_{-a}^{a} |\rho_n(x)|^{q-p} |\rho_n(x)|^p dx \leqslant M^{q-p} \int_{-a}^{a} |\rho_n(x)|^p dx$$

for $q > p$ and any $a > 0$. Letting $a \to \infty$, we get

$$\|\rho_n\|_q^q \leqslant M^{q-p} \|\rho_n\|_p^p, \quad 1 < p < q,$$

because ρ_n belongs to all the spaces $L^p(\mathbb{R})$, $p > 1$.

Using the estimate (9) and the transformation

$$(q-p)\frac{p'q'}{q} = pq\left(\frac{1}{p} - \frac{1}{q}\right)\frac{p'q'}{q} = p(p' - q'),$$

we obtain

$$\|\rho_n\|_q^{q'} \leqslant M^{(q-p)q'/q} \|\rho_n\|_p^{pq'/q} \leqslant \left(\frac{\kappa_p}{2\pi}(1 + c_p)^p\right)^{p'-q'} \|\rho_n\|_p^{p(p'-q')+pq'/q}.$$

Observe that

$$p(p' - q') + \frac{pq'}{q} = pq'\left(\frac{p'}{q'} - 1 + \frac{1}{q}\right) = pp'\left(1 - \frac{1}{p'}\right) = p'.$$

Thus, we have proved the following result:

Theorem 2. *For any simple partial fraction ρ_n without poles on $\mathbb{R}$:*

$$\|\rho_n\|_q^{q'} \leqslant \left(\frac{\kappa_p}{2\pi}(1+c_p)^p\right)^{p'-q'} \|\rho_n\|_p^{p'}, \quad 1<p<q\leqslant\infty. \tag{10}$$

For $q=\infty$, the estimate (10) coincides with the result of Chunaev and Danchenko (7), because of the equality $p(p'-1)=p'$. However, for any $q<\infty$, Theorem 2 is stronger than (7), since in this case

$$p(p'-q') = p'(p'-q')/(p'-1) < p'$$

and, therefore,

$$(1+c_p)^{p(p'-q')} < (1+c_p)^{p'}.$$

Even more, in contrast to (7), the estimate (10) has the following important property: the left-hand side of the estimate tends to the right-hand side as $q\to p$.

Our next purpose is to establish some (q,∞) Nikol'skii inequalities for differences of simple partial fractions. Let Θ be a weak norm of the Hilbert transform, i.e., the smallest possible value of a constant C in the Kolmogorov inequality

$$m(\{x\in\mathbb{R} : |Hf(x)|\geqslant\delta\}) \leqslant C\|f\|_1/\delta,$$

where f is any real-valued summable function and $m(E)$ denotes the Lebesgue measure of a set $E\subset\mathbb{R}$. Recall that [7, p. 338]

$$\Theta = \frac{\pi^2/8}{1-3^{-2}+5^{-2}-\dots} = 1.347\dots$$

It was proved in [10] that if r is a real rational function of degree n and $\mu(r,\delta) := m(\{x\in\mathbb{R} : |r'(x)/r(x)|\geqslant\delta\})$, then, for any $\delta>0$,

$$\mu(r,\delta) \leqslant 2\pi\Theta\cdot n/\delta, \tag{11}$$

where the constant $2\pi\Theta$ cannot be replaced by a smaller value. Note that (11) can be formulated as follows: for any real rational function r of degree n and $\delta>0$ there is a set $E=E(r,\delta)\subset\mathbb{R}$, such that $m(E)\leqslant\delta$ and

$$|r'(x)| \leqslant 2\pi\Theta\cdot\frac{n}{\delta}\,|r(x)|, \quad x\in\mathbb{R}\backslash E.$$

The last estimate was first obtained (with a bigger factor $C \ln n$ instead of $2\pi\Theta$) by Gonchar [6] and used by him in the proof of the converse theorem, mentioned in Section 1 above.

Estimates of the quantity $\mu(r,\delta)$ are well-known in the case of complex polynomials $r = P$ by the works of Macintyre and Fuchs, Govorov and Grushevskii and others (see details and references in [10]). For example, the famous result by Macintyre and Fuchs (1940) is

$$\mu(P,\delta) \leqslant 2e \cdot n/\delta, \quad n = \deg P \quad (\delta > 0).$$

The best possible result [11] for *real* polynomials P of degree n is

$$\mu(P,\delta) \leqslant \pi \cdot n/\delta \quad (\delta > 0). \tag{12}$$

Using (11) and (12), we easily establish the following extension of theorem 3 of the paper [5], where the case of complex polynomials $r = P$ is considered:

Theorem 3. *Let $1 < q \leqslant \infty$ and let E be an arbitrary bounded or unbounded segment of $\mathbb{R}$. Then, for any real rational function r of degree n without poles and zeros on E we have*

$$\|R\|^{q'}_{L^q(E)} \leqslant \left(2\pi\Theta \cdot nq'\right)^{q'/q} \|R\|_{L^\infty(E)}, \quad 1/q + 1/q' = 1,$$

where $R(x) = r'(x)/r(x)$. Moreover, if $r(x) = P(x)$ is a real polynomial of degree n, i.e., $R(x) \equiv \rho_n(x)$ is a real-valued simple partial fraction, then the constant $2\pi\Theta$ in this estimate can be replaced by π.

Proof. Set $M = \|R\|_{L^\infty(E)}$. By the assumptions of the theorem, we have $M < \infty$ and $R \in L^q(E)$ for all $q > 1$. Next, we have [7, p. 233]

$$\|R\|^q_{L^q(E)} = q \int_0^M \tilde{\mu}(\delta)\delta^{q-1} d\delta, \quad \tilde{\mu}(\delta) := m(\{x \in E : |R(x)| \geqslant \delta\}).$$

But $\tilde{\mu}(\delta) \leqslant \mu(r,\delta)$, hence, by (11):

$$\|R\|^q_{L^q(E)} \leqslant q \int_0^M 2\pi\Theta \cdot n \cdot \delta^{q-2} d\delta = 2\pi\Theta \cdot nq' M^{q-1},$$

and the first assertion of the theorem follows. Analogously, the second assertion follows from (12). □

Corollary. *Let* $n = 2, 3, \ldots$ *and* $g(x) = -2x/(x^2 + 1)$. *There is a real polynomial* Q *of degree* $n - 1$, *such that for every* $1 < q \leqslant \infty$

$$\left\| g - \frac{1}{n}\frac{Q'}{Q} \right\|_q^{q'} < \frac{c_q}{\sqrt{n}}, \quad c_q := 4(\pi + 1)(3\pi q')^{q'/q}, \quad 1/q + 1/q' = 1.$$

Proof. Recall that $Hg(x) = 2/(x^2 + 1)$ (see Section 2), therefore,

$$Hg \in L, \quad \|Hg\|_1 = 2\pi, \quad Hg \in \mathrm{Lip}_1\{2; \mathbb{R}\}.$$

By applying Proposition 3, we get existence of a real polynomial Q of degree $n - 1$, such that

$$\|\Delta\|_\infty < \frac{4(\pi + 1)}{\sqrt{n}} \equiv \frac{c_\infty}{\sqrt{n}} \qquad \left(\Delta(x) := \frac{1}{n}\frac{Q'(x)}{Q(x)} - g(x)\right).$$

Now consider the logarithmic derivative $R(x) := h'(x)/h(x)$, where

$$h(x) := Q(x)(x^2 + 1)^n$$

is a real polynomial of degree $(n - 1) + 2n < 3n$. But $R(x) \equiv n\Delta(x)$; hence, by Theorem 3, we have

$$\|\Delta\|_q^{q'} = n^{-q'}\|R\|_q^{q'} \leqslant n^{-q'}(\pi \cdot \deg h \cdot q')^{q'/q}\|R\|_\infty \leqslant$$

$$\leqslant n^{1-q'}(\pi \cdot 3n \cdot q')^{q'/q}\|\Delta\|_\infty = n^{1-q'+q'/q}(3\pi q')^{q'/q}\|\Delta\|_\infty,$$

where $1 - q' + q'/q = 0$. Thus, the result follows from this and the previous estimate of $\|\Delta\|_\infty$. □

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References

[1] Aleksandrov A. B., *Norm of the Hilbert transformation in a space of Hölder functions.* Funct. Anal. Appl., 1975, vol. 9, no. 2, pp. 94–96. DOI: https://doi.org/10.1007/BF01075444

[2] Borodin P. A., Kosukhin O. N., *Approximation by the simplest fractions on the real axis.* Moscow Univ. Math. Bull., 2005, vol. 60, no. 1, pp. 1–6. https://zbmath.org/?q=an%3A1101.41016

[3] Chunaev P. V., Danchenko V. I., *Quadrature formulas with variable nodes and Jackson–Nikolskii inequalities for rational functions.* J. Approx. Theory, 2018, vol. 228, pp. 1 – 20.
DOI: https://doi.org/10.1016/j.jat.2017.12.004

[4] Danchenko V. I., *Estimates of the distances from the poles of logarithmic derivatives of polynomials to lines and circles.* Sb. Math., 1995, vol. 82, no. 2, pp. 425 – 440.
DOI: https://doi.org/10.1070/SM1995v082n02ABEH003573

[5] Danchenko V. I., Dodonov A. E., *Estimates for L_p-norms of simple partial fractions.* Russian Math. (Iz. VUZ), 2014, vol. 58, no. 6, pp. 6 – 15.
DOI: https://doi.org/10.3103/S1066369X14060024

[6] Gonchar A. A., *Converse theorems concerning best approximations by rational functions.* Izv. Akad. Nauk SSSR Ser. Mat., 1961, vol. 25, pp. 347 – 356 (in Russian).
DOI: https://mathscinet.ams.org/mathscinet-getitem?mr=125382

[7] King F. W., *Hilbert transforms. Vol. 1.* Encyclopedia of mathematics and its applications (vol. 124). Cambridge Univ. Press, Cambridge, 2009.
DOI: https://doi.org/10.1017/CBO9780511721458

[8] Kober H., *A note on Hilbert's operator.* Bull. Amer. Math. Soc., 1942, vol. 48, pp. 421 – 427.
DOI: https://doi.org/10.1090/S0002-9904-1942-07688-9

[9] Komarov M. A., *Approximation by linear fractional transformations of simple partial fractions and their differences.* Russian Math. (Iz. VUZ), 2018, vol. 62, no. 3, pp. 23 – 33.
DOI: https://doi.org/10.3103/S1066369X18030040

[10] Komarov M. A., *Distribution of the logarithmic derivative of a rational function on the line.* Acta Math. Hungar., 2021, vol. 163, no. 2, pp. 623 – 639. DOI: https://doi.org/10.1007/s10474-020-01102-w

[11] Komarov M. A., *Metric property of a real polynomial.* Bull. London Math. Soc., 2021, vol. 53, no. 2, pp. 336 – 346.
DOI: https://doi.org/10.1112/blms.12421

[12] Petrushev P. P., Popov V. A., *Rational approximation of real functions.* Encyclopedia of mathematics and its applications (vol. 28). Cambridge Univ. Press, Cambridge, 1987.
DOI: https://doi.org/10.1017/CBO9781107340756

[13] Protasov V. Yu. *Approximation by simple partial fractions and the Hilbert transform.* Izv. Math., 2009, vol. 73, no. 2, pp. 333 – 349.
DOI: https://doi.org/10.1070/IM2009v073n02ABEH002449

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